ELEMENTS of HADRONIC MECHANICS

Volume II:

THEORETICAL FOUNDATIONS

Ruggero Maria Santilli

ELEMENTS of
HADRONIC MECHANICS

Volume II:

THEORETICAL FOUNDATIONS

Ruggero Maria Santilli
The Institute for Basic Research
P. O. Box 1577
Palm Harbor, FL 34682, USA

Second Edition

– 1995 –

EXCERPTS FROM THE REVIEWS

A. Jannussis (Univ. of Patras, Greece): *"Hadronic Mechanics supersedes all theories to date."*
(opening address of the International Conference on the Frontiers of Physics. Olympia, Greece, 1993)

H. P. Leipholz (Univ. of Waterloo, Canada): *"Santilli's studies are truly epoch making."*

J. V. Kadeisvili (Intern. Center of Phys., Kazakhstan): *"Santilli's Lie-isotopic and Lie-admissible generalizations of the algebraic, geometric and analytic foundations of Lie's theory are of clear historical proportions."*

A. U. Klimyk (Inst. for Theor. Phys., Ukraine): *"The three books on Hadronic Mechanics are the most authoritative for a study of the Lie-isotopic and Lie-admissible generalizations of Lie's theory and their many applications."*

D. F. Lopez (Univ. of Campinas, Brasil): *"Santilli succeeded, first, in reaching a structural generalization of the available mathematics as a pre-requisite for his generalization of current physical theories. These achievements are unprecedented in the history of physics."*

A. O. E. Animalu (Univ. of Nsukka, Nigeria): *"Because of its beauty, mathematical consistency and range of applicability vastly beyond quantum mechanics, if we deny the historical character of Hadronic Mechanics we exit the boundaries of science."*

T. L. Gill (Howard Univ., Washington, D. C.): *"The three volumes on Hadronic Mechanics represent the most important contribution to physics in the last fifty years."*

Dedicated to my spiritual guide and friend

Father DOMINIC CALARCO, S. X., Dr. Miss.

*because of his teaching that
the problems of contemporary societies,
including problems in the current condition of physical research,
are of a primary ethical nature.*

TABLE OF CONTENTS

PREFACE

These three volumes are the first books written on a generalization of quantum mechanics under the name of *hadronic mechanics* which I proposed back in 1978 when at Harvard University under support from the U. S. Department of Energy and then studied by a number of mathematicians, theoreticians and experimentalists.

The primary objective of the new mechanics is to reach a quantitative treatment in a form suitable for experimental verifications of the old legacy that the strong interactions in general, and the hadronic structure in particular, have a nonlocal component due to the experimentally established mutual penetration and overlapping of the wavepackets-wavelengths-charge distributions of hadrons, as symbolically represented below
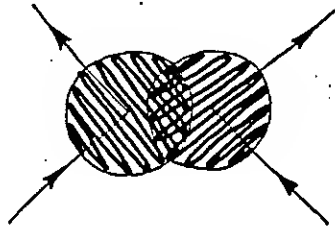
under the condition that the representation preserves all the essential physical characteristics of quantum mechanics, such as observability, causality, etc.

I have presented the mathematical foundations of hadronic mechanics in Volume I. The scope of this Volume II is the identification of the theoretical foundations of the new mechanics, with particular reference to the basic physical laws for an axiomatically correct treatment of the above physical conditions. Applications and experimental verifications will be studied in Volume III.

The fundamental hypothesis of hadronic mechanics is the generalization of the basic unit of the enveloping operator algebra of quantum mechanics into a form with an arbitrary, integro-differential dependence

$$I \;=\; \text{diag.} \,(1,\, 1,\, \dots) \quad \rightarrow \quad \hat{1}(t,\, r,\, p,\, \dot{p},\, \psi,\, \partial\psi,\, \partial\partial\psi,\, \dots).\qquad\qquad (1)$$

The description of physical systems in hadronic mechanics then requires *two* operators, the Hamiltonian $H = K + V$ for the representation of all interactions derivable from a potential V, and the generalized unit $\hat{1}$ for the representation of all interactions and internal effects which are not derivable from a potential or a Hamiltonian by conception.

A first simple example of generalized unit is given by the diagonal and positive–definite form in three dimension $\hat{1} = \text{Diag.} \,(b_1^{-2},\, b_2^{-2},\, b_3^{-2})$ which permits a direct representation of non*spherical* charge distributions of hadrons, such as ellipsoids with semiaxes b_k^2, as well as all their infinitely possible *deformations*. In turn, these capabilities of hadronic mechanics (which are manifestly absent in quantum mechanics) permit the first exact–numerical representation on record of the total magnetic moment of the deuteron, tritium and of few–body nuclei.

An example illustrating the nonlocal–integral character of the theory is *Animalu's generalized unit* $\hat{1} = \exp\{N\!\int dv\psi^\dagger(r)\phi(r)\}$ for the electrons of the Cooper pair in superconductivity with wavefunctions ψ and ϕ, which permits the first representation on record of their *attractive interactions* in a way remarkably in agreement with experimental data. Numerous additional examples will be identified during the course of our analysis, such as the generalized unit permitting the first numerical representation on record of the experimental data on the Bose–Einstein correlation as originating from non*local interactions* in the interior of the $p\text{-}\bar{p}$ fireballs.

As studied in Vol. I, the generalization of the unit requires, for evident reason of compatibility, a consequential generalization of the entire mathematical formalism of quantum mechanics into a new formalism admitting of $\hat{1}$ as the correct left and right unit. This includes a generalization of: fields of real and complex numbers; vector, metric and pseudo–metric spaces; Euclidean, Minkowskian and Riemannian geometries; ordinary functions (e.g., trigonometric functions), special functions (e.g., spherical functions), transform (e.g., Fourier transform), distributions (e.g., Dirac delta distribution); Banach and Hilbert spaces; Lie algebras, Lie groups and Lie symmetries; transformations and representation theories; classical Hamiltonian mechanics; etc.

Generalized units (1) are classified into *Hermitean generalizations* $\hat{1} = \hat{1}^\dagger$, characterizing the *Lie–isotopic branch of hadronic mechanics,* and non*hermitean generalizations* $\hat{1} \neq \hat{1}^\dagger$, characterizing the more general *Lie–admissible branch of hadronic mechanics.*

The former methods are used for the treatment of closed–isolated systems of particles with Hamiltonian and nonhamiltonian internal interactions verifying conventional total conservation laws, including the reversibility of the center–of–mass trajectory. The latter formulations are used for the

characterization of open-nonconservative systems in irreversible conditions under the most general known, external, nonlinear-nonlocal-nonhamiltonian interactions.

Each of the above two branches is then classified into *Kadeisvili's five different classes* depending on the primary topological characteristics of the generalized unit. This illustrates the rather diversified structure of hadronic mechanics for the characterization of a hierarchy of physical systems with increasing complexity and methodological needs. By comparison, quantum mechanics has one single structure, as well known.

By using the above diversified structure, in this volume I establish the *direct universality* of hadronic mechanics, i.e., the capability of the theory of representing all possible operator systems (universality) directly in the frame of the experimenter (direct universality). As we shall see, this includes a representation of gravitational singularities represented via the zeros of the generalized units.

I should stress that the above direct universality refers to *systems* and does not include various other methods for the treatment of the same systems. As an example, hadronic mechanics represents all possible systems described by the so-called q-deformations (and actually much more as we shall see), but *hadronic mechanics and q-deformations are structurally inequivalent,* e.g., because defined on different fields, different Hilbert spaces, etc.

As we shall see, hadronic mechanics permits a variety of novel interpretation of existing data, applications and predictions which are simply beyond any hope of quantitative treatment with quantum mechanics. In particular, the new mechanics permits novel structure models of nuclei, hadrons and stars with a variety of new predictions all verifiable with current technology, such as: a quantitative formulation of an*tigravity;* the prediction of a *space-time machine* based on the alteration of the un*its* of space and time; an apparent new form of subnuclear energy I called *hadronic ene*rgy; a new technology at distances smaller than 10^{-13} cm I called *hadronic technology;* and others. In this volume I present the *theoretical foundations* permitting these predictions. Their treatment and experimental verification will be considered in Vol. III.

The a*lgebraic origin* of hadronic mechanics can be traced back to A. A. Albert (Trans. Amer. Math. Soc. **64**, 552 (1948)) who introduced the notion of *Lie-admissible algebras* as generally nonassociative algebras U with elements a, b, .. and abstract product ab which are such that the attached algebras U⁻ with product $[a, b]_U = ab - ba$ are Lie. Albert introduced the above notion for the primary purpose of studying the so-called *noncommutative Jordan algebras* with realization of the product

$$(a, b) = \lambda \, a \, b - (1 - \lambda) \, b \, a , \qquad (2)$$

where ab is associative and λ is a scalar, which do not possess a well defined content of Lie algebra (i.e., there is no finite value of λ under which product (b) is Lie). As such, their possible physical relevance is unknown at this time.

The notion of Lie-admissible algebras U used in these volume was introduced by R. M. Santilli in his Ph. D. studies (Lett. Nuovo Cimento **51**, 570 (1967)) according to the *dual condition* that U⁻ is Lie a*nd* that Lie algebras are contained in the classification of U. This turns Lie-admissible algebras into genuine coverings of Lie algebras. Santilli [loc. cit.] then introduced the realization

$$(a , b) = p\,a\,b - q\,b\,a,\qquad(3)$$

where ab is associative and p, q are scalars.

Subsequently, Santilli (Hadronic J. **1**, 223, 574 and 1279 (1978)) introduced the so-called *general realization of Lie-admissible algebras*

$$(A , B) = A\,R\,B - B\,S\,A,\qquad(4)$$

where R, S, R±S are nonsingular operators, whose attached antisymmetry product

$$[A , B]_U = (A , B) - (B , A) = A\,T\,B - B\,T\,A,\quad T = R + S,\qquad(5)$$

does satisfy the Lie algebra axioms but it is not conventionally Lie and characterizes instead Lie-isotopic algebras. In this way I reached in 1978 the central dynamical equations of for closed-conservative and open-nonconservative systems, respectively [loc. cit.]

$$i\,dA / dt = [A , H]_U = A\,T\,H - H\,T\,A,\qquad(6)$$

$$i\,dA / dt = (A , H)_U = A\,R\,H - H\,S\,A,\qquad(7)$$

which are at the foundations of hadronic mechanics.

The *classical analytic origin* of hadronic mechanics can be traced back to G. D. Birkhoff (*Dynamical systems*, A.M.S., Providence, RI (1927)) who identified the following generalization of Hamilton's equations

$$\Omega_{\mu\nu}(a)\,\frac{da^{\nu}}{dt} = \frac{\partial H}{\partial a^{\mu}},\qquad(8)$$

$$\Omega_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu,\quad \det\Omega \neq 0,\quad a = \{\,r^k, p_k\,\},\quad k = 1, 2, 3,\quad \mu, \nu = 1, 2, ..., 6,$$

which preserves the abstract axioms of Hamiltonian mechanics, thus being an *isotopy* of the latter, because the underlying two form $\Theta = \Omega_{\mu\nu}da^{\mu}\wedge da^{\nu}$ is an exact, nowhere degenerate and therefore symplectic as the canonical two-form, although of the most general possible exact type. Similarly, the brackets characterized by the contravariant tensor $\Omega^{\mu\nu} = |\,(\Omega_{\alpha\beta})^{-1}|^{\mu\nu}$ do verify the Lie

axioms, although they characterize the more general Lie–isotopic algebras.

Some fifty six years following their appearance, I conducted systematic studies of the largely forgotten Eq.s (8), which I called *Birkhoff's equations* (R. M. Santilli, *Foundations of Theoretical Mechanics*, Vol. II: *Birkhoffian Generalization of Hamiltonian mechanics*, Springer–Verlag, Heidelberg–New York (1983)) and prove them to be directly universal for all possible local–differential Newtonian systems verifying certain topological conditions.

However, the emerging step–by–step generalization of Hamiltonian mechanics, which I called *Birkhoffian mechanics,* resisted all attempts for quantization into a form usable for practical applications. This forced me to conduct a second laborious study of the *classical foundations* of hadronic mechanics (R.M. Santilli, Hadronic J. Suppl. **4B**, issue no. 1 (1988); *Isotopic Generalizations of Galilei's and Einstein's Relativities*, Vol.s I and II, Second Edition, Ukraine Academy of Sciences, Kiev (1994)) which resulted in the construction of: A) the more general *Birkhoff–isotopic mechanics* with basic equations

$$\Omega_{\mu\alpha}(a)\, T^{\alpha}{}_{\nu}(t, a, \dot{a}, ...) \; \frac{da^{\nu}}{dt} \; = \; \frac{\partial H}{\partial a^{\mu}},\qquad (9)$$

where T is a nowhere degenerate *symmetric* matrix; B) the *symplectic isotopic geometry* with nowhere–degenerate exact two–isoforms $\hat{\theta} = \Omega_{\mu\alpha}T^{\alpha}{}_{\nu}da^{\mu}\wedge da^{\nu}$; and C) the classical realization of the Lie–isotopic algebra characterized by the contravariant tensor $\hat{1}^{\mu}{}_{\alpha}\,\Omega^{\alpha\nu} = |\,(\Omega_{\rho\sigma}(a)\,T^{\sigma}{}_{\tau})^{-1}\,|^{\mu\nu}$.

The above studies permitted the identification of the *classical origin of the fundamental quantity of hadronic mechanics, the generalized unit* $\hat{1}$. In turn, this permitted the identification of unique and unambiguous generalized methods for the mapping of classical into operator theories, and established the direct universality of hadronic mechanics as originating at the classical level. In fact, the Newtonian systems mapped into quantum mechanics must be restricted to admit a meaningful Hamiltonian, while such restriction is un–necessary for hadronic mechanics which provides the operator image of the most general possible Newtonian systems which are arbitrarily nonlinear, nonlocal–integral and nonhamiltonian.

The *statistical origin* of hadronic mechanics can be traced back to the studies by I. Prigogine and his Bruxelles School (see, e.g., Cl. George et al., Hadronic J. <u>1</u>, 520 (1978)) which are based on a non*unitary* formulation of conventional quantum statistics. Subsequently, J. Fronteau, Tellez–Arenas and R. M. Santilli (Hadronic J. <u>3</u>, 130 (1979)) formulated a generalization of statistical mechanics with a Lie–admissible structure. More recently, A. Jannussis and R. Mignani (Physica <u>A187</u>, 575 (1992)) proved that the nonunitary irreversible statistics of Prigogine's school has an essential Lie–admissible structure.

The above results permitted hadronic mechanics to identify the origin

of irreversibility in the ultimate level of the structure of matter, that of elementary particles in open-nonconservative conditions, such as a proton in the core of a star considered as external. This elementary origin of irreversibility is studied in this volume at the nonrelativistic, relativistic and gravitational levels. Macroscopic irreversibility is then a mere collection of *elementary* nonunitary-irreversible systems.

These advances resolves recently identified inconsistencies of the conventional conception of irreversibility, such as those treated by the so-called *No-Reduction Theorems* which establish the impossibility of reducing a macroscopic body in irreversible conditions with continuously decaying angular momentum to an ideal collection of quantum mechanical particles in reversible conditions and with conserved angular momentum (and viceversa). The only way I know of resolving these inconsistencies is by identifying the origin of irreversibility in the ultimate elementary level of matter with a consequential, *necessary* generalization of quantum mechanics.

It is important in these introductory notes to indicate the reasons why hadronic mechanics was constructed via the Lie-isotopic and Lie-admissible methods rather than other generalized approaches existing in the literature. Consider, for instance, the so-called *q-deformations* by L. C. Biedenharn (J. Phys. **A22**, L873 (1989)), A. J. MacFarlane (J. Phys. **A22**, L4581 (1989)) and many others, with generic product

$$(a , b) = a b - q b a ,$$ (10)

which, on one side, generalize quantum mechanics while, on the other side, are treated via conventional quantum mechanical methods (e.g., conventional fields, metric spaces, Hilbert spaces, etc.).

Even though I originated the q-deformations some twenty two years before Biedenharn, MacFarlane and the others,[1] I was forced to abandon them by the late 1970's because of a considerable number of rather serious problematic aspects of physical character, such as: lack of form-invariance under the time evolution of the theory; loss of Hermiticity, and therefore of observability, under the time evolution (see below why); loss of the measurement theory because of the lack of invariance of the assumed unit under the time evolution; lack of uniqueness of generalized operations such as exponentiation, with consequential

---

[1] It should be noted that Biedenharn and MacFarlane did not quote in their 1989 papers on q-deformations the historical paper by Albert (Trans. Amer. Math. Soc. **64**, 552 (1948)) on the essential Lie-admissible character of their deformations, or the preceding more general (p, q)-number and (R, S)-operator-deformations by R.M. Santilli (Lett. Nuovo Cimento **51**, 570 (1967); Hadronic J. **1**, 574 (1978)), or any of the related literature on hadronic mechanics which, by 1989, was rather considerable. The quotation of the above prior literature was then ignored in the subsequent vast literature in q-deformations as the reader can verify.

ambiguities in the related generalized physical laws (such as q–uncertainties); lack of validity of the q–special functions at all times; and others.

In fact, the time evolution of q–deformations (10) is evidently noncanonical and therefore *nonunitary*. This implies: the lack of conservation in time of the basic unit of the theory, the conventional form I = diag. $(1, 1, 1, ...)$, because nonunitary transforms are such that, by definition, $I' = UIU^\dagger \neq I$; the lack of form–invariance of the theory,

$$U(ab - qba)U^\dagger = a'Rb' - b'Sa', \tag{11}$$

$$R = (UU^\dagger)^{-1}, \quad S = qR, \quad a' = UAU^\dagger, \quad b' = UbU^\dagger. \tag{12}$$

The other problematic aspects then follow.

Even though transforms (11) have the (R, S)–*operator*–structure (7), they are unacceptable for hadronic mechanics because again not form–invariant. In fact, a second nonunitary transform $WW^\dagger = \hat{T}^{-1} \neq I$ establishes their lack of form–invariance,

$$W a'Rb'W^\dagger - Wb'Sa'W^\dagger = a''\hat{T}R'\hat{T}b'' - b''\hat{T}S'\hat{T}a'', \tag{13}$$

thus leaving all original problematic aspects essentially unchanged.

The only way I know to achieve an *axiomatic Lie–admissible theory*, that is, a theory possessing the same axiomatic properties of quantum mechanics (form–invariance, Hermiticity–observability at all times, invariance of the basic unit, uniqueness of the various operations, validity of functional analysis at all times, etc.) is by reformulating the theory according to the basic axioms of the genotopic branch of hadronic mechanics studied in this volume (the most general possible branch over genofields, genospaces, genohilbert spaces, etc.).

Another line of inquiry which I had to abandon for the construction of hadronic mechanics is that of the so–called *nonlinear theories*. I am here referring to theories studied by R. W. Hasse (J. Math. Phys. **16**, 2005 (1975)), N. Gisin (J. Phys. **A 14**, 2259 (1981), .H.–D. Doebner and G. A. Goldin (Phys. Lett. **A 162**, 397 (1992)) and several others with a *nonlinearity in the wavefunction* represented with the conventional Schrödinger's equation

$$i \partial_t \psi(t, r) = H(t, r, p, \psi, \psi\dagger, ...) \psi(t, r), \tag{14}$$

which also generalize quantum mechanics, yet are treated via conventional quantum methods.

In fact, the above approach generally represents open–nonconservative systems and, as such, it is expected to have nonhermitean Hamiltonians and *nonunitary time evolutions,* in which case they have the same problematic aspects of the q–deformations. Irrespective of whether the Hamiltonian is

Hermitean or not, Eq. (14) have the additional problematic aspects caused by the evident loss of the superposition principle.

The only way known to this author of resolving the above problematic aspects is via the axioms of hadronic mechanics which imply the factorization of all nonlinear contributions $H(t, r, p, \psi, \psi^\dagger, ...) = H_0(t, r, p)T(\psi, \psi^\dagger, ...)$, and then the reconstruction of the entire formalism of quantum mechanics with respect to the generalized unit $1 = T^{-1}$.

Yet another line of research which I had to be abandoned for the construction of hadronic mechanics is the *theory with nonassociative Lie-admissible envelope* submitted by S. Weinberg (Ann. Phys. **194**, 336 (1989)) according to the following basic equation

$$ i \frac{dA}{dt} = A \hat{\times} H - H \hat{\times} H = \frac{\partial A}{\partial \psi_k} \frac{\partial H}{\partial \psi_k^\dagger} - \frac{\partial H}{\partial \psi_k} \frac{\partial A}{\partial \psi_k^\dagger}, \qquad (15) $$

where the product $A\hat{\times}H$ characterizes a nonassociative Lie-admissible algebra.[2] Note that the equations are also nonlinear in the wavefunctions as Eq.s (14).

The reasons why this latter approach had to be abandoned are known since the early studies of general Lie-admissible theories. In fact, *when used as the enveloping algebra* $A\hat{\times}H$ of the time evolution $idA/dt = A\hat{\times}H - H\hat{\times}A$[3] *nonassociative* Lie-admissible algebras admit no unit at all, thus preventing any applicability of the measurement theory. Also they admit no known exponentiation, thus preventing the achievement of a consistent generalized symmetry as well as consistent generalized physical laws dependent on the exponentiations (such as Gaussians, the uncertainties, etc.). Moreover, Weinberg's theory violates the *No-Quantization Theorem* by S. Okubo (Hadronic J. **5**, 1667 (1989)) according to which the Heisenberg's type and the Schrödinger-type formulations are inequivalent for all theories with nonassociative envelopes.

By looking now in retrospective, the only known generalizations of quantum mechanics which are axiomatic in the sense indicated earlier can be essentially derived as follows. First, let us recall the axiomatic structure of quantum mechanics as embedded, say, in its fundamental commutation rules

---

[2] S. Weinberg abstained from quoting in his 1989 paper the contributions by Albert of 1948, Santilli of 1967 and 1978 quoted earlier, or other contributions in hadronic mechanics (which were rather numerous by 1989), in order to identify the *nonassociative Lie-admissible character of the envelope* of his his equations. All subsequent papers on Eq.s (15) also did not quote the above essential literature, as the reader can verify.

[3] Recall that in quantum mechanics *the envelope is associative* with conventional product AH while *the brackets of the time evolution are nonassociative*, i.e., $idA/dt = AH - HA$ = nonassociative-Lie. I therefore refer in the text to to the problematic aspects suffered by nonassociative Lie-admissible *envelopes* with product $A\hat{\times}H$ and *not* to theories with Lie-admissible *brackets* $idA/dt = (A, H) = ARH - HSA$ = nonassociative Lie-admissible.

$$r p - p r = i \hbar 1, \tag{16}$$

formulated on a conventional Hilbert space $\mathcal{K}$ over a complex field. As it is well known, the enveloping operator algebra ξ with elements r, p and their polynomial combinations is *associative* with conventional product rp and fundamental unit $1 = \text{diag.} (1, 1, ...)$, $1A = A1 = A, \forall A \in \xi$. The time evolution of the theory is unitary, i.e., represented by the operator U such that $UU^\dagger = U^\dagger U = 1$. This implies: the invariance of the basic unit at all times, $1' = U1U^\dagger \equiv 1$, with consequential applicability of the measurement theory at all times; the preservation of the Hermiticity of the Hamiltonian $H = UHU^\dagger$ at all times with consequential observability at all times; the form-invariance of the theory,

$$U(r p - p r)U^\dagger = r'p' - p'r' = i\hbar U1U^\dagger = i\hbar 1, \tag{17}$$

and the remaining conventional axiomatic properties.

The Lie-isotopic generalization of quantum mechanics preserving all the above axiomatic properties can be constructed as follows. First, recall from Vol. 1 that the Lie-isotopic algebras are a non*unitary* image of conventional Lie algebras. One can therefore subject rule (16) to a nonunitary transformation $UU^\dagger = \hat{1} \neq 1$, for which we have

$$U(r p - p r)U^\dagger = r'Tp' - p'Tr' = i\hbar U1U^\dagger = i\hbar\hat{1} = i\hbar T^{-1}, \tag{18}$$

where one should note that $\hat{1} = UU^\dagger$ and $T = (UU^\dagger)^{-1}$ are *Hermitean*.

This first step renders ne*cessary* the following isotopic generalizations: 1) the enveloping algebra ξ with product AB is lifted into the form $\hat{\xi}$ with isoassociative product $A*B = ATB$, T fixed; 2) the Lie product $[A, B]_{\vee\xi} = AB - BA$ is lifted into the Lie-isotopic product $[A, B]_{\hat{\xi}} = ATB - BTA$; and 3) the fundamental unit 1 is lifted into the generalized quantity $\hat{1}$ of Eq. (1) which is indeed the correct left and right unit of the theory, $\hat{1}*A = T^{-1}TA \equiv A*\hat{1} \equiv A, \forall A \in \hat{\xi}$.

However, the above formulation is still insufficient because under an additional nonunitary time evolution $WW^\dagger = D \neq 1$, the generalized unit $\hat{1}$ is not preserved, $\hat{1}' = W\hat{1}W^\dagger \neq \hat{1}$, and the Lie-isotopic product is not form-invariant. Also, the envelope is now *isoassociative* with Hermiticity condition on $\mathcal{K}$ $H^\dagger = TH^\dagger T^{-1} \neq H^\dagger$, $T = (UU^\dagger)^{-1}$, and general loss of Hermiticity-observability according to *Lopez's* Lemma II.3.C.1 of p. 122 (D. F. Lopez, Hadronic J. 16, 429 (1993)), etc.

The *only* solution I know resolving *all* these problematic aspects is that along the axioms of the isotopic branch of hadronic mechanics. In this case, nonunitary transforms $WW^\dagger \neq 1$, can always be reformulated in the isounitary form

$$W = \hat{W}\, T^{1/2}, \quad W\, W^\dagger = \hat{W}\, T\, \hat{W}^\dagger = W * \hat{W}^\dagger = \hat{W}^\dagger\, T\, \hat{W} = \hat{W}^\dagger * W = \hat{1} = \hat{1}^\dagger = T^{-1}, \quad (19)$$

yielding the *fundamental isocommutation rule* of hadronic mechanics

$$r\, T\, p - p\, T\, r = i\, \hbar\, T^{-1}, \tag{20}$$

first identified by Santilli (Hadronic J. Suppl. **4B**, 1 (1989)).

It is then easy to prove the form-invariance of: the generalized unit $\hat{1}' = \hat{W}*\hat{1}*\hat{W}^\dagger = \hat{W}\,T\,T^{-1}\,T\,\hat{W}^\dagger = \hat{1}$; the isoassociative product $\hat{W}*(A*B)*\hat{W}^\dagger = A'*B'$; the Lie product; and, consequently the fundamental isocommutation rules

$$\hat{W}*(r*p - p*r)*\hat{W}^\dagger = r'*p' - p'*r' = i\,\hbar\,\hat{W}*\hat{1}*\hat{W}^\dagger = i\,\hbar\,\hat{1}. \tag{21}$$

The preservation of all other axiomatic features of quantum mechanics, including the Hermiticity–observability at all times, is then consequential, as we shall see. As a matter of fact, hadronic and quantum mechanics emerge as coinciding at the abstract, realization free level, as one can see from the abstract identity of Eq.s (16) and (20).

It should be noted that all structures which deviate from Eq. (20) violate one of the other axiom of hadronic mechanics, therefore resulting in problematic aspects for physical applications. For instance, structures with commutation rules of the generalized Lie type

$$r\, p - p\, r = i\,\hbar\, F(q, ...), \quad r\, T(q, ...)\, p - p\, T(q, ..)\, r = i\,\hbar\, F(q, ...), \quad F \neq T^{-1}, \tag{22}$$

do not possess an axiomatic structure and, as such, are afflicted by the problematic aspects indicated earlier. In fact, they are not form-invariant even when expressed in isofields, isospaces and isohilbert spaces, thus suffering all the shortcomings of the conventional q-deformations.

Note that this is the fate also for the so-called *quantum groups* because they preserve the conventional quantum–Lie brackets but generalize their eigenvalues, thus implying nonunitary transforms with all the problematic aspects of q-deformations.

The achievement of an axiomatic structure for the more general Lie-admissible formulations is evidently more complex. In this case the dynamical equations describe irreversible systems and therefore require the selection of a given direction in time. In fact, Eq. (7) can be written $idA/dt = ARH - HSA = A<H - H>A$, where $>$ represents motion forward to future time, and $<$ represents motion forward from past times. The generalized unit is necessarily nonhermitean, thus requiring two different units one for the isoproduct $>$ and the other for the conjugate product $<$.

The Lie–admissible theory which is axiomatic in the above sense to my

best knowledge at this writing is characterized by the equations

$$i\, dA\,/\,dt = A\,R\,H - H\,S\,A \; = A < H \; - \; H > A \; = \quad \begin{cases} i\,\hbar\,1^> = \; i\,\hbar\,S^{-1} \\ \text{or} \\ i\,\hbar\,^<1 \; = \; i\,\hbar\,R^{-1} \end{cases} \qquad (23)$$

$$R = S^\dagger, \qquad\qquad\qquad (24)$$

formulated over a dual generalization of the entire quantum mechanical formalism, one per each direction of time (the genofields, genospaces and genohilbert spaces mentioned earlier), which were first identified by Santilli (Hadronic J. **1**, 574 (1978) and Hadronic J. Suppl. **4B**, 1 (1989)).

I should stress that the above structures are the *only* axiomatically consistent formulations which I know at this writing. In fact, structures of the type

$$r\,R\,p \; - \; p\,S\,r \; = \; i\,\hbar\,R^{-1} \text{ or } = i\,\hbar\,S^{-1} \text{ but } \; R \neq S^\dagger, \qquad (25)$$

$$r\,R(q,\ldots)\,p \; - \; p\,S(q,\ldots)\,r \; = \; i\,\hbar\,T(q,\ldots)\,, \quad T \neq R^{-1} \text{ and } T \neq S^{-1}\,, \qquad (26)$$

violate one of another axiom of hadronic mechanics therefore resulting in one or another problematic aspect for physical applications.

The following additional comments are recommendable in these introductory words. Hadronic mechanics studies physical conditions fundamentally different then those of quantum mechanics. In fact, the latter studies the motion of point-like particles in the homogeneous and isotropic vacuum, such as an electron of an atomic cloud (exterior problem), while the former studies the more general class of extended-nonspherical particles moving within inhomogeneous and anisotropic physical media, such as a proton in the core of a star (interior problem). In particular, hadronic mechanics has no impact for the atomic structure because, by construction, it recovers quantum mechanics identically for all mutual distances greater than 1 fm (10^{-13} cm). The differences in the mathematical structures between quantum and hadronic mechanics should therefore be interpreted as a representation of said physical differences.

A most insidious aspect in the study of hadronic mechanics is the rather widespread tendency of appraising it via the mathematical methods of quantum mechanics. This attitude leads to a host of misrepresentations and inconsistencies which often remain undetected, such as assuming that the the magnitude of the angular momentum is $J^2 = JJ$ with respect to the unit I (rather than the correct form $J^2 = JT(x, \dot{x}, \psi, \partial\psi,\ldots)J$ with respect to the unit T^{-1}) which, for hadronic mechanics, violates linearity and all basic axioms of the theory. The appraisal of hadronic mechanics via the formalism of quantum mechanics is equivalent to the appraisal of quantum mechanics via the formalism of Newtonian mechanics.

In particular, the transition from Newtonian to quantum mechanics did

imply certain necessary mathematical generalizations, most notably the use of infinite–dimensional Hilbert spaces, although fundamental mathematical notions such as numbers, angles, vector spaces, trigonometry, special functions, integral transforms, etc., remained common to both classical and quantum disciplines.

In the transition from quantum to hadronic mechanics the *totality* of its mathematical structure must be generalized in a simple yet effective way, as indicated earlier. In view of this occurrence, I have attempted to render this volume self–sufficient for a first study of hadronic mechanics. However, its technical knowledge can only be acquired following a study of the mathematical foundations of Vol. I.

The theoretical foundations of hadronic mechanics are the result of the efforts of numerous scholars identified in the various chapters. Among the mathematicians I mention here the contributions to the Lie–isotopic theory by A. U. Klimyk, D. S. Sourlas and G. F. Tsagas, and those to the Lie–admissible theory by H. C. Myung. Among the physicists who participated in the earlier study of the theory besides myself, I mention the contributions by A. O. E. Animalu, A. K. Aringazin, G. D. Brodimas, G. Eder, J. Fronteau, M. Gasperini, A. Kalnay, A. Jannussis (and other associates), R. Mignani, M. Mijatovic, M. Nishioka, S. Okubo, A. Tellez–Arenas, B. Veljanosky and others. The experimentalists who contributed to hadronic mechanics will be identified in Volume III.

In closing allow me to indicate that a primary objective of hadronic mechanics is the introduction of two, sequential, Lie–isotopic and Lie–admissible generalizations of the Galilean, special and general relativities for nonlinear, nonlocal–integral and nonhamiltonian systems in closed–reversible and open–irreversible conditions, respectively. Readers with the personal conviction that current relativities have a final character for all possible physical conditions existing in the Universe are discouraged from inspecting these volumes. On the contrary, readers with the "young minds of all ages" (mentioned in the preface of Volume I) may find the content of these volume stimulating.

All true scientists (in Einstein's definition) are encouraged to participate in the laborious scientific process of trial and error toward truly fundamental advances, not in marginal talks in academic corridors, but in the only way physical knowledge really advances, via publications.

Summer 1994

Ruggero Maria Santilli
The Institute for Basic Research
P.O.Box 1577, Palm Harbor, FL 34682 USA
Fax 1–813–934 9275
E–mail: i b r r m s @ p i n e t . a i p . o r g

PREFACE TO THE SECOND EDITION

In this second edition I have corrected a number of misprints and errors of the first edition which were kindly brought to my attention by a number of readers.

I have also updated a number of references, particularly those on key issues of hadronic mechanics, according to a number of papers recently appeared in various Journals.

I have finally updated a number of important aspects, such as: the prediction of the isodual theory that gravity is reversed for *elementary* antiparticles, such as the positrons, but bound states of particles and antiparticles, such as the positronium, are attracted in the field of Earth; the isominkowskian geometry permits a symbiotic representation of both the Minkowskian and the Riemannian geometries, which is at the foundation of the isotopic unification of gravity and relativistic quantum mechanics presented in the preceding edition of this volume; the indication (without treatment at this time) of different operator expressions of hadronic mechanics with the use of the isodifferential calculus with a nontrivial formulation of the isodifferentials $\hat{d}x = \hat{1}dx$, where $\hat{1}$ is the isounit; more adequate transformations of the Lie–admissible equations under genounitary transforms; and other aspects.

Any additional comment by interested colleagues would be sincerely appreciated.

Ruggero Maria Santilli
Palm Harbor, FL
Fall 1995

Permanent address:
The Institute for Basic Research
P. O. Box 1577, Palm Harbor, FL 34682, U.S.A.
E-mail: ibrrms@pinet.aip.org
Fax: 1–813–934 9275

ACKNOWLEDGMENTS

xxvi

1: CLASSICAL FOUNDATIONS

1.1: STATEMENT OF THE PROBLEM

We begin the physical studies of this second volume by pointing out that:

1) Hadronic mechanics possesses a well defined classical image with unique interconnecting maps in a way fully parallel to the known classical foundations of quantum mechanics;

2) The primary difference between the classical foundations of quantum and hadronic mechanics is that the former are of potential–Hamiltonian type while the latter admit potential–Hamiltonian as well as nonpotential–nonhamiltonian forces;

3) The classical foundations of quantum mechanics admit a limited class of Newtonian systems, while those of hadronic mechanics are *directly universal*, that is, admitting of all possible classical systems (universality) in the frame of the experimenter (direct universality).

The above classical universality then sets the foundations of the corresponding direct universality of hadronic mechanics for all possible operator systems.

The classical foundations of quantum mechanics, the familiar *Hamiltonian mechanics*[1] are well known (see ref.s [1] and quoted literature). The classical foundations of hadronic mechanics have been studied in:

* Monographs [2] on the integrability conditions for the existence of a potential V, a Hamiltonian H or a Lagrangian L, the so-called *conditions of variational selfadjointness* (SA) and ensuing *Birkhoffian generalization of Hamiltonian mechanics, or Birkhoffian mechanics* for short;

* Monographs [3] on the *Hamilton–isotopic mechanics* and related covering called *Birkhoff–isotopic mechanics,* which possesses a Lie–isotopic structure (Ch.

---

[1] The term "Hamiltonian mechanics" is misleading because referred to the so-called *truncated analytic equations* which are not those originally conceived by Hamilton with *external terms* (see Sect. 1.7.1). Nevertheless, the term is now of general use and will be kept in this volume to denote the conventional canonical mechanics without external terms.

1.4); and

   * Monographs [4] on the still more general *Hamilton–admissible mechanics* and related covering called *Birkhoff–admissible mechanics* which possesses a Lie–admissible structure (Ch. 1.7).

In this chapter we shall outline the main structural lines of these classical studies with particular reference to those profiles which are at the classical foundations of hadronic mechanics. Particular attention is due to the *Hamilton–isotopic mechanics because it is the classical image of the Lie–isotopic branch of hadronic mechanics,* and *to the Hamilton–admissible mechanics because it is the classical image of the Lie–admissible branch* (Fig. 1.1.1).

It should be noted that we know today unique and unambiguous operator maps of Hamilton–isotopic mechanics into the Lie–isotopic hadronic mechanics, called *isoquantization,* and of Hamilton–admissible mechanics into the Lie–admissible hadronic mechanics called *genoquantization.*[2] We also know the operator image of the more general Birkhoff, Birkhoff–isotopic and Birkhoff–admissible mechanics but this latter knowledge is merely formal at this writing.

CLASSICAL MECHANICS		**HADRONIC MECHANICS**
HAMILTON–ISOTOPIC MECHANICS	isoquantization →	**LIE–ISOTOPIC BRANCH**
HAMILTON–ADMISSIBLE MECHANICS	genoquantization →	**LIE–ADMISSIBLE BRANCH**
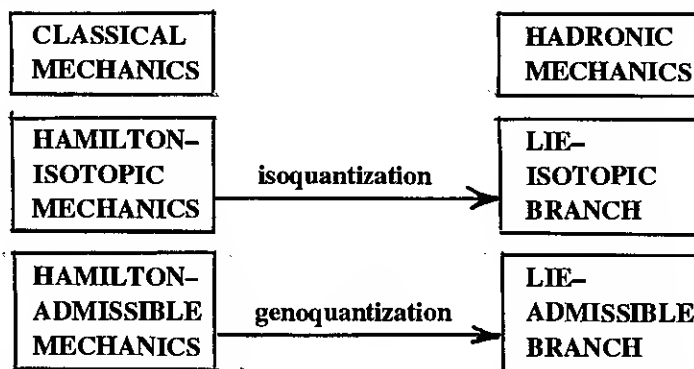
FIGURE 1.1.1: A schematic view of the classical foundations of hadronic mechanics.

It is important to know from the outset the physical differences between the classical foundations of quantum and hadronic mechanics. As well known, quantum mechanics was conceived for systems (such as the atomic structure)

---

[2] It appears recommendable for the reader to get accustomed from the beginning with the fact that the term "quantum" is conceptually misleading and technically inappropriate within the context of hadronic mechanics. Names such as "quantization", "quantum of energy", "quantized particles", etc., do have a correct meaning, conceptually and technically, but only for quantum mechanics, that is, for point–particles in vacuum. In the transition to the covering hadronic mechanics, as indicated from the Preface and studied throughout this volume, the very notion of quantum of energy must be generalized into the integral isounit, $\hbar \to \hat{\hbar} = \hbar\hat{1}$, representing exchanges of energy for extended wavepackets totally immersed within *hadronic media,* i.e., media composed of wavepackets of other particles with a density of the order of that of hadrons.

which are isolated from the rest of the Universe, admit only conservative internal forces and are characterized by only one operator, the Hamiltonian H or Lagrangian L. These systems are called *closed Hamiltonian* (or *variationally selfadjoint*) *systems* [2]. Classical Hamiltonian mechanics then provides the ideal classical foundations of quantum mechanics because it characterizes precisely closed Hamiltonian systems of N particles in Euclidean space and related total conserved quantities which we can write as

$$m_a \ddot{r}_{ka} = F^{SA}_{ka}(t, r, \dot{r}), \quad k = 1, 2, 3 \, (=x, y, z), a = 1, 2, ..., N, \quad (1.1.1a)$$

$$F^{SA}_{ka}(t, r, \dot{r}) = \frac{d}{dt} \frac{\partial V(t, r, \dot{r})}{\partial \dot{r}^k_a} - \frac{\partial V(t, r, \dot{r})}{\partial r^k_a} \quad (1.1.1b)$$

$$E = H = K + V, \quad K = \sum_a p_a^2 / 2 m_a, \quad P_k = \sum_a p_{ak}, \quad (1.1.1c)$$

$$M_k = \sum_a \epsilon_{kij} r_{ai} p_{aj}, \qquad G_k = \sum_a (m_a r_{ak} - t \, p_{ak}) \quad (1.1.1.d)$$

On the contrary, hadronic mechanics was conceived for systems with local–differential–potential as well as nonlocal–integral–nonpotential internal forces; that is, with forces which, by central assumption, *are not* representable with the single quantity H. As studied in detail in Vol. II of ref. [2], the latter systems can also be closed–isolated, thus verifying the same total conservation laws (1.1.1.c) and (1.1.1d), in which case they are called *closed nonhamiltonian* (or *variationally nonselfadjoint*) *systems* [2] and can be written jointly with the closure conditions and ensuing total conserved quantities (see Appendix II.1.B for more details) as

$$m \ddot{r}_k = F^{SA}_k(t, r, \dot{r}) + F^{NSA}_k(t, r, \dot{r}, \ddot{r}, ...) . \quad (1.1.2a)$$

$$\sum_{a=1,...,N} F^{NSA}_a = 0, \; \sum_{a=1,...,N} r_a \cdot F^{NSA}_a = 0, \; \sum_{a=1,...,N} r_a \times F^{NSA}_a = 0, \quad (1.1.2b)$$

$$E = H = K + V, \quad K = \sum_a p_a^2 / 2 m_a, \quad P_k = \sum_a p_{ak}, \quad (1.1.2c)$$

$$M_k = \sum_a \epsilon_{kij} r_{ai} p_{aj}, \qquad G_k = \sum_a (m_a r_{ak} - t \, p_{ak}) \quad (1.1.2d)$$

Then conventional Hamiltonian mechanics loses any validity as the classical counterpart of hadronic mechanics, in favor of the suitable generalized mechanics.

In summary, a primary physical difference of the classical foundations of quantum and hadronic mechanics is that the former are patterned along contemporary analytic trends, those representing systems with only one quantity H or L (*variationally selfadjoint interactions*), while the latter are patterned along the original analytic conception by Lagrange and Hamilton (Sect. 1.7.1), according to which the systems of our physical reality *cannot* be solely represented with one quantity H or L, but require 3N additional external terms F^{NSA}_k (*variationally*

nonselfadjoint interactions.

In the language of these volumes we can say that:

1) the classical foundations of quantum mechanics are given by *exterior dynamical systems* (Ch. 1.1); i.e., systems of particles which can be effectively approximated as being point-like when moving within the homogeneous and isotropic vacuum; while

2) the classical foundations of hadronic mechanics are given by *interior dynamical systems* (Ch. 1.1); i.e., systems of particles which are extended and therefore deformable, while moving within inhomogeneous and anisotropic physical media.

An objective of this chapter is to illustrate that the hadronic representation of systems with the *two* quantities, the Hamiltonian $H = K + V$ (or Lagrangian $L = K - V$) and the isounit $\hat{1}$ (Ch.s 1.2, 1.4, 1.7) is patterned precisely along the original conception by Lagrange and Hamilton to such an extent as to preserve even the number $(1 + 3N)$ of independent quantities. In fact, the independent elements of the isounit $\hat{1}$ (e.g., its diagonal terms) are precisely $3N$.

The original analytic equations with external terms are rewritten in the Hamilton-isotopic form for closed nonhamiltonian systems, or in the Hamilton-admissible form for open nonhamiltonian systems, because the analytic brackets with external terms violate the conditions for the existence of any algebra, let alone Lie algebras (Sect. 1.7.1).

The additional knowledge recommendable from the outset pertains to the reasons why only one mechanics is sufficient for the classical image of quantum mechanics, while hadronic mechanics requires two different, yet complementary mechanics.

Closed variationally selfadjoint systems are composed by collections of particles each one in a stable orbit, as majestically illustrated by the Solar systems. Under these conditions, one mechanics only with totally antisymmetric brackets is evidently sufficient to represent the stability of both the system as a whole and each of its constituents [1].

The situation for the more general variationally nonselfadjoint systems (I.1.2) is fundamentally different. In fact, when studied from the outside as a whole the systems are closed as the Hamiltonian ones, thus requiring a mechanics with *totally antisymmetric brackets* as an evident necessary condition for the conservation of the total energy The nonhamiltonian internal forces then requires that such brackets are of the generalized *Lie-isotopic* type, thus yielding in a unique way (up to isoequivalence) the Hamilton-isotopic mechanics [3].

However, global stability is achieved in systems (I.1.2) via a collection of particles each of which is in unstable conditions. We merely have internal exchanges of energy and other physical quantities but always such to satisfy total conservation laws. While the emphasis in the exterior global treatment is in the total conservation laws, the emphasis for the study of each individual constituent is instead in the characterization of the most general possible *time-rate-of-variation* of the energy, angular momentum and other physical quantities when

considering the rest of the system as external. The latter requirement *prevents the general use of antisymmetric brackets,* yielding in a unique way (also up to equivalence) the Hamilton–admissible mechanics [4].

UNIQUENESS OF THE CLASSICAL FOUNDATIONS
OF QUANTUM M͏
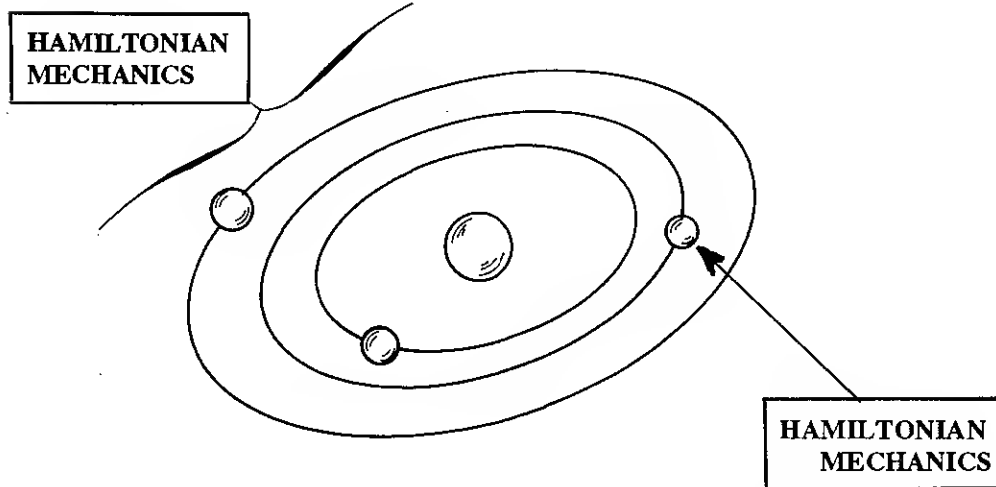
FIGURE 1.1.2: A schematic view of the effectiveness of Hamiltonian mechanics for the characterization of closed conservative systems (such as the Solar system) as a whose as well as of their individual constituents.

DUALITY OF THE CLASSICAL FOUNDATIONS
OF HADRONIC MECHANICS
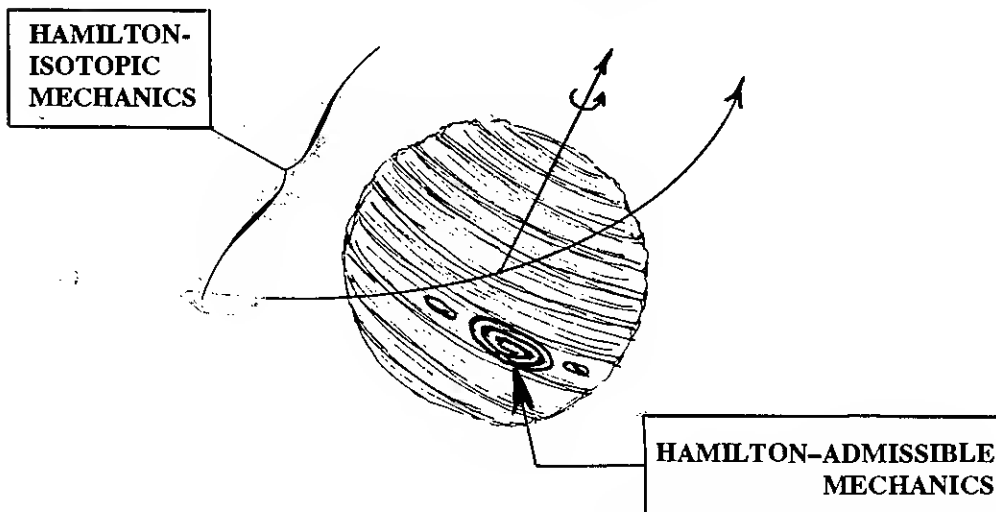
FIGURE 1.1.3: A view of the dual classical foundations of hadronic mechanics here

schematically applied to Jupiter, the Hamilton–isotopic mechanics for the characterization of total conservation laws under nonpotential internal forces, and the Hamilton–admissible mechanics for the characterization of the time–rate–of–variation of physical quantities (e.g., Jupiter's vortices) under the most general known external forces of linear or nonlinear, local or nonlocal and potential or nonpotential type.

The general use of the dual mechanics evidently has its own exceptions. A first one is given by *two–body closed* nonhamiltonian *systems* or by *three–body systems in restricted or Lagrangian configuration* [3]. In fact, as we shall see in details later on, the orbits of these systems must be necessarily stable, even though the forces remain generalized. In this case the Lie–isotopic formulations are sufficient for the representation of both the systems as a whole and each of their constituents.[3]

Another exception is due to the existence of the so-called *indirect analytic representations of open–nonconservative systems via Lie or Lie–isotopic formulations* [2] because techniques of the *inverse problem* [2] do indeed permit the computation of a Hamiltonian H(t, r, p). However, such an approach is not directly universal and misleading on various counts (e.g., because the generator of the time evolution *cannot* be the energy, see below), by establishing again the validity of the original analytic conception of Lagrange and Hamilton.

To begin, the truncated analytic equations can only represent *local–differential interactions* (because of their symplectic character), thus eliminating *ab initio* the interaction of primary relevance for hadronic mechanics, those of nonlocal–integral type.

Second, a single Hamiltonian H(t, r, ṙ) can represent only a rather limited class of local–differential nonconservative systems, called *nonessentially nonselfadjoint* [2]. The direct universality of the original analytic vision by Lagrange and Hamilton (Sect. 1.7.1) for all linear and nonlinear, local and nonlocal, selfadjoint and nonselfadjoint interactions is then manifestly preferable.

Third, the representation of nonpotential forces via the Hamiltonian is misleading, particularly for operator forms, because of the loss of the direct physical significance of the algorithms at hand (see the studies in this respect of ref.s [2]). As an example, the Hamiltonian

$$H = \tfrac{1}{2} e^{\gamma t} p^2, \qquad p = e^{-\gamma t}\dot{r}, \qquad m = 1, \qquad \gamma > 0, \tag{1.1.3}$$

can indeed represent a particle with quadratic damping, $m\ddot{r} = -\gamma\dot{r}^2$ (Vol. 1 of ref. [2], p. 209). But then, as one can see, the quantities $H = \tfrac{1}{2} e^{\gamma t} p^2$ and $p = e^{-\gamma t}\dot{r}$ lose their original physical meaning as *physical energy* and *physical linear momentum,* respectively, in favor of purely mathematical meanings as *canonical Hamiltonian* and *canonical momentum,* respectively.

[3] As we shall see, this is the case in particular for quark theories which can be indeed effectively treated only via the Lie–isotopic formulations.

The insidious character at the operator level is then evident. Owing to the canonicity of H and p, quantum mechanical structures can indeed be formally used. The point is that basic physical laws such as Heisenberg's uncertainties $\Delta r \Delta p \geq \frac{1}{2}\hbar$, even though *mathematically correct* for system (1.1.3), lose their original physical meaning, trivially because the mathematical algorithm "p" is no longer the physical linear momentum.

In order to avoid these insidious occurrences, unless otherwise stated, all investigations of this volume are based on the following

Fundamental assumption 1.1: *All analytic representations must be "direct"; i.e., existing in the r-frame of the experimenter with all algorithms at hand have a direct physical meaning, such as* p = mṙ, H = K + V, K = p²/2m, M = r×p, *etc.*

Under the above basic assumption, Hamiltonian mechanics can only represent exterior systems with action-at-a-distance, potential-conservative forces (see Vol. II of ref. [2] for technical details and examples). On the contrary, the covering Hamilton-isotopic and Hamilton-admissible mechanics can provide the desired direct representations with both, action-at-a-distance potential forces represented by the Hamiltonian H = K + V, and contact, nonpotential-nonhamiltonian forces represented by a generalization of the algebraic and geometric structure of the theory.

By no means do the above comments exhaust all the foundational elements. An additional important aspect is that exterior classical systems (1.1.1) have a sole dependence on time t, coordinates r and velocities ṙ, while interior classical systems (1.1.2) have additional dependencies on the acceleration r̈, density μ, temperature τ, index of refraction n and any needed interior quantity. This latter aspect is studied in Appendix II.1.C.

In this chapter we outline the essential elements of

Hamiltonian mechanics (see, e.g., ref.s [1] and quoted literature),

Birkhoffian mechanics (Vol. II of ref. [2]),

Hamilton-isotopic and **Birkhoff-isotopic mechanics** (Vol. II of ref. [3]),

Hamilton-admissible and **Birkhoff-admissible mechanics** (Vol. II of ref. [4], and

Nambu's mechanics [5].

The above mechanics are used for the *classical characterization of particles*. The operator form of Nambu's mechanics (studied in Ch. II.2.6) is an intriguing realization of hadronic mechanics used for our studies of quark theories.

We shall also outline the

isodual Hamiltonian mechanics (Vol. II of ref. [3]),

isodual Hamilton-isotopic mechanics [loc. cit.], and

isodual Hamilton-admissible mechanics [loc. cit.].

The latter mechanics are used for the *classical characterization of antiparticles*.

Needless to say, this chapter is a mere elementary outline. A technical knowledge of the above classical theories can only be acquired via a study of the original extended treatments [2,3,4,5].

1.2: HAMILTONIAN MECHANICS AND ITS ISODUAL

Let $E(r,\delta,R)$ be the conventional three-dimensional Euclidean space (Sect. I.3.4) with local coordinates $r = \{r^k\}$, $k = 1, 2, 3$, and metric $\delta = $ diag. (1, 1, 1) over the reals $R = R(n,+,\times)$ (Sect. I.2.5). Let $T^*E(r,\delta,R)$ be its six-dimensional cotangent bundle or phase space (App. I.5.A) with local coordinates for a point-particle of (non-null) mass m $\{r, p\} = \{r^k, p_k\}$, where p_k is the physical linear momentum, $p_k = m\dot{r}_k$.

Consider a system of N particles (also of non-null masses) in $T^*E(r,\delta,R)$ characterized by Latin subindex a = 1, 2, ..., N, with corresponding charts $\{r_a{}^k, p_{ak}\}$. We then introduce the unified notation in $T^*E(r,\delta,R)$ familiar from Vol. I,

$$a = \{a^\mu\} = \{r_a{}^k, p_{ak}\}, \quad \mu = 1, 2, ..., 6N, \quad k = 1, 2, 3, \quad a = 1, 2, ..., N, \quad (1.2.1)$$

The fundamental quantity of classical *Hamiltonian mechanics* (see, e.g., ref.s [1]) is the first-order, *canonical action*

$$A^\circ = \int_{-\infty}^{+\infty} dt \, [\, p_{ak} \, \dot{r}_a{}^k - H(t, r, p) \,] = \int_{-\infty}^{+\infty} [\, R^\circ{}_\mu(a) \, da^\mu - H(t, a) \, dt \,],$$

$$(1.2.2a)$$

$$R^\circ = \{ p, 0 \}, \quad (1.2.2b)$$

from which the entire mechanics can be built.

One recognizes in the integrand the fundamental, 6N-dimensional, *canonical one-form* of the symplectic geometry (App. I.5.A)

$$\theta = p \, dr \equiv R^\circ \, da, \quad (1.2.3)$$

or, more properly, the fundamental, (6N+1)-dimensional, *contact Hamiltonian one-form*

$$\xi^\circ = p \, dr - H \, dt \equiv \tilde{R}^\circ \, d\tilde{a}, \quad \tilde{R}^\circ = (R^\circ, -H), \quad \tilde{a} = (a, t). \quad (1.2.4)$$

The exterior derivative of one-form (1.2.3) then characterizes the familiar *fundamental symplectic two-form* (App. I.5.A)

$$\omega = d\theta = \omega_{\mu\nu} \, da^\mu \wedge da^\nu = 2 \, dr \wedge dp, \quad (1.2.5a)$$

$$\omega_{\mu\nu} = \frac{\partial R^\circ{}_\nu}{\partial a^\mu} - \frac{\partial R^\circ{}_\mu}{\partial a^\nu} = \begin{pmatrix} 0_{3N\times 3N} & -1_{3N\times 3N} \\ 1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix}_{\mu\nu} \quad (1.2.5b)$$

where $\omega_{\mu\nu}$ is the *canonical symplectic tensor*.

The application of conventional variational techniques to action (1.2.1) yields the (truncated) *covariant Hamilton's equations*

$$\omega_{\mu\nu}\,\dot{a}^{\nu} = \frac{\partial H(t, a)}{\partial a^{\mu}}, \tag{1.2.6}$$

The contravariant tensor

$$\omega^{\mu\nu} = (|\omega_{\rho\sigma}|^{-1})^{\mu\nu} = \begin{pmatrix} 0_{3N\times3N} & 1_{3N\times3N} \\ -1_{3N\times3N} & 0_{3N\times3N} \end{pmatrix}^{\mu\nu} \tag{1.2.7}$$

is the *canonical Lie tensor* and characterizes the the *canonical Lie brackets* (*Poisson brackets*) among two (sufficiently smooth, regular and bounded) functions C(a) and D(a) on $T^{*}E(x,\delta,R)$

$$[C, D] = \frac{\partial C}{\partial a^{\mu}}\,\omega^{\mu\nu}\,\frac{\partial D}{\partial a^{\nu}} = \frac{\partial C}{\partial r_a{}^k}\frac{\partial D}{\partial p_{ak}} - \frac{\partial D}{\partial r_a{}^k}\frac{\partial C}{\partial p_{ak}}, \tag{1.2.8}$$

The *fundamental (canonical) brackets* of the theory can then be written in the unified notation

$$(\,[a^{\mu}, a^{\nu}]\,) = (\,\omega^{\mu\nu}\,) = \begin{pmatrix} 0_{3N\times3N} & 1_{3N\times3N} \\ -1_{3N\times3N} & 0_{3N\times3N} \end{pmatrix} \tag{1.2.9}$$

The *canonical Hamilton–Jacobi equations* can also be derived via known procedures form the basic action (1.2.2), and can be written in unified notation

$$\frac{\partial A^{\circ}}{\partial t} + H = 0, \qquad \frac{\partial A^{\circ}}{\partial a^{\mu}} = R^{\circ}{}_{\mu}, \quad \mu = 1, 2, ..., 6N, \tag{1.2.10}$$

The last set of equations can be written in the disjoint r– and p–notation

$$\frac{\partial A^{\circ}}{\partial r_a{}^k} = p_{ak}, \qquad \frac{\partial A^{\circ}}{\partial p_{ak}} \equiv 0. \tag{1.2.11}$$

We recover in this way the known property that the *canonical action is independent from the linear momentum* .

The rest of the theory follows via known procedures [1]. For instance, a sufficiently smooth and invertible transformation

$$a \to a' = a'(a) = \{\,r'(r, p), p'(r, p)\,\}, \tag{1.2.12}$$

is called *canonical* when it preserves the values of the canonical tensor, e.g.,

$$\omega_{\mu\nu} \to \omega'_{\mu\nu} = \frac{\partial a'^{\rho}}{\partial a^{\mu}} \; \omega_{\rho\sigma} \frac{\partial a'^{\sigma}}{\partial a^{\nu}} \equiv \omega^{\mu\nu} , \qquad (1.2.13)$$

Action (1.2.2a) and related Hamilton's equations then preserve their form under canonical transformations.

The most celebrated example of physical applications of Hamiltonian mechanics is the direct representation of the N-body system in vacuum (*classical exterior dynamical problem* of Vol. 1) via the Hamiltonian on T*E(r,δ,R)

$$H = K + V = \sum_{a,k} \{ p_a^2 / 2m_a + V(r_{ab}) \} , \qquad (1.2.14a)$$

$$r_{ab} = [(r^i_a - r^i_b) \, \delta_{ij} \, (r^j_a - r^j_b)]^{\frac{1}{2}} . \qquad (1.2.14b)$$

which characterizes *closed selfadjoint/Hamiltonian systems* (ref. [2], Vol. 11, Ch. 6); that is, systems isolated from the rest of the universe, with conservative equations of motion

$$\dot{a} = \begin{pmatrix} r_{ak} \\ \dot{p}_{ak} \end{pmatrix} = \Xi = \begin{pmatrix} p_{ak} / m_a \\ F_{ak}^{SA}(r) \end{pmatrix} \qquad \begin{matrix} (1.2.15a) \\ \\ (1.2.15b) \end{matrix}$$

where SA stands for *variational selfadjointness*, i.e., the verification of the integrability conditions for the existence of potential V(r) [2].

Systems (1.2.15) then admit the *direct Hamiltonian representation* in terms of Eq.s (1.2.6)

$$\omega_{\mu\nu} \Xi^{\nu} = \partial_{\mu} H. \qquad (1.2.16)$$

The systems then satisfy the invariance under the 10-parameter *Galilei symmetry* G(3.1), with consequential conservation of the ten *total Galilean quantities* [1]

$$E = H, \qquad P_a = \sum_a p_{ak} , \qquad (1.2.17a)$$

$$M_k = \sum_a \epsilon_{kij} \, r_{ai} \, p_{aj} , \qquad G_k = \sum_a (m_a \, r_{ak} - t \, p_{ak}) . \qquad (1.2.17b)$$

which are the generators of G(3.1). A typical example is our Solar system in Newtonian approximation.

For the remaining methodological aspects of Hamiltonian mechanics we refer the interested reader for brevity to the literature in the field [1].

A new antiautomorphic formulation of Hamiltonian mechanics was introduced by this author in Vol. 11, ref. [3] under the name of *isodual Hamiltonian mechanics*. It is based on the transition from the basic unit +1 to its *isodual unit* $1^d = -1$, with the following reformulations:

1) the field of real numbers R(n,+,×) is mapped into the *isodual isofield* $F^d(n^d,+,×^d)$ (Sect. 1.2.5.B), with *isodual numbers* $n^d = n1^d = -n$, where the generic

product is of the isodual form $n^d \times^d m^d = -n^d m^d = -nm$, the conventional norm $| n |$ > 0 of $R(n,+,\times)$ is mapped into the *isodual norm* $| n^d |^d = - | n | < 0$ and *the magnitude of all quantities is therefore negative definite*;

2) the Euclidean space $E(r,\delta,R)$ is mapped into the *isodual Euclidean space* $E^d(r,\delta^d,R^d)$ with *isodual Euclidean metric* $\delta^d = -\delta$ and *isodual invariant* $r^{2d} = (r^i \delta^d_{ij} r^j)1^d = r^2 \equiv r^i \delta_{ij} r^j$;

3) the Hamiltonian H is mapped into the *isodual Hamiltonian* $H^d = -H$ (Sect. I.5.4.F) while the covariant analytic equations are mapped into the *isodual Hamilton's equations*

$$\omega^d_{\mu\nu} \times^d da^{\nu d}/^d d^d t = \partial^d_\mu H^{d}/^d \partial a^{d\mu} \equiv -\omega_{\mu\nu} \dot{a}^\nu = -\partial_\mu H, \qquad (1.2.18)$$

where $/^d = -/$ represents the *isodual quotient.*

4) the underlying geometry is mapped into the *isodual symplectic geometry* (Sect. I.5.4.E) with *isodual canonical form*

$$\theta^d = R^\circ_\mu \times^d da^{\mu d} = -\theta, \quad \omega^d = \omega^d_{\mu\nu} \times^d da^{\mu d} \wedge^d da^{\nu d} = -\omega; \qquad (1.2.19)$$

5) the underlying algebra is mapped into the *isodual Lie algebra* (Sect. I.4.4) with *isodual generators* $X_k^d = -X_k$ and *isodual Poisson brackets*

$$[A, B]^d = (\partial^d_\mu A^d) \times^d \omega^{d\mu\nu} \times^d (\partial^d_\nu B^d) = -[A, B]. \qquad (1.2.20)$$

The conventional Hamiltonian mechanics will be used in these volumes to represent *classical particles in exterior conditions in vacuum*, while the isodual form will be used to represent *classical antiparticles in exterior conditions in vacuum*. The reader should be aware that isodual formulations are nontrivial, e.g., because they imply that *antiparticles moves backward in time and possess* a *negative–definite energy* although refereed to a *negative–definite unit.*

In this volume we shall show the physical consistency of such a novel representation of antiparticles and identify some of its far reaching implications, such as the first theoretical formulation of *antigravity* of Sect. II.8.7 or the *space–time machine* of Sect. II.9.7.

1.3: BIRKHOFFIAN MECHANICS AND ITS ISODUAL

The *Birkhoffian generalization of Hamiltonian mechanics* or *Birkhoffian mechanics* for short (Vol. II, ref. [2]), was built via an infinite number of local–differential isotopies of Hamiltonian mechanics; that is isotopies which preserve not only the axiomatic structure of the theory, but also the original local–differential character, thus including its underlying symplectic geometry.

As a result, Hamiltonian and Birkhoffian mechanics coincide by conception

and construction at the abstract, realization–free level. Hamiltonian mechanics is characterized by the simplest possible realization of the symplectic geometry, the canonical one. Birkhoffian mechanics is instead characterized by its most general possible realization, thus permitting a substantial broadening of the manifestly limited representational capabilities of Hamiltonian mechanics.

The generalized mechanics was called "Birkhoffian" for certain historical reasons reviewed in detail in Vol. 11 of ref.s [2].

The fundamental quantity is again the action on $T^*E(r,\delta,R)$ which is also of first–order, but this time it is given by by the most general possible first–order form usually called *Pfaffian action*

$$A = \int_{-\infty}^{+\infty} [\, R_\mu(a)\, da^\mu - B(t, a)\, dt\,], \qquad R \neq R° = \{\, p\,, 0\,\}, \quad (1.3.1)$$

where B is called the *Birkhoffian* because it is generally different than the Hamiltonian H, from which the entire mechanics can also be built. The most salient generalization is therefore given by the replacement of the canonical 6N–quantities $R° = \{\, p, 0\}$ with the Pfaffian quantities

$$R \equiv \{\, P(r, p)\,, Q(r, p)\,\}\,. \tag{1.3.2}$$

The integrand of action (1.3.1) is also a *Birkhoffian one–form*

$$\Theta = R\, da\,, \tag{1.3.3}$$

or, more properly, the (6N+1)–dimensional *contact Birkhoffian one–form*

$$\xi = \tilde{R}\, d\tilde{a}\,, \quad \tilde{R} = (\,R\,, -B)\,, \quad \tilde{a} = (\,a\,, t)\,. \tag{1.3.4}$$

The geometric equivalence of Hamiltonian and Birkhoffian mechanics is then seen from the fact that, at the abstract, coordinate–free level, all distinctions between one–forms θ and Θ (or $\tilde{\theta}$ and $\tilde{\Theta}$) cease to exist.

Despite that, the physical differences between Hamiltonian and Birkhoffian mechanics are significant. In fact, the variation of action (1.3.1) yields the equations

$$\Omega_{\mu\nu}(a)\, \dot{a}^\nu = \frac{\partial B(t, a)}{\partial a^\mu}\,, \tag{1.3.5}$$

called by this author for certain historical reasons the *covariant Birkhoff's equations* [2], where $\Omega_{\mu\nu}(a)$ is the *general, exact, symplectic tensor* (App. 1.5.A)

$$\Omega_{\mu\nu} = \frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\,, \qquad \det.(\Omega_{\mu\nu}) \neq 0\,, \tag{1.3.6}$$

which now is no longer of the simple off-diagonal form (1.2.6), but has a general structure in 6N×6N dimension. It is evident that Eq.s (1.3.5) constitute a generalization of the truncated Hamilton's equations, yet preserving their abstract analytic-geometric-algebraic axioms.

The contravariant form,[4]

$$\Omega^{\mu\nu} = (|\Omega_{\rho\sigma}|^{-1})^{\mu\nu},$$ (1.3.7)

is also a Lie tensor and characterizes the most general possible, classical, local-differential, and regular *Lie-isotopic brackets* [5] (also called *generalized Poisson brackets* [1])

$$[C, D]^* = \frac{\partial C}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial D}{\partial a^{\nu}},$$ (1.3.8)

whose form in the disjoint r- and p-variables is rather complex and will be ignored.

The *fundamental Birkhoffian brackets* of the theory can then be written in the unified notation

$$[a^{\mu}, a^{\nu}]^* = \Omega^{\mu\nu}(a), \qquad \mu, \nu = 1, 2, \ldots, 6N.$$ (1.3.9)

The *Birkhoffian Hamilton-Jacobi equations* can also be derived via known procedures from action (1.3.1), and can be written (Vol. II, ref.s [2])

$$\frac{\partial A}{\partial t} + B = 0, \qquad \frac{\partial A}{\partial a^{\mu}} = R_{\mu},$$ (1.3.10)

The last set of equations can be written in the disjoint notation via Eq.s (1.3.2)

$$\frac{\partial A}{\partial r_a{}^k} = P_{ak}(r, p), \qquad \frac{\partial A}{\partial p_{ak}} \equiv Q_a{}^k(r, p);$$ (1.3.11)

namely, *the Pfaffian action A is dependent on the linear momenta* p. As we shall see in the next chapter, this has important consequences for operator theories.

One can then see from the above lines the isotopic character of the map from Hamilton to Birkhoff's mechanics. Geometrically, we have the map from the Hamiltonian to the Birkhoffian two-form within fixed local coordinates

---

[4] Note that contravariant form (1.3.7) *always* exists for Birkhoffian mechanics, owing to the *symplectic* and therefore nondegenerate character of the two-form, det $\Omega \neq 0$.

[5] As the reader recalls from App. I.5.A, this is due to the fact that the symplectic form Ω is exact, $\Omega = d\Phi$, and therefore closed, $d\Omega = d(d\ \Omega) \equiv 0$ (Poincaré lemma).

$$\omega = \omega_{\mu\nu} \, da^{\mu} \wedge da^{\nu} = d\,\theta \;\; \rightarrow \;\; \Omega = \Omega_{\mu\nu}(a) \, da^{\mu} \wedge da^{\nu} = d\,\Theta . \qquad (1.3.\ 12)$$

The isotopy then follows from the preservation of the original axioms (exact and closed character of the two-form) within the same local chart r (no transformation of the frame of the experimenter).

Algebraically, the isotopy can be seen via the map of the brackets also within fixed local coordinates

$$[\, C, D \,] \;\; \rightarrow \;\; [\, C, D \,]^{*} \qquad\qquad (1.3.13)$$

which preserves the Lie algebra axioms.

Finally, the isotopy can be seen analytically from the map

$$A^{\circ} = \int_{-\infty}^{+\infty} (\, R^{\circ}_{\mu} \, da^{\mu} - H \, dt) \;\; \rightarrow \;\; A = \int_{-\infty}^{+\infty} (\, R_{\mu} da^{\mu} - B \, dt), \qquad (1.3.14)$$

which preserves the derivability of the analytic equations from a first-order action.

Despite the preservation of the underlying geometric, algebraic and analytic axioms, the transition from Hamilton to Birkhoff mechanics is not physically trivial. To clarify this point we first recall that the transition is characterized by *noncanonical transformations* a $\rightarrow$ a′(a) i.e., transformations which *do not preserve Hamilton's equations,* and actually map Hamilton's tensor $\omega^{\circ}{}_{\mu\nu}$ = const. into Birkhoff's one $\Omega_{\mu\nu} = \Omega_{\mu\nu}(a)$

$$\omega_{\mu\nu} \;\; \rightarrow \;\; \omega'_{\mu\nu} = \frac{\partial a^{\rho}}{\partial a^{\mu}} \, \omega_{\rho\sigma} \, \frac{\partial a^{\sigma}}{\partial a^{\nu}} = \Omega_{\mu\nu}(a'), \qquad (1.3.15)$$

Then *Birkhoff's equations (1.3.5) are the image of Hamilton's equations (1.2.6) under noncanonical transformations.*

Second, in Hamilton mechanics a physical system is represented via the Hamiltonian H *only.* On the contrary, the representation of physical systems in Birkhoff's mechanics requires the knowledge of (1+6N)-quantities, the Birkhoffian B as well as the Pfaffian functions R_{μ}. The latter are reducible in most cases to 3N independent functions [loc. cit.], thus preserving the number (1+3N) of independent quantities in the original Lagrange and Hamilton equations.

We therefore encounter for the first time in this volume *the capability of representing interactions with quantities other than the Hamiltonian,* which is the reason for the very construction of the isotopies and genotopies of Vol. 1. In App. 11.1.A we illustrate this capability via *the representation of the Lorentz force with the algebraic tensor of the theory,* rather than the traditional *minimal coupling rule* within the context of the Hamiltonian. Once this capability is illustrated for the electromagnetic interactions, we have as a natural consequence the extension to the contact nonpotential component of the strong

interactions due to mutual penetration of hadrons (App. II.1.B).

The Birkhoffian action (1.3.1) admits a degree of freedom (absent in Hamiltonian mechanics) called *Birkhoffian gauge transforms* (Vol. II, ref.s [2], p. 62)

$$R_\mu \rightarrow R'_\mu = R_\mu + \partial G / \partial a^\mu , \qquad B \rightarrow \dot{B} = B - \partial G / \partial t , \qquad (1.3.16)$$

under which the Birkhoffian B can be reduced to the total physical energy H = K + V. The equations of motion in first order form, after elimination of the integrating factors to verify the conditions of variational selfadjointness (see both ref.s [2] for details) are of the general class

$$\dot{r}_a^k = p_a^k / m_a , \qquad \dot{p}_{ak} = F_{ak}^{SA}(t, r, p) + F_{ak}^{NSA}(t, r, p). \qquad (1.3.17)$$

The forces F_{ak}^{SA} are then represented via the potential V(t, r, p) in the Birkhoffian B = H = K + V, and the forces F_{ak}^{NSA} are represented via the Pfaffian functions R_μ in the general symplectic tensor $\Omega_{\mu\nu}(a)$.

When dealing with Birkhoffian mechanics, we shall call the generator of the time evolution the *Birkhoffian* and use the corresponding symbol B, when it *does not* represent the physical energy K + V. If this is the case, we shall use the term *Hamiltonian* and the symbol H.

A subclass of the general class of systems (1.3.17) verifies the *conventional* total conservation laws (1.2.17) under closure conditions (1.1.2b) on the nonselfadjoint forces, i.e.,

$$\sum_{ak} F_{ak}^{NSA} \equiv 0, \qquad \sum_{ak} r_{ak} \times F_{ak}^{NSA} \equiv 0, \qquad \sum_{ak} r_{ak} \wedge F_{ak}^{NSA} \equiv 0. \qquad (1.3.18c)$$

Since the number of equations (1.3.17) is higher than that of conditions (1.3.18), algebraic (unconstrained) solutions in the nonselfadjoint forces always exist. Otherwise, Eq.s (1.3.18) can be interpreted as bona-fide *subsidiary constraints* to equation of motion (1.3.17) (see ref.s [2], Vol. II, Ch. 6 and Appendix II.1.B of this volume for details).

As indicated in Sect. II.1.1, the ensuing systems are called *closed nonhamiltonian (or variationally nonselfadjoint)* in the sense that the systems are isolated from the rest of the universe as the conservative systems (1.2.15) are, but cannot be *entirely* represented with the Hamiltonian because the internal forces are a combination of Hamiltonian and nonhamiltonian forces.

In Hamiltonian mechanics one customarily assigns the Hamiltonian H and (rarely) checks the explicit form of the equations of motion. In Birkhoffian mechanics one can assign instead (1+6N) functions, the Birkhoffian B (or H) and the 6N Pfaffian functions R_μ. The equations of motion can then be computed accordingly.

However, the most important approach in Birkhoffian mechanics is the *inverse Newtonian problem*; i.e., one first assigns the nonhamiltonian equations of motion in their first-order (vector-field) form

$$\dot{a} = \Xi(a) = \begin{pmatrix} p_{ak} / m_a \\ F_{ak}{}^{SA}(t, a) + F_{ak}{}^{NSA}(t, a) \end{pmatrix} \qquad (1.3.19)$$

and then computes the $(1+6N)$ functions B (or H) and R_μ verifying the *direct analytic representation*

$$(\partial_\mu R_\nu - \partial_\nu R_\mu) \; \Xi^\nu = \partial_\mu B , \quad \partial_\mu = \partial / \partial a^\mu . \qquad (1.3.20)$$

according to one of the several methods given in Vol. II of ref.s [2].

A technical knowledge of this inverse problem is necessary for a full understanding of hadronic mechanics. Recall that quantum mechanics can effectively and unambiguously represent only systems whose classical images are conservative. Hadronic mechanics represents instead systems whose classical images are nonhamiltonian. The understand of nonhamiltonian classical system is therefore important for the primitive Newtonian meaning of a nonlocal-nonhamiltonian representation of the strong interactions (App. II.1.B).

We here limit ourselves to quote the following

Theorem 1.3.1 (Direct Universality of Birkhoffian Mechanics, ref. [2], Vol. II, Sect. 4.5): *All local-differential, analytic, regular, finite-dimensional, unconstrained or holonomic, nonhamiltonian (variationally nonselfadjoint) systems in their first-order form (1.3.19) always admit a direct analytic representation (1.3.20) in terms of Birkhoff's equations in a star-shaped neighborhood of the local variables a = {r, p} where r represents the frame of the experimenter and p represents the physical linear momentum .*

Note that the above theorem implies the *direct universality of the symplectic geometry for local Newtonian systems.* For numerous additional profiles and examples, the reader may inspect ref.s [2].

The construction of the *isodual Birkhoffian mechanics* on the isodual Euclidean space $E^d(r,^d,R^d)$ over the isodual real field $R^d(n^d,+,\times^d)$ along the lines of the isodual Hamiltonian mechanics of the preceding section, is left to the interested reader.

The reader should be finally aware that the Birkhoffian mechanics and its isodual generally used in an indirect way in these volumes. In fact, we shall fist use the techniques of ref.s [2] to compute a Birkhoffian representation of a given nonhamiltonian system, and then factorize such a representation into the Hamilton-isotopic or Hamilton-admissible form. The construction of the operator form of the Birkhoffian representation will not be used owing to the insufficient knowledge of the latter at this writing (see next chapter).

1.4: HAMILTON-ISOTOPIC AND BIRKHOFF-ISOTOPIC MECHANICS AND THEIR ISODUALS

As indicated in Sect. 11.1, the Birkhoff's mechanics is strictly local-differential, while a central objective of hadronic mechanics is the study of nonlocal interactions, thus requiring the identification of a classical mechanics which is also of integral character.

For these and other reasons, following the construction of the Birkhoffian mechanics [2], this author was forced to construct a second generalized mechanics presented in ref.s [3] under the name of *Hamilton-isotopic* (or *isohamilton) mechanics* with most general form known as *Birkhoff-isotopic* (or *isobirkhoff) mechanics.*

The new mechanics was constructed, this time, via general, nonlocal-integral isotopies of Hamiltonian mechanics. This implies that the underlying symplectic geometry is now inapplicable in favor of the covering *isosymplectic geometry* (Sect. 1.5.4). In turn, this implies the abandonment of the conventional 6-dimensional unit of the phase space, $1 = \text{diag.}\ (1, 1, .., 1)$, in favor of an *isounit* $\hat{1}$ with an arbitrary, integro-differential dependence on all possible (or otherwise needed) local quantities

$$1 \;\rightarrow\; \hat{1}(t, r, \dot{r}, \ddot{r}, ...),\qquad\qquad (1.4.1)$$

hereon assumed of *Kadeisvili's Class I*; i.e., sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive-definite (Sect. 1.1..5). Still in turn, lifting (1.4.1) implies the abandonment of the notion of ordinary real numbers in favor of the covering *isoreal numbers* and *isofields* (Sect. 1.2.5):

$$R(n,+,\times) \;\rightarrow\; \hat{R}(\hat{n},+,*),\quad \hat{n} = n\hat{1},\qquad\qquad (1.4.2)$$

where the multiplication is subjected to the compatible lifting

$$n \times m = n\,m \;\rightarrow\; \hat{n} * \hat{m} = \hat{n}\,T\,\hat{m} = (n\,m)\hat{1},\quad \hat{1} = T^{-1}.\qquad (1.4.3)$$

The fundamental carrier space is therefore the three-dimensional *isoeuclidean space* (Sect. 1.3.4) also of Class I, here assumed for simplicity in the diagonal form

$$\hat{E}(r,\hat{\delta},\hat{R}):\quad \hat{\delta} = T(t, r, \dot{r}, \ddot{r}, ...)\delta,\quad \hat{1} = T^{-1},\qquad\qquad (1.4.4a)$$

$$T = \text{diag.}\ (b_1{}^2, b_2{}^2, b_3{}^2),\quad b_k = b_k(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...) > 0, k = 1, 2, 3,\qquad (1.4.4b)$$

$$r^2 = (x^1\,b_1{}^2\,x^1 + x^2\,b_2{}^2\,x^2 + x^3\,b_3{}^2\,x^2)\hat{1}\ \in\ \hat{R}(\hat{n},+,*).\qquad (1.4.4c)$$

The *isophase space* is then given by the *isocotangent bundle* $T^*\hat{E}(r,\hat{\delta},\hat{R})$ with the

same local chart (1.2.1). Its basic geometric notion is the *iso-one-form*

$$\hat{\theta}^\circ = p * dr = p\, T\, dr = p_{ai}\, T^i_{\ j}(t, r, \dot{r}, \ddot{r} \ldots)\, dr^j \ . \tag{1.4.5}$$

In order to do dynamics, isospace (1.4.4) must be enlarged to include a representation of time. This must be also done in the isotopic form, resulting in the structure

$$\hat{E}(t,\hat{R}_t) \times \hat{E}(r,\delta,\hat{R}(\hat{n},+,\times)) \ , \quad \hat{1}_t = T_t^{-1} > 0 \ , \tag{1.4.6}$$

where $\hat{1}_t$ (T_t) is the *time isounit* (*time isotopic element*) and, as such, it is independent of the *space isounit* $\hat{1}$ (*space isotopic element* T). The emerging (1+6N)-dimensional geometry is called *isocontact geometry* and admits the basic one-isoform

$$\hat{\xi}^\circ = p * dr - H \odot dt = p\, T\, dr - H\, T_t\, dt \ , \tag{1.4.7}$$

which can also be written in unified notation

$$\hat{\xi}^\circ = \tilde{R}^\circ * da = \tilde{R}^\circ_\mu\, \hat{T}^\mu_{\ \nu}\, d\tilde{a}^\nu \ , \quad \mu, \nu = 1, 2, \ldots, 6N + 1, \tag{1.4.8a}$$

$$\tilde{R}^\circ = \{\, R^\circ , - H \,\} \ , \quad \hat{T}_{6N \times 6N} = \text{diag.}\, (T, T_t) \ . \tag{1.4.8b}$$

Note that the isocontact one-form $\hat{\xi}^\circ$ coincides at the abstract level with the canonical, contact one-form ξ°, trivially, because at the coordinate-free level one loses any distinction between the conventional and isotopic products. This ensures the preservation of the original axioms of the contact geometry, as well as the embedding of all nonlocal terms in the product itself.

With these preliminaries in mind, the Hamilton-isotopic mechanics can be built [3] from the *isocanonical action*

$$\hat{A}^\circ = \int_{-\infty}^{+\infty} \hat{\theta}^\circ = \int_{-\infty}^{+\infty} \tilde{R}^\circ * da = \int_{-\infty}^{+\infty} (\, R^\circ_\mu\, T^\mu_{\ \nu}\, da^\nu - H\, T_t\, dt). \tag{1.4.9}$$

The use of variational techniques then yields the *covariant Hamilton-isotopic equations*

$$\hat{\omega}_{\mu\nu}\, \dot{a}^\nu = \omega_{\mu\alpha}\, T_2^{\alpha}{}_\nu(t, a, \dot{a}, ..)\, \dot{a}^\nu = \frac{\partial H(t, a)}{\partial a^\mu} \ , \tag{1.4.10}$$

where $\omega_{\mu\nu}$ is the canonical symplectic tensor, and the isotopic element T_2 is given by (see Sect. 1.5.4)

$$T_{2\nu}{}^\mu = (\, b^\mu\,)^2 (\, b_\nu\,)^2 + \omega^{\mu\rho} (R^\circ_\nu\, \frac{\partial b^2_\nu}{\partial a^\rho} - R^\circ_\rho\, \frac{\partial b^2_\rho}{\partial a^\nu}) . \tag{1.4.11}$$

The *contravariant Hamilton-isotopic equations* are then given by

$$\dot{a}^{\mu} = \hat{1}_2{}^{\mu}{}_{\alpha}\omega^{\alpha\nu} \; \frac{\partial H}{\partial a^{\nu}} \qquad \left\{ \begin{array}{l} \dot{r}^i = \hat{1}_2{}^i{}_j \; \dfrac{\partial H}{\partial p_j} \\[2ex] \dot{p}_i = -\hat{1}_{2i}{}^j \; \dfrac{\partial H}{\partial r_j} \end{array} \right. \qquad (1.4.12a)$$

$$\hat{1}_{2\mu}{}^{\nu} = \left(\left| T_2 \right|^{-1} \right)_{\mu}{}^{\nu} . \qquad (1.4.12b)$$

and characterizes the the *classical Lie–isotopic brackets* on $T^*\hat{E}(r,\hat{\delta},\hat{R})$

$$[C \, \hat{,} \, D] = \frac{\partial C}{\partial a^{\mu}} \; \hat{1}_2{}^{\mu}{}_{\alpha}(t, a, \dot{a}, \ddot{a}, \mu, \tau, n, \,..)\omega^{\alpha\nu} \; \frac{\partial D}{\partial a^{\nu}} =$$

$$= \frac{\partial C}{\partial r_a{}^i} \hat{1}_2{}^i{}_j(t, r, p, \dot{p}, ...) \; \frac{\partial D}{\partial p_{aj}} - \frac{\partial C}{\partial r_a{}^i} \hat{1}_2{}^i{}_j(t, r, p, \dot{p}, ...) \frac{\partial D}{\partial p_{aj}}, \qquad (1.4.13)$$

whose verification of the Lie–isotopic axioms is ensured by the validity of the isopoincaré lemma (Sect. I.5.4.D).

The *fundamental Lie–isotopic brackets* of the theory can then be written in the unified notation

$$\left(\, [a^{\mu} \, \hat{,} \, a^{\nu}] \, \right) = \left(\; \hat{1}_2{}^{\mu}{}_{\alpha}\omega^{\alpha\nu} \; \right) = \begin{pmatrix} 0_{3N\times 3N} & 1_{3N\times 3N} \\ -1_{3N\times 3N} & 0_{3N\times 3N} \end{pmatrix} . \qquad (1.4.14)$$

The *isocanonical Hamilton–Jacobi equations* can also be derived via known procedures from action (1.4.9), and can be written in unified notation

$$\frac{\partial \hat{A}^{\circ}}{\partial t} + H T_t = 0, \qquad \frac{\partial \hat{A}^{\circ}}{\partial a^{\mu}} = R^{\circ}{}_{\alpha} T^{\alpha}{}_{\mu}, \quad \mu = 1, 2, ..., 6N. \quad (1.4.15)$$

The last set of equations can be written in the disjoint r– and p–notation

$$\frac{\partial \hat{A}^{\circ}}{\partial r_a{}^k} = p_{ai} T^i{}_k, \qquad \frac{\partial \hat{A}^{\circ}}{\partial p_{ak}} \equiv 0, \quad a = 1, 2, ..., N, \quad k = 1, 2, 3 \quad (1.4.16a$$

The remarkable property of the isotopies is that the Hamilton–isotopic mechanics does preserve the independence of the isoaction from the linear momentum as in the original mechanics,

$$\frac{\partial A^{\circ}}{\partial p_{ak}} \equiv 0, \qquad (1.4.17)$$

even though the isotopic element have an arbitrary dependence on the velocities, accelerations and other quantities.[6]

---

[6] The reader may recall from Vol. I that this is due to the fact that contractions in

The basic invariance of the Hamilton–isotopic equations is given by the *Galilei–isotopic* (or *isogalilean*) *symmetry* $\hat{G}(3.1)$ and related *Galilei–isotopic* (or *isogalilean*) *relativity* [3] on $\hat{E}(t,\hat{R}_t) \times T^*\hat{E}(r,\hat{\delta},\hat{R})$. The reader will recall from Vol. 1 that the isotopies of Lie imply the preservation of the original generators and parameters (with the understanding that they must now be properly written in the new isospace). Thus, the isosymmetry $\hat{G}(3.1)$ is characterized by the ten generators

$$H = \sum_{a,k} \{ p_a^2 / 2m_a + V(\hat{r}_{ab})\} = \sum_{a,k} \{ p_a \hat{\delta}^{ij} p_{aj} / 2m_a + V(\hat{r}_{ab})\} ($$

$$\hat{r}_{ab} = [(r^i_a - r^i_b) \hat{\delta}_{ij}(r^j_a - r^j_b)]^{\frac{1}{2}} , \quad P_a = \sum_a p_{ak} , \quad M_k = \sum_a \epsilon_{kij} r_{ai} p_{aj} ,$$

$$G_k = \sum_a (\hat{m}_a * r_{ak} - \hat{t} \odot p_{ak}) \equiv \sum_a (m_a r_{ak} - t p_{ak}) , \quad \hat{m} \in \hat{R}, \hat{t} \in \hat{R}_t \qquad (1.4.18)$$

All infinitely possible isogalilean symmetries of Class I are locally isomorphic to the conventional symmetry, $\hat{G}(3.1) \approx G(3.1)$. The imposition of the isogalilean symmetry to nonconservative systems implies their "closure" into an isolated form verifying the conservation of all quantities (1.4.18), thus eliminating the need for the subsidiary constraints (1.1.2b). Thus, *the isogalilean symmetry characterizes closed nonhamiltonian (nonselfadjoint) systems* [3].

The classical isogalilean symmetry is the true, ultimate foundation of hadronic mechanics from which the latter can be derived via technical procedures. What is most crucial for the understanding of hadronic mechanics is the capability of the isogalilean symmetry and related relativity of providing a fully *geodesic* representation of trajectories within physical media which actually coincide with the trajectories in vacuum when written in isospace (see the *isoriemannian geometry* of Ch. 1.5)..

We can say in figurative terms that *physical media disappear when represented via the isogalilean relativity,* thus reaching the desired unity between interior and exterior problems. As we shall see, exactly the same situation occurs at the operator level.

To avoid a prohibitive length, we have to refer the reader to both volumes of ref.s [3], whose knowledge is hereon assumed.

As anticipated in Sect. 11.1.1, it is also important to understand that the isogalilean symmetry can also provide a form–invariant description of open–nonconservative *systems.* As a simple example, consider an extended test particle moving within a physical medium with nonlinear, nonlocal and noncanonical equations of motion

$$m \ddot{r} + \gamma \dot{r}^2 \int_\sigma d\sigma \, \mathcal{F}(\sigma, ...) = 0 , \qquad (1.4.19)$$

---

isospace can be formally written in exactly the same way as in conventional spaces, e.g., $r_k r^k = \hat{\delta}_{ki} r^i r^k = T_{ki} r^i r^k$. The derivative, say, with respect to a covariant variable is then insensitive with respect to the isotopic element because absorbed in the contravariant part.

where: r represents the center–of–mass trajectory; $-\gamma \dot{r}^2$ represents a drag force quadratically dependent on the speed $\dot{r}$; σ is the surface of the body; $\mathcal{F}$ is a kernel; and the integral over σ represents the correction to the trajectory due to the shape of the body (e.g., of a missile).

Equations (1.4.19) are structurally outside the representational capability of both, Hamilton's and Birkhoff's mechanics because of their integral character. Yet, their representation with Hamilton–isotopic mechanics is direct and immediate. The fundamental point is the identification of a generalized isounit which represents the *totality* of the nonlocal terms, and the construction of the isocanonical representation with respect to that unit.

A simple representation with Hamilton–isotopic equations (1.4.12) in six dimensions is given by [3]

$$\hat{1}_2 = \text{Diag.}\{ \hat{S} e^{\gamma \, r \int_\sigma d\sigma \, \mathcal{F}(\sigma)}, \ \hat{S} e^{\gamma \, r \int_\sigma d\sigma \, \mathcal{F}(\sigma)} \}, \tag{1.4.20a}$$

$$\hat{S} = \text{diag.}(b_1^2, b_2^2, b_3^2), \quad b_k > 0, \quad k = 1, 2, 3, \tag{1.4.20b}$$

$$H = p^2 / 2m = p * p / 2m = p_i \, T_{2ij} \, p_j / 2m, \quad T_2 = \hat{S} e^{\gamma \, r \int_\sigma d\sigma \, \mathcal{F}(\sigma)}. \tag{1.4.20c}$$

Note that, *in addition to the "direct representation" of the system in the coordinates r of the experimenter, the Hamilton–isotopic mechanics provides a direct characterization of the original nonspherical shape as well as all its infinitely possible deformations via the isotopic element* $T = \text{diag.}(b_1^2, b_2^2, b_3^2)$, *which is also structurally outside the technical capabilities of Hamiltonian and Birkhoffian mechanics* (see ref.s [3].

Recall from Ch..s II.3 and II.5 the following important application of the isotopies. The conventional Euclidean geometry has the trivial unit $1 = \text{diag.}$ (1, 1, 1) and, consequently, it can only represent perfectly spherical and rigid objects. On the contrary, the isoeuclidean geometry can represent all possible actual shapes of the text body considered, jointly with all its infinitely possible deformations, which is permitted precisely by the fundamental lifting $1 \to \hat{1}$.

System (1.4.19) also provides an example of the integro–differential topology underlying the Hamilton–isotopic mechanics (see Sect. 1.1.3) which is everywhere local–differential except at the isounit. In particular, the local–differential part represents the center–of–mass trajectory, while the nonlocal–integral part represents the contributions/corrections due to shape, contact–nonpotential interactions and other nonhamiltonian effects.

Finally, system (1.4.19) provides a direct illustration of the isogalilean symmetry $\hat{G}(3.1)$. In fact, the system is manifestly *noninvariant* under the Galilei transformations. Nevertheless, the system is invariant under the Galilei–isotopic transformations constructed with isotopic element in Eq.s (1.4.20c) while being locally isomorphic to the original symmetry $G(3.1)$ (because $\hat{1}_2 > 0$).

We finally note that Eq.s (1.4.19) *do not* constitute a closed nonhamiltonian system because they are one–dimensional. In fact, it takes a minimum of two constituents to have a closed nonhamiltonian system.

In Vol. I we have introduced a new conjugation called *isoduality* for the representation of antiparticles. It is based on isounits of Kadeisvili's Class II (same isounits of Class I, but now negative–definite). The assumption of a negative–definite unit then implies the consequential assumption of new numbers, called *isodual isonumbers*, new fields and the *isodual isofields* (Sect. I.2.5.B)

$$\hat{R}^d\,(\hat{n}^d,+,*^d\,): \quad \hat{n}^d \;=\; n\,\hat{1}^d\,, \quad \hat{n}^d\,*^d\,\hat{m}^d \;=\; \hat{n}^d\,T^d\,\hat{m}^d \;=\; (\,n\,m\,)\,I^d\,, \quad (1.4.21a)$$

$$\hat{1}^d = -\hat{1}, \quad T^d = -T, \quad \hat{1}^d = (T^d)^{-1}, \quad (1.4.21b)$$

new spaces, the *isodual isoeuclidean spaces*,

$$\hat{E}^d(r,\hat{\delta}^d,\hat{R}^d\,): \quad \hat{\delta}^d \;=\; T^d\,\delta \;=\; -\delta\,, \quad \hat{1}^d \;=\; (T^d)^{-1} \;=\; -\hat{1}\,. \quad (1.4.22)$$

new symmetry, the *isodual isogalilean symmetry* $\hat{G}^d(3.1)$, etc.

Recall also that in this isodual universe the magnitude of quantities (the *isodual norm*) is *negative definite*. The *isodual Hamilton–isotopic mechanics* can therefore be defined as the image of the original mechanics under isoduality.

To begin, the Hamiltonian is now negative–definite, as one can see from the case with only kinetic energy

$$H^d \;=\; p^{2d}\,*^d\,p^d\,/^d\,2^d\,\times^d\,\hat{m}^d \;=\; (\,p\,*^d\,p\,)\hat{1}^d\,/^d\,2\,m\,\hat{1}^d \;=\;$$

$$=\; p\,*^d\,p\,/\,2m \;=\; -p\,*\,p\,/\,2m \;=\; -H \qquad (1.4.23)$$

(we have tacitly applied here the isodual isooperations from Vol. I), and the same property persists under the addition of potentials. At any rate, the magnitude of the energy is given by the isodual norm (Ch. 1.2)

$$|\,H^d\,|^d \;=\; -|\,H\,|\,. \qquad (1.4.24)$$

Also, time moves in the forward direction in our universe, while it moves in the backward direction in the isodual universe, evidently because in the isodual isofield $\hat{R}^d(t,\hat{1}^d_t\,),\hat{1}_t < 0$,

$$|\,t^d\,|^d \;=\; -|\,t\,|\,. \qquad (1.4.25)$$

With an understanding of these preliminaries, the *contravariant isodual Hamilton–isotopic equations* were introduced by this author in ref. [3] and can be write

$$\dot{a}^{d\mu} = \hat{1}_2^{d}{}_{\mu\alpha}\omega^{d\alpha\nu}\,\frac{\partial^d H^d}{\partial^d a^\nu} \quad \left\{ \begin{array}{l} -\dot{r}^i \;=\; -\hat{1}^d{}_2{}^i{}_j\,\dfrac{\partial H^d}{\partial p_j} \\[3mm] -\dot{p}_i \;=\; +\hat{1}^d{}_{2i}{}^j\,\dfrac{\partial H^d}{\partial r_j} \end{array} \right. \qquad (1.4.26)$$

the *isodual Lie–isotopic brackets* on $T^*\hat{E}^d\,(x,\hat{\delta}^d,\hat{R}^d\,)$ are

$$[C \hat{,} D]^d = \frac{\partial C}{\partial a^\mu} \, \hat{1}^d{}_2{}^\mu{}_\alpha(t, a, \dot{a}, ...) \, \omega^{\alpha\nu} \frac{\partial D}{\partial a^\nu} = -[C \hat{,} D], \qquad (1.4.\,27)$$

while the *isodual isocanonical Hamilton–Jacobi equations* are

$$\frac{\partial^d \hat{A}^\circ {}^d}{\partial t^d} d + H^d T_t{}^d = 0, \quad \frac{\partial^d \hat{A}^{\circ d}}{\partial a^{d\mu}} d = R^{\circ d}{}_\alpha T^d{}_\mu{}^\alpha, \quad \mu = 1, 2, ..., 6N, \quad (1.4.28a)$$

We then have the following important result[3]

Universal invariance under isoduality. *All physical laws are invariant under isoduality.*

This can be seen in a number of ways, e.g., via the property that the Hamilton–isotopic equations change sign under isoduality although they are now referred to a negative–definite isounit[7]

$$\omega_{\mu\alpha} T_2{}^\alpha{}_\nu \frac{da^\nu}{dt} = \frac{\partial H}{\partial a^{d\nu}} = -\omega^d{}_{\mu\alpha} T_2{}^d{}^\alpha{}_\nu \frac{d^d a^\nu}{dt^d} = -\frac{\partial^d H^d}{\partial a^{d\nu}}, \qquad (1.4.29)$$

This implies the following:

Proposition 1.4.1 [3]: *The classical equations of motion for particles in a given isospace, and the equations for antiparticles in the isodual isospace coincide .*

This property is mathematically trivial from Eq.s (1.4.29) but physically nontrivial. For instance, it implies that *the trajectory of an antiparticle of charge –e in isoeuclidean space under an external magnetic field coincides with the trajectory of a particle of charge –e in Euclidean space under the same external field* (see later on Sect. 11.8.7).

To state it in different terms, the trajectories of particles and antiparticles under the same external magnetic field are manifestly different *when represented in our space.* An important objective of isodual theories is to make that difference disappear when the antiparticle is represented in isodual spaces. In actuality, the entire conjugation from particles to antiparticles, including the change of sign of the charge, is represented by isoduality, as we shall see.

These aspects will be better studied at the operator relativistic level of Ch.s 11.7–11.9. What is important at this point is that the basic concepts are purely classical [3] and that hadronic mechanics merely provides their operator realization.

Note that the isodual mechanics cannot be even conceived within the

---

[7] Note that in the conjugation $E(r, \hat{\delta}, \hat{R}) \to E^d(r, \hat{\delta}^d \hat{R}^d)$ the local coordinates are unchanged. This implies that $m^d \dot{r}^d \equiv m\dot{r}$, which corresponds precisely to Hamilton's equations in the *covariant,* rather than contravariant form.

mathematical context of Hamiltonian and Birkhoffian mechanics, because of an essential dependence on the *generalization of the basic unit.*

We now introduce, apparently for the first time, the following important

Theorem 1.4.1 (Direct Universality of Hamilton-isotopic mechanics):

All local–differential,[8] closed, variationally nonselfadjoint and regular Newtonian systems with conserved total energy H *admit a direct representation in a star–shaped neighborhood of their variables via the Hamilton–isotopic equations.*

Proof: Theorem 11.1.3.1 and the gauge degrees of freedom (11.1.3.16) establish the existence of a Birkhoffian representation of the systems considered with H as the generator of the time evolution and a given Birkhoff tensor $\Omega_{\mu\nu}(a)$. The above theorem then follows from the decomposition

$$(\Omega_{\mu\nu}) = (\omega_{\mu\alpha})(T^{\alpha}{}_{\nu}) = (\hat{\omega}_{\mu\nu}), \quad (T^{\alpha}{}_{\nu})^{t} = (T^{\alpha}{}_{\nu}), \qquad (1.5.30)$$

where T is symmetric, with corresponding contravariant forms

$$(\Omega^{\mu\nu}) = (1^{\mu}{}_{\alpha})(\omega^{\alpha\nu}), \quad (1^{\mu}{}_{\alpha})^{t} = (1^{\alpha}{}_{\nu}), \qquad (1.5.31a)$$

$$(\Omega^{\mu\nu}) = (\Omega_{\alpha\beta})^{-1}, \quad (\omega^{\mu\nu}) = (\omega_{\alpha\beta})^{-1}, \quad (1^{\mu\alpha}) = (T^{\alpha}{}_{\nu})^{-1}, \qquad (1.3.31b)$$

with consequential reduction of Birkhoff's into the Hamilton–isotopic equations

$$\Omega_{\mu\nu}(a) \, \dot{a}^{\nu} = \omega_{\mu\alpha} \, T^{\alpha}{}_{\nu}(a) \, \dot{a}^{\nu} = \partial_{\mu} H , \qquad (1.5.32)$$

and reformulation of the theory over an isofield $\hat{R}(\hat{n},+,*)$ with isounit $\hat{1} = T^{-1}$. **q.e.d.**

Numerous examples can be derived from the various applications of Vol. 11, ref.s [2]. For instance, a first class of example occurs when T is a scalar function, i.e.

$$\Omega_{\mu\nu}(a) = \omega_{\mu\nu} T(a) , \qquad (1.4.33)$$

restricted by the condition of Ω being a symplectic two–form (see, e.g., [loc. cit.], p. 102). Additional examples will be presented during the course of our analysis when studying closed two– and three–dimensional nonhamiltonian systems.

We finally note that the most general formulation of the type under consideration here is given by the *Birkhoff–isotopic* (or *isobirkhoffian*) *mechanics* [3] which is based on the most general possible isoaction

$$\hat{A} = \int_{-\infty}^{+\infty} \hat{\Theta} = \int_{-\infty}^{+\infty} \tilde{R} * da = \int_{-\infty}^{+\infty} (R_{\mu} \, T^{\mu}{}_{\nu} \, da^{\nu} - B \, T_{t} \, dt), \qquad (1.4.34)$$

with *covariant Birkhoff–isotopic equations*

---

[8] The case of nonlocal–integral or discontinuous forces is unknown at this moment,

$$\Omega_{\mu\nu} \, \dot{a}^\nu \;=\; \Omega_{\mu\alpha}(a) \, T_2{}^\alpha{}_\nu(t, a, \dot{a}, ..) \, \dot{a}^\nu \;=\; \frac{\partial B(t, a)}{\partial a^\mu}, \qquad (1.4.35)$$

where $\Omega_{\mu\nu}(a)$ is Birkhoff's tensor and B the Birkhoffian, with *general Lie-isotopic brackets*

$$[C \, \hat{,} \, D]^* \;=\; \frac{\partial C}{\partial a^\mu} \, 1_2{}^\mu{}_\alpha(t, a, \dot{a}, ...) \Omega^{\alpha\nu}(a) \, \frac{\partial D}{\partial a^\nu}, \qquad (1.4.36)$$

fundamental Lie–isotopic brackets

$$[\, a^\mu \, \hat{,} \, a^\nu \,]^* \;=\; 1_2{}^\mu{}_\alpha \, \Omega^{\alpha\nu}, \qquad (1.4.37)$$

and *general isocanonical Hamilton–Jacobi equations*

$$\frac{\partial \hat{A}}{\partial t} + B \, T_t = 0, \qquad \frac{\partial \hat{A}}{\partial a^\mu} = R_\alpha T^\alpha{}_\mu, \quad \mu = 1, 2, ..., 6N, \quad (1.4.38a)$$

The last set of equations can be written in the disjoint r– and p–notation

$$\frac{\partial \hat{A}^\circ}{\partial r_a{}^k} = P_{ai}(r, p) \, T^i{}_k(t, r, p, \dot{p}, ...), \qquad \frac{\partial \hat{A}^\circ}{\partial p_{ak}} \equiv Q_a{}^i(r, p) \, T_i(t, r, p, \dot{p}, ...). \quad (1.4.40a)$$

In summary, the primary difference between the Hamilton–isotopic and the Birkhoff–isotopic mechanics is the transition from the canonical Pfaffian functions $R^\circ = \{p, 0\}$ to the general ones $R = \{P, Q\}$. This implies the replacement of the canonical tensor ω with the Birkhoffian one everywhere in the theory. The general loss of direct physical meaning of the Hamiltonian H then follows, thus suggesting the use of the Birkhoffian B.

The *isodual Birkhoff–isotopic mechanics* can be derived accordingly. The reader should keep in mind that this volume is dedicated to the operator image of the Hamilton–isotopic mechanics and not to that of the Birkhoff–isotopic mechanics for technical reasons indicated in the next chapter.

We close this section with a clarification of the terms "non–first–order Lagrangians" or "isolagrangians" used in Vol. 1 and in the rest of our analysis. As well known, *first–order Lagrangians* are given by the familiar expressions with a dependence on time, coordinates and their first–order time derivative, $L = L(t, r, \dot{r})$. The systems we are studying are instead of arbitrary order (App. 11.1.C). This would require *second–order Lagrangians* $L = L(t, r, \dot{r}, \ddot{r})$ for which the conventional canonical formalism is lost, as it is well known.

At this point the reader can see the effectiveness of the isotopic methods. In fact, by embedding all second– and higher order, nonhamiltonian terms in the isotopic elements T and T_t, a Hamiltonian becomes fully definable. In turn, this

permits the definition of a *first-order isolagrangian*; that is, a Lagrangian $\hat{L}(t, r, \dot{r})$ which is of first-order in the isospace $\hat{E}(r,\delta,\hat{R})$, but which becomes of second-order $\hat{L}(t, r, \dot{r}) \equiv L(t, r, \dot{r}, \ddot{r})$ when projected in the original space $E(r,\delta,R)$.

An illustration is provided by the kinetic energy which can be written in isospace

$$\hat{L}(\dot{r}) \;=\; \tfrac{1}{2} * \hat{m} * \dot{r}^2 \;=\; \tfrac{1}{2} m \, \dot{r} * \dot{r}, \tag{1.4.41}$$

which, when projected in the ordinary space, becomes

$$\hat{L}(\dot{r}) \;\equiv\; L(t, r, \dot{r}, \ddot{r}, ...) \;=\; \tfrac{1}{2} m \, \dot{r} \; T(t, r, \dot{r}, \ddot{r}, ...) \, \dot{r} \;. \tag{1.4.42}$$

Thus, the isotopies permit the treatment of nonhamiltonian, acceleration-dependent systems under the preservation of the axioms of Hamiltonian systems.

This basic property can be studied via the *isolegendre transform*. Consider an isolagrangian which is regular,

$$\hat{\det}. \; \left(\frac{\partial^2 L}{\partial r_a{}^i * \partial r_a{}^j} \right) \;\neq\; 0 \;. \tag{1.4.43}$$

where we have used the notion of isodeterminant (Ch. I.6). Recall from functional isoanalysis (Ch. I.6) that the isotopic form of the partial derivative is given by

$$\frac{\hat{\partial} L}{\hat{\partial} \dot{r}} \;=\; \hat{1} \frac{\partial L}{\partial \dot{r}} \;. \tag{1.4.44}$$

Then we can introduce the *isocanonical momentum* via the rule

$$p_{ak} \;=\; \hat{\partial} L \hat{/} \hat{\partial} \dot{r}_a{}^k \;=\; m_a \, \dot{r}_{ak} \; T\hat{1} \;=\; m_a \, \dot{r}_{ak} \;. \tag{1.4.45}$$

The isolegendre transform can then be defined via the rule

$$\hat{L}(t, r, \dot{r}) \;\to\; \hat{H} \;=\; p * \dot{r} \;-\; \hat{L}(t, r, \dot{r}) \;=\; \hat{H}(t, r, p) \;. \tag{1.4.46}$$

In fact, for the case of the kinetic energy we have

$$\hat{L}(\dot{r}) \;=\; \tfrac{1}{2} * \hat{m} * \dot{r}^2 = \tfrac{1}{2} m \, \dot{r} * \dot{r} \;\to\; \hat{H} = p^2 \hat{/} 2 * \hat{m} \;=\; p * p \, / \, 2 \, m \;. \tag{1.4.47}$$

The same situation occurs for more general Lagrangians and Hamiltonians.

In the following we shall continue to use the symbols L and H in general because these functions must be defined on their appropriate carrier spaces, thus automatically implying their definition in isospace when it occurs.

The Birkhoff–isotopic mechanics is excessively general for our needs. In fact, the Hamilton–isotopic mechanics is directly universal for all potential forces

represented with the Hamiltonian $H = K(p) + V(t, r, p)$, as well as for all nonpotential forces of interest for hadronic mechanics, which are represented via the isounit $\hat{1}_2(t, r, p, \dot{p}, ..)$. As a result, the Hamilton–isotopic mechanics is assumed as the fundamental classical discipline for the representation of *particles in stable interior dynamical conditions*. The isodual Hamilton–isotopic mechanics is then assumed as the fundamental discipline for a*ntiparticles in stable interior dynamical conditions*.

1.5: HAMILTON-ADMISSIBLE AND BIRKHOFF–ADMISSIBLE MECHANICS AND THEIR ISODUALS

The primary emphasis of the preceding section was the preservation of conventional conservation laws under generalized interior dynamics.

In this section we study a generalization of Hamilton's mechanics which is structurally nonconservative; that is, it implies the nonconservation of all possible Hamiltonians $H = K + V$.

As familiar from Ch. 1.7, the most effective mechanics for this purpose is that originally proposed by Hamilton which is not the "Hamiltonian mechanics" of current general knowledge, but rather that based on the equations with *external terms*

$$\dot{r}_k = \frac{\partial H(t, r, p)}{\partial p_k} , \quad \dot{p}_k = - \frac{\partial H(t, r, p)}{\partial r_k} + F_k^{NSA}(t, r, p, \dot{p}, ...) , \quad (1.5.1)$$

which imply the *desired* nonconservation of the energy

$$d H / dt = v_k F_k^{NSA} \neq 0 . \tag{1.5.2}$$

However, Eq.s (1.5.1) do not possess a consistent algebraic structure (because their brackets violate the scalar and distributive laws, see Sect. 1.7.2). For this reason, Hamilton's original equations were rewritten by this author in the *identical* form [4]

$$\dot{a}^\mu = S^{\circ\mu\nu}(t, a, \dot{a}, ...) \frac{\partial H(t, a)}{\partial a^\nu} , \tag{1.5.3}$$

$$S^{\circ\mu\nu} = \omega^{\mu\nu} + s^{\mu\nu} , \quad (s^{\mu\nu}) = \text{diag.} (0, F^{NSA} / (\partial H / \partial p)) ; \tag{1.5.4}$$

namely, the algebraic tensor $S^{\circ\mu\nu}$ is composed of a totally antisymmetric part, the canonical Lie tensor $\omega^{\mu\nu}$, plus a totally symmetric part, the new tensor $s^{\mu\nu} = s^{\nu\mu}$.

Eq.s (1.5.3) were called by this author *Hamilton-admissible equations* because their underlying brackets

$$(C, D) = \frac{\partial C}{\partial a^{\mu}} \, S^{\circ \mu \nu}(t, a, \dot{a}, ...) \, \frac{\partial D}{\partial a^{\nu}} \, . \qquad (1.5.5)$$

characterize a *Lie-admissible algebra* in the sense that the attached antisymmetric brackets are Lie (see Ch. I.7 for details)

$$(C, D) - (D, C) = 2[C, D] \, . \qquad (1.5.6)$$

The above occurrence is nontrivial. It was thought for over a century that Lie algebras were the only nonassociative algebras appearing in physics. This author pointed out beginning in 1967 (see the historical notes of Sect. I.7.3) the Lie-admissible structure of the true Hamilton's equations. This established the physical relevance of one of the largest known class of nonassociative algebras in the most fundamental aspect of dynamics, the brackets of the classical time evolution [4].

More recent advances [3] have required a reformulation of Eq.s (I.5.3) into a form suitable for their operator image, which is called *Hamilton-admissible* (or *genohamiltonian*) *mechanics* as referred to in these volumes. These latter advances can be expressed by writing the Lie-admissible tensor $S^{\circ \mu \nu}$ in a way as close as possible to the Lie-isotopic one; i.e., via the a factorization of the canonical-Lie tensor $\omega^{\mu \nu}$ time a tensor here denoted $\hat{1}^{>}$

$$S^{\circ \mu \nu} = \hat{1}^{> \mu}{}_{\alpha} \omega^{\alpha \nu}, \qquad (1.5.\ 7)$$

where the quantity $\hat{1}^{>}$ called the *space genounit* (see next chapter) is given by

$$\hat{1}^{>}{}_{\alpha}{}^{\nu} = I_{\alpha}{}^{\nu} + s^{\beta \alpha} \omega_{\alpha \nu}, \qquad (S^{\circ \mu \nu}) = (\hat{1}^{> \mu}{}_{\alpha})(\omega^{\alpha \nu}) =$$

$$= \begin{pmatrix} 0 & I \\ -I & s \end{pmatrix} = \begin{pmatrix} I & 0 \\ s & I \end{pmatrix}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \qquad (1.5.8)$$

and all quantities have $3N \times 3N$-dimension. In addition, we have the *time genounits* $\hat{1}_{t}^{>}$ and $^{<}\hat{1}_{t}$ which are independent from the space ones.

As one can see, in the transition from the Lie-isotopic to the Lie-admissible formulations, the isounit changes from a totally symmetric form $\hat{1}$ to a form which is neither totally antisymmetric nor totally symmetric

$$\hat{1}: \hat{1}_{\alpha}{}^{\beta} \equiv \hat{1}_{\beta}{}^{\alpha} \quad \rightarrow \quad \hat{1}^{>}: \hat{1}^{>}{}_{\alpha}{}^{\beta} \neq \pm \hat{1}^{>}{}_{\beta}{}^{\alpha} \, . \qquad (1.5.9)$$

The assumption of the quantity $\hat{1}^{>}$ ($^{<}\hat{1}$) as the basic genounit for motion forward in time (backward in time), implies a further generalization of isonumbers and isofields into *genonumbers* and *genofields* $\hat{R}^{>}(\hat{n}^{>}, +, ^{>})$

($^<\hat{R}(^<\hat{n},+,^<)$) for the space part, with corresponding genofields for the time part $\hat{R}_{\uparrow>}$ and $\hat{R}_{<\uparrow}$, as well of isospaces into genospaces $\hat{E}^>(r,\hat{\delta}^>,\hat{R}^>)$ and $^<\hat{E}(r,^<\hat{\delta},^<\hat{R})$, often represented for simplicity with the unified notations $^<\hat{R}^>(^<\hat{n}^>,+,<>)$ and $^<\hat{E}^>(r,^<\hat{\delta}^>,^<\hat{R}^>)$ and $^<\hat{R}_t^>$, where the selection of only one direction of the multiplication is evidently understood (see Vol. 1 for details).

The Hamilton–admissible mechanics therefore has two separate *genocotangent bundles* and related basic *genoactions*:

$$\hat{R}_{\uparrow>} \times T^*\hat{E}^>(x,\hat{\delta}^>,\hat{R}^>): \quad \hat{A}^{\circ>} = \int_{-\infty}^{+\infty} (R^\circ > da - H >_t dt^>) , \qquad (1.5.10a)$$

$$\hat{R}_{<\uparrow} \times T^{*<}\hat{E}(x,^<\hat{\delta},^<\hat{R}) : \quad ^<\hat{A}^\circ = \int_{-\infty}^{+\infty} (da < R^\circ - {}^<td <_t H) . \qquad (1.5.10b)$$

The use of variational techniques then permits the identification of two separate *Hamilton–admissible equations* , one for motion forward in time and one for motion backward in time

$$\dot{a}^{>\mu} = 1^>{}_2{}^\mu{}_\alpha \omega^{\alpha\nu} \frac{\partial H}{\partial a^\nu} \quad \left\{ \begin{array}{l} \dot{r}^{>i} = 1^>{}_2{}^i{}_j \dfrac{\partial H}{\partial p_j} \\[3mm] \dot{p}^>{}_i = -1^>{}_{2ij} \dfrac{\partial \hat{H}}{\partial r_j} \end{array} \right. \qquad (1.5.11a)$$

$$^<\dot{a}^\mu = {}^<1_2{}^\mu{}_\alpha \omega^{\alpha\nu} \frac{\partial H}{\partial a^\nu} \quad \left\{ \begin{array}{l} {}^<\dot{r}^i = {}^<1_2{}^i{}_j \dfrac{\partial H}{\partial p_j} \\[3mm] {}^<\dot{p}_i = -{}^<1_{2i}{}^j \dfrac{\partial H}{\partial r_j} \end{array} \right. \qquad (1.5.11b)$$

which are expressed in terms of the genotime derivatives (Ch. 1.7)

$$da^> / dt^> = 1_t^> da / dt, \quad d^<a / d^<t = {}^<1_t da / dt , \qquad (1.5.12)$$

and space genounits

$$1^>{}_{2\mu}{}^\nu = (|T_2^>{}^\alpha{}_\beta|^{-1})_\mu{}^\nu , \quad {}^<1_{2\mu}{}^\nu = (|{}^<T_2{}^\alpha{}_\beta|^{-1})_\mu{}^\nu , \qquad (1.5.13)$$

interconnected by time reversal $1^> = (^<1)^t$.

We therefore have the *forward Lie–admissible brackets*

$$(C, D)^> = \frac{\partial C}{\partial a^\mu} \, 1^>{}_2{}^\mu{}_\alpha(t, a, \dot{a}, ...) \, \omega^{\alpha\nu} \frac{\partial D}{\partial a^\nu} , \qquad (1.5.14)$$

and the *backward Lie–admissible brackets*,

$$^<(C,D) = \frac{\partial C}{\partial a^\mu} \; {}^{\triangleleft}\mathbb{1}_2{}^\mu{}_\alpha(t, a, \dot{a}, ...) \, \omega^{\alpha\nu} \; \frac{\partial D}{\partial a^\nu} , \qquad (5.15)$$

whose verification of the general Lie–admissibility is trivial (App. 1.4.A and Sect. 1.7.6).

Similarly, we have two *fundamental, forward and backward Lie-admissible brackets*:

$$(\, (a^\mu \,\hat{,}\, a^\nu \,)^> \,) = (\, \mathbb{1}_2{}^{>\mu}{}_\alpha \, \omega^{\alpha\nu} \,) = \begin{pmatrix} 0_{3N\times3N} & 1^>_{2,3N\times3N} \\ -1^>_{2,3N\times3N} & s^>_{3N\times3N} \end{pmatrix}. \qquad (1.5.16a)$$

$$(\, {}^<(a^\mu \,\hat{,}\, a^\nu \,) \,) = (\, {}^{\triangleleft}\mathbb{1}_2{}^\mu{}_\alpha \, \omega^{\alpha\nu} \,) = \begin{pmatrix} 0_{3N\times3N} & {}^{\triangleleft}\mathbb{1}_{2,3N\times3N} \\ -{}^{\triangleleft}\mathbb{1}_{2,3N\times3N} & {}^<s_{3N\times3N} \end{pmatrix} \qquad (1.5.16b)$$

The *genocanonical Hamilton–Jacobi equations* can be derived via known procedures form the basic action, and can be written in unified notation

$$\frac{\partial^<\hat{A}^{\circ>}}{\partial^<t^>} + H^<T^>{}_t = 0, \qquad \frac{\partial^<\hat{A}^{\circ>}}{\partial a^\mu} = R^\circ{}_\alpha{}^<T^{>\alpha}{}_\mu . \qquad (1.5.17a)$$

As one can see, the genoaction $^<\hat{A}^{\circ>}$ is also independent from the linear momenta.

The construction of analytic representations of given equations of motion via the Hamilton–admissible equations is elementary indeed and so is their direct universality[9]. Given nonhamiltonian vector–field for motion forward in time, one represents all potential forces F^{SA} with the Hamiltonian $H = K + V$, while all nonhamiltonian forces $F^{>NSA}$ are simply represented by the s–tensor via the rules

$$s^> = \text{diag.} (F^{>NSA} / (\partial H^>/\partial p)). \qquad (1.5.18)$$

The conjugate system moving backward in time is then represented by

$$^<s = \text{diag.} (\, {}^<F^{NSA} / (\partial^<H/\partial p)). \qquad (1.5.19)$$

Note that the above direct universality holds under a strict implementation of Basic Assumption 1.1.1, the direct physical meaning of all algorithms at hand, where $p = m\dot{r}$, $H = K + V = E$, $M = r^\times p$, etc.

Also, the direct universality of the Hamilton–admissible mechanics illustrates the corresponding direct universality of the Hamilton–isotopic mechanics. In fact, the latter discipline can be considered an evident particular case of the former when the genounits are symmetric.

The above results then establish the following

---

[9] This direct universality includes nonlocal–integral and nonhamiltonian as well as discontinuous (e.g., impulsive) forces, thus yielding a mechanics of Kadeisvili Class V.

Theorem 1.5.1 (Direct Universality of Hamilton-admissible Mechanics): *All Newtonian systems with nonconserved energy* H *can be directly represented via the Hamilton-admissible equations with the energy* H = K + V *as the generator of time evolution.*

In Ref.s [2] we established the direct universality of a first-order variational principle for the characterization of local-differential analytic systems. The above results permit us to extend such direct universality according to the following:

Corollary 1.5.1A: *All possible nonconservative Newtonian systems are representable via a genovariational principle in a sufficiently smooth neighborhood of their variables.*

As we shall see, the use of the genotopic quantization techniques will permit the achievement of a unique and unambiguous operator image of all possible Newtonian systems.

For the local-differential and analytic subcase, the above theorem can also be proved via the direct universality of Birkhoffian mechanics (Theorem 1.1.3.1). In fact, we have the following property here presented for the first time:

Proposition 1.5.1 (Reduction of Birkhoffian into Hamilton-admissible mechanics): *Birkhoff's equations can always be reduced to the Hamilton-admissible equations in each selected direction of time,*

$$\Omega_{\mu\nu}\, \mathring{a}^{\nu} \equiv \omega_{\mu\alpha}\, {}^{<T>}{}^{\alpha}{}_{\nu}\, {}^{<\mathring{a}>\nu} = \partial_{\mu}\, {}^{<H>}. \qquad (1.5.20)$$

Proof. A nondegenerate totally antisymmetry matrix $(\Omega_{\mu\nu})$ can be always decomposed into the product of a totally antisymmetry matrix $(\omega_{\alpha\beta})$ and a matrix $({}^{<T>\alpha}{}_{\beta})$ which is neither antisymmetric nor symmetric,

$$(\Omega_{\mu\nu}) = (\omega_{\mu\alpha})\, ({}^{<T>\alpha}{}_{\nu}), \quad ({}^{<T>\alpha}{}_{\nu})^{t} \neq \pm\, ({}^{<T>\alpha}{}_{\nu}), \quad (1.5.21)$$

with corresponding contravariant forms

$$(\Omega^{\mu\nu}) = ({}^{<T>\mu}{}_{\alpha})(\omega^{\alpha\nu}), \quad ({}^{<T>\mu}{}_{\alpha})^{t} \neq \pm\, ({}^{<T>\mu}{}_{\alpha}), \quad (1.5.22a)$$

$$(\Omega^{\mu\nu}) = (\Omega_{\alpha\beta})^{-1}, \quad (\omega^{\mu\nu}) = (\omega_{\alpha\beta})^{-1}, \quad ({}^{<T>\mu}{}_{\alpha}) = ({}^{<T>\alpha}{}_{\nu})^{-1}. \quad (1.5.2b)$$

Proposition 1.5.1 then follows. **q.e.d.**

An explicit example of the above reduction is given in App. II.1.A via the representation of the external Lorentz force via both the Birkhoffian and the Hamilton–admissible representations.

The fundamental invariance of the Hamilton–admissible equations is given by the *Galilei–admissible* (or *genogalinean*) *symmetry* [3] which will be studied in detail in Ch. II.7. Its function is, this time, *the characterization of time–rate–of–variations of physical quantities via a symmetry of the equations of motion.* In fact, the generators of $\widehat{G}$(3.1) remain the conventional Galilean ones, as for the Galilei–isotopic symmetry $\widehat{G}$(3.1). However, while these generators are conserver for the latter, they are not for the former by construction.

The classical (and, as we shall see, operator) Lie–isotopic formulations represent *closed–isolated* systems with generalized internal structure, while the Lie–admissible formulations represent *open–nonconservative* systems interacting with the most general known external forces.

Also, the Lie–isotopic formulation are structurally *time–reversal invariant*, i.e., reversible for a reversible Hamiltonian. The formulations are therefore ideally suited to represent, e.g., the reversible center–of–mass trajectory of Jupiter under generalized internal structure (Fig. I.1.3).

On the contrary, the Lie–admissible formulations are *structurally irreversible*, i.e., irreversible irrespective of whether the Hamiltonian is reversible or not. As such, the formulations are ideally suited to represent open–irreversible internal systems for both classical and operator settings, as we shall see.

The *isodual Hamilton–admissible mechanics* is given by the isodual map of the preceding structures.

A more general formulation is given by the *Birkhoff–admissible* (or *genobirkhoffian*) *mechanics* and its isodual [3] which are essentially given by the replacement of the canonical tensor $\omega^{\mu\nu}$ with the Birkhoff tensor $\Omega^{\mu\nu}(a)$. For these and numerous other properties, we refer the interested reader to monographs [3,4].

Owing to:

1) the remarkable simplicity in the construction of analytic representations for nonhamiltonian systems,

2) their direct universality for all infinitely possible systems (including discontinuous–impulsive forces), and

3) the strict implementation of Assumption I.1.1. on the direct physical meaning of all algorithms at hand,

the Hamilton–admissible mechanics is our fundamental classical discipline for the representation of *particles in open–irreversible interior conditions*, while the isodual Hamilton–admissible mechanics is assumed as the fundamental classical discipline for the representation of *antiparticles in open–irreversible interior conditions.*

1.6: NAMBU'S MECHANICS AND ITS ISODUAL

In 1973 Nambu [5] introduced a new mechanics, now called *Nambu's mechanics*, which is based on a novel time evolution different than the Birkhoffian, Hamilton–isotopic and Hamilton–admissible time evolutions.

Let $N \geqq 1$ and $n \geqq 2$ be two positive integers, and let

$$x_{ak}, \quad a = 1, 2, ..., N, \quad k = 1, 2, ..., n \tag{1.6.1}$$

be local coordinates, where N represents the number of the "particles" and k represents the dimension of the space. A system in Nambu's mechanics is characterized by $(n - 1)$ –"Hamiltonians", usually denoted H_1, H_2, ..., H_{n-1}.

The time evolution of a given quantity $F = F(x)$, is characterized by *Nambu's equation of motion*,

$$\frac{dF}{dt} = \{F, H_1, H_2, ..., H_{n-1}\} := \sum_a \frac{\partial(F, H_1, H_2, ..., H_{n-1})}{\partial(x_{a1}, x_{a2}, ..., x_{an})}. \tag{1.6.2}$$

where $\partial(......) / \partial(......)$ stands for the *Jacobian*.

The "fundamental commutation rules" can then be written $\{x_{a1}, x_{a2}, ..., x_{an}\}$ = 1.

As one can see, for $n = 2$, Nambu's mechanics is trivially equivalent to the conventional Hamiltonian mechanics, thus having the conventional Lie structure. However, for $n \geqq 3$, Nambu's mechanics is structurally different than all preceding mechanics outlined in this chapter. The best way to see this is from the algebraic viewpoint.

Recall that for any product (or brackets) to characterize an algebra as commonly understood, it must first verify the left and right scalar and distributive laws, and then verify additional axioms specific for the algebra at hand, such as Lie, Lie–isotopic or Lie–admissible (Sect. 1.2.4 or App. 1.4.A).

While all preceding mechanics of this chapter are based on brackets which do characterize a well defined algebra, Nambu's mechanics does not for $n \geqq 3$; that is, the generalized brackets are no longer bilinear and, as such, cannot characterize any algebra, whether Lie or nonlie.

Several connections of Nambu's mechanics with other mechanics has been studied in the literature.[10] That important for this volume is the connection between Nambu's and Birkhoff's mechanics [6]. Recall that the latter is "directly universal" for local–differential systems. Thus, Nambu's equations of motion

---

[10] In regard to the literature in the field it should be noted that studies on Nambu's mechanics via the use of transformations of the local coordinate, while useful for other purposes, are inapplicable for this Volume because of their loss of "direct representations". In fact, Nambu's mechanics will be isoquantized in the next chapter under the condition of preserving the local coordinates.

must have a form directly representable in an *identical* way via Birkhoff's equations.

This form can be readily found to be [6]

$$\{F, H_1, H_2\} \equiv [F, H]^* = \frac{\partial F}{\partial x_{ai}} \Omega^{ij} \frac{\partial H}{\partial x_{aj}} . \tag{1.6.3a}$$

$$H = H_1 + H_2, \qquad \Omega^{ij} = \epsilon^{ijk} \partial H / \partial x_a^k . \tag{1.6.3b}$$

It is an instructive exercise for the interested reader to prove that brackets (1.6.4) with realization (1.6.6) are indeed Lie–isotopic. However, the underlying space is *odd.* Thus, we have a case of the so–called *degenerate Lie–isotopic brackets.*

This essentially means that, while the algebraic character is fully defined, including isoexponentiation, isosymmetries, etc., the geometric counterpart cannot be introduced unless one retorts to complex procedures involving subsidiary constraints and the like.

This aspect is important to understand that, while a generalization of the conventional methods of symplectic quantization is possible for the operator map of the other mechanics, the same procedure is not possible for Nambu's mechanics. The occurrence explains the inability to reach an operator version of the mechanics despite numerous attempts, prior to the application of the isotopic techniques.

1.7: THE NO-"NO-INTERACTION THEOREM"

One of the most problematic aspect of classical relativistic mechanics is the so-called *"No–interaction theorem"* which essentially states that a theory invariant under the Lorentz group O(3.1) is equivalent to a theory of free particles (see the reviews in ref.s [1]).

By comparison, a theory invariant under the isotopies of the Lorentz symmetry cannot be made equivalent to a free theory, thus establishing its capability to represent nontrivial interactions beginning at the classical level. This latter property can be expressed via the following theorem introduced in App. IV.C, ref. [3].

Theorem 1.7.1 (No-"No-Interaction Theorem" [3]): *A theory invariant under the Lorentz–isotopic symmetry $\hat{O}(3.1)$ possesses irreducible interactions which persist even under the elimination of the potential in the Hamiltonian via the transformation theory.*

The main lines are so simple to appear trivial. In the conventional

canonical case, given a Hamiltonian H = K + V verifying the needed continuity and regularity conditions, there exist canonical transformations a = {r,p} → a′ = {r′(t, r, p), p′(t, r, p)} under which the potential is null, and we shall write

$$H(r, p) = K(p) + V(r) \rightarrow H'(p') = K(p'), \quad V(r') \equiv 0 . \tag{1.7.1a}$$

$$\omega^{\mu\nu} \rightarrow \omega'^{\mu\nu} \equiv \omega^{\circ\mu\nu} . \tag{1.7.1b}$$

In the transition to the Hamilton–isotopic formulations the situation is essentially the same; that is, there also exist isocanonical transformations under which the potential is rendered null. Jointly, however, the transformations are such to preserve the underlying Hamilton–isotopic equations by definition,

$$H(r, p) = K(p) + V(r) \rightarrow H'(p'), \quad V'(r') \equiv 0 , \tag{1.7.2a}$$

$$\hat\omega^{\mu\nu} \rightarrow \hat\omega'^{\mu\nu} \equiv \hat\omega^{\mu\nu}, \quad \hat\omega = \omega\hat{1} . \tag{1.7.2b}$$

The point is that the Lie–isotopic tensor is a direct representative of interactions (see App. 11.1.A below) because of its functional dependence

$$\hat\omega^{\mu\nu} = \hat\omega^{\mu\nu}(t, r, p, \dot p, ...) . \tag{1.7.3}$$

While all the interactions represented with the potential can be rendered null, those represented by the Lie–isotopic tensor itself cannot be rendered null by the very definition of isocanonical transformations, and they are therefore irreducible as per Theorem 1.7.1.

The first physical significance of the theorem is for *conventional* electromagnetic interactions. In fact, as shown in App. 11.1.A, they can be represented via the Lie–isotopic tensor, thus permitting their classical nontrivial formulation. The relevance of Theorem 1.7.1 for the nonhamiltonian studies of these volumes is self–evident.

One of the most significant recent applications of the above "No–No–Interaction Theorem" has been done by T. Gill and his associates [16] via a *proper–time relativistic dynamics* which does permit interactions. We regret the inability to review these studies at this time.

APPENDIX 1.A: CLASSICAL LIMIT OF ELECTROMAGNETIC INTERACTIONS REPRESENTED VIA GENERALIZED UNITS

As well known, the classical limit of the electromagnetic interactions is given by the *Lorentz force* [1] which can be written in terms of the charge e, electric field E_k, magnetic field M_k, and potentials $\{\phi, A_k\}$, k = 1, 2, 3 (= x, y, z) in the familiar form

$$m \ddot{r}_k = F_k{}^{SA, \, elm} = e \, (\, E_k - \epsilon_{kij} M_i \, \dot{r}_j \,)^{SA} =$$

$$= e \, [\, (- \partial_k \phi - \partial_t A_k) - \delta^{mn}{}_{kj} (\partial_m A_n) \, \dot{r}^j \,]^{SA} , \qquad (1.A.1)$$

where SA stands for *variational selfadjointness* (verification of the integrability conditions for the existence of a potential) [2] and

$$\partial_k = \partial / \partial r^k, \qquad \partial_t = \partial / \partial t , \qquad \delta^{mn}{}_{kj} = \begin{pmatrix} \delta^m{}_k & \delta^m{}_j \\ \delta^n{}_k & \delta^n{}_j \end{pmatrix} . \qquad (1.A.2)$$

As equally well known, interactions (1.A.1) are traditionally represented with the *minimal coupling rule* within the context of the truncated Hamilton's equations (1.2.6)

$$H_{Lor} = (P - e \, A)^2 + e \, \phi , \qquad P = m \dot{r} + e \, A . \qquad (1.A.3)$$

The ensuing analytic–geometric–algebraic representation is therefore that in terms of a conventional Hamiltonian vector-field characterized by the canonical symplectic structure and canonical realization of Lie algebras [1].

We shall now first study the representation of the Lorentz force (1.A.1) with the *Birkhoffian analytic, geometric and algebraic formulations* (Sect. 11.1.3) in lieu of the Hamiltonian form (1.A.3), first introduced in Ref.s [2], Vol. 11, pp. 98–101. We shall then reinterpret this result in terms of the *hamilton–admissible mechanics* apparently for the first time in this appendix, to reach a *representation of the classical Lorentz force via the genounit of the theory*, rather than the Hamiltonian.

The latter result is important for various aspects of these volumes. First, it shows *the capability of the isounit to represent conventional potential (SA) forces.* The result therefore illustrates a novel and intriguing degree of freedom of isotopic formulations, which is completely absent in classical and quantum Hamiltonian mechanics and it is given by the shifting of the representation of potential forces from the Hamiltonian to the isounit. Finally, the result is at the foundation of the representation of the classical limit of the strong interactions which, as we shall see in the next appendix, are nonhamiltonian (NSA), thus requiring the *necessary* use of the isounit.

The *Birkhoffian representation of the Lorentz force* [loc. cit.] is given by

$$B_{Lor} = p^2 / 2m + e \phi , \qquad p = m \dot{r} , \qquad a = \{ a^\mu \} = \{ r, p \} , \qquad (1.A. \, 4a)$$

$$\Omega_{\mu\nu} = \partial_\mu R_\nu - \partial_\mu R_\nu , \qquad R = \{ R_\mu \} = \{ p + e A, \, 0 \} , \qquad \mu, \nu = 1, 2, ..., 6, \qquad (1.A.4b)$$

and can be explicitly written in terms of covariant Birkhoff's equations (1.3.5) [loc. cit., p. 100]

$$
\begin{pmatrix} e\{\partial_j A_i - \partial_i A_j\}_{3\times3}) & -1_{3\times3} \\ 1_{3\times s} & 0 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} e\{\partial_i \phi - \partial_t A_i\} \\ p/m \end{pmatrix} . \tag{1.A.5}
$$

In this latter case the Lorentz force is represented via: a *Birkhoffian vector-field*, an exact, nowhere degenerate and therefore symplectic, but *noncanonical* two-form (1.3.4) and the noncanonical-Birkhoffian, Lie-isotopic tensor (1.3.7) in the realization

$$
(\Omega^{\mu\nu}) = ([\| \Omega_{\alpha\beta} \|^{-1}]^{\mu\nu}) = \begin{pmatrix} e\{\partial_j A_i - \partial_i A_j\}_{3\times3} & -1_{3\times3} \\ 1_{3\times3} & 0 \end{pmatrix}^{-1} =
$$

$$
= \begin{pmatrix} 0_{3\times3} & 1_{3\times3} \\ -1_{3\times3} & e\{\partial_j A_i - \partial_i A_j\}_{3\times3} \end{pmatrix} . \tag{1.A.6}
$$

It is an instructive exercise for the interested reader to prove that the following *Lorentz brackets,*

$$
[A,B]^* = \frac{\partial A}{\partial a^\mu} \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^\nu} =
$$

$$
= \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} + \frac{\partial A}{\partial p_i} e\{\partial_j A_i - \partial_i A_j\} \frac{\partial B}{\partial p_j} , \tag{1.A.7}
$$

satisfy the axioms of classical Lie-isotopic algebras (Ch. 1.4).

The Hamilton-isotopic reformulation (Sect. 11.1.4) of the above Birkhoffian representation is straightforward. It is based on the following isotopic realization of Theorem 11.1.5.1,

$$
\Omega_{\mu\nu} = \omega_{\mu\alpha} T^{>}_{elm}{}^{\alpha}{}_{\nu} , \tag{1.A.8a}
$$

$$
(T^{>}_{elm}{}^{\alpha}{}_{\nu}) = \begin{pmatrix} 1_{3\times3} & 0_{3\times3} \\ -e\{\partial_j A_i - \partial_i A_j\}_{3\times3} & 1_{3\times3} \end{pmatrix} , \tag{1.A.8b}
$$

with contravariant forms:

$$
\Omega^{\mu\nu} = T^{>}_{elm}{}^{\mu}{}_{\alpha} \omega^{\alpha\nu} , \tag{1.A.9a}
$$

$$
(T^{>}_{elm}{}^{\mu}{}_{\nu}) = \begin{pmatrix} 1_{3\times3} & 0_{3\times3} \\ e\{\partial_j A_i - \partial_i A_j\}_{3\times3} & 1_{3\times3} \end{pmatrix} , \tag{1.A.8b}
$$

$$
(\Omega^{\mu\nu}) = (\Omega_{\alpha\beta})^{-1} , \quad (\omega^{\mu\nu} = (\omega_{\alpha\beta})^{-1} , \quad (T^{>}_{\alpha}) = (T^{>\alpha}{}_{\nu})^{-1} , \tag{1.A.9c}
$$

where $\omega_{\mu\nu}$ is the canonical form (1.2.5.b) and $\omega^{\mu\nu}$ its Lie counterpart (1.2.7). It is easy to see that the use of the above tensors in Eq.s (1.4.10) or (1.4.12) yields Eq.s (1.A.1) or (1.A.5) identically. Note that the isotopic elements and isounits are nowhere degenerate as well as *unimodular*.

Note that the quantity (1.A.9b) *is not* symmetric. As such, it *does not*

characterize Hamilton–isotopic formulations, but rather those of *Hamilton–admissible* type. The occurrence can be expressed by saying that *the Lorentz force is represented with the genounit of the theory*. This result is fully expected because the electromagnetic field is *external*, thus calling for the Hamilton–admissible and not the Hamilton–isotopic mechanics. The above representation of the Lorentz force is therefore a verification of the consistency of the new disciplines.

Needless to say, when the system is "closed" into a many–body system with the inclusion of the external interactions, we do expect the emergence of the Hamilton–isotopic representation, as we shall see in the various examples of closed nonhamiltonian systems studied later on.

The Lorentz force can therefore be represented with the following progressions of analytic representations in the same coordinates $a = \{a^\mu\}$:

Hamiltonian representation:

$$\delta A^\circ = \delta \int (R^\circ \, da - H_{Lor.} \, dt) = \delta \int (\mathbf{P} \, d\mathbf{r} - H_{Lor.} \, dt) = 0 , \quad (1.A.10a)$$

$$R^\circ = \{ \mathbf{P}, 0 \} , \quad \mathbf{P} = m\dot{\mathbf{r}} + e\mathbf{A} , \quad H_{Lor} = (\mathbf{P} - e\mathbf{A})^2 / 2m + e\phi; \quad (1.A.10b)$$

Birkhoffian representation:

$$\delta A = \delta \int (R \, da - B_{Lor.} \, dt) = \delta \int (\mathbf{p} + e\mathbf{A}) \, d\mathbf{r} - B_{Lor.} \, dt) = 0 , \quad (1.A.11a)$$

$$R = \{ (\mathbf{p} + e\mathbf{A}), 0 \} , \quad \mathbf{p} = m\dot{\mathbf{r}}, \quad B_{Lor.} = \mathbf{p}^2 / 2m + e\phi; \quad (1.A.11b)$$

Hamilton-admissible representation:

$$\delta A^\diamond = \delta \int (R^\circ > da - \hat{H}_{Lor.} \, dt) = 0 , \quad (1.A.12a)$$

$$R^\circ = \{ \mathbf{p}, 0 \} , \quad \mathbf{p} = m\dot{\mathbf{r}} , \quad H_{Lor.} = \mathbf{p}^2 / 2m + e\phi , \quad (1.A.12b)$$

$$R^\circ > da \; := R^\circ \, T^>_{elm} \, da . \quad (1.A.12c)$$

Note that the actions of the Hamiltonian and Birkhoffian representations coincide, while the action of the Hamilton–admissible representation of the *same* Lorentz force is different from the preceding ones. Also, the Hamilton–admissible representation directly expresses the external character of the interactions, which is not equally transparent in the other representations.

According to our approach, the transition from the *particle* of charge +e to the corresponding *antiparticle* with charge –e is representable via the isodual theories. Thus, each of the above three different representations of the Lorentz force admits its own isodual form. For example, the *isodual Hamiltonian representation of the Lorentz force* is given by

$$\delta A^{\circ \, d} = \delta \int (R^{\circ \, d} \, x^d \, d^d a - H^d_{Lor.} \, x^d dt^d) = - \delta A^\circ , \quad (1.A.13a)$$

$$R^{\circ\,d} = \{ -P, 0 \}, \quad H^d_{Lor} = p^d \times^d p^d /^d 2^d \times^d m^d + e^d \times^d \phi^d = - H_{Lor}. \quad (1.A.13b)$$

The interested reader can then compute the isodualities of the remaining representations as well as other aspects not investigated at this writing, such as the possible existence of Birkhoff–isotopic and Birkhoff–admissible representations of the Lorentz force.

It is also instructive to verify that the Hamiltonian (relativistic) representation of the Lorentz force is restricted by the No–Interaction Theorem, as well known, while the Birkhoffian and Hamilton–admissible representations are not and, as such, they can be assumed as a sound basis for operator versions.

APPENDIX 1.B: CLASSICAL LIMIT OF STRONG INTERACTIONS REPRESENTED VIA GENERALIZED UNITS

As well known, contemporary theories of strong interactions admit no known classical image. On the contrary, our studies are based on the following

Fundamental assumption 1.B.1: *The strong interactions admit a well defined classical limit given by contact, zero range, nonhamiltonian, nonselfadjoint (NSA) forces with general equations of motion*

$$m \ddot{r}_k = F_k^{SA}(t, r, \dot{r}) + F_k^{NSA}(t, r, \dot{r}, \ddot{r}, ...). \quad (1.B.1)$$

The above assumption explains the vast nature of classical studies [2,3,4] as the foundations of their operator counterpart.

To see the physical origin of the above assumption recall that:

1) hadrons are not point–like, but all have a finite charge radius of ≈ 1 fm $(10^{-13}$ cm);

2) hadrons are not ideal spheres with points in them, but the densest objects measures in laboratory until now; and

3) strong interactions do not have an infinite range like the electromagnetic ones, but have a finite range of the order of the size of all hadrons.

The above experimental evidence establishes that a necessary condition to activate the strong interactions is that hadrons enter into conditions of mutual penetration, thus experiencing forces of *contact–nonhamiltonian and nonlocal–integral type.* At the classical limit, the range of 1 fm becomes null, resulting in contact, zero range, nonhamiltonian forces F_k^{MSA}.[11]

---

[11] Countless examples of zero–range forces exist in our Newtonian environment, such as the contact–resistive forces experienced by a missile in our atmosphere. When readers are first exposed to interactions with "zero range" they generally try to circumvent them with artificial constructions, because *zero–range interactions are instantaneous by conception thus being beyond the capabilities of Einstein's special and general relativities* (see Ch.s

Classical nonselfadjoint systems can be studied according to the following two different yet compatible viewpoints:

I) Classical, closed-conservative treatment possessing a Lie-isotopic structure [3]. It is generally believed that total conservation laws can occur only under *conservative* internal forces. This belief has been disproved in ref.s [2,3,4] because total conservation laws can also be achieved under nonconservative internal forces. In this latter case, we merely have internal exchanges of energy and other quantities, but always in such a way to verify the *conventional* total conservation laws.

As anticipate in Sect. I.I.I, conventional, conservative, *closed Hamiltonian (selfadjoint) systems* achieve global stability via the stability of the orbits of each constituent. On the contrary, the more general *closed nonhamiltonian (nonselfadjoint) systems* achieve global stability via a collection of orbits each of which is unstable. See Fig. II.1.1.3 for an example.

The analytic representation of closed nonhamiltonian systems has been studied in details in ref.s [2,3]. We here recall that a systems of N particles with nonnull masses m_a, $a = 1, 2, ..., N$, and equations of motion of type (I.B.1) is "closed" when characterized by the differential equations (Vol. II, pp. 235–236, ref.s [2])

$$m \ddot{r}_{ka} = F_{ka}^{SA}(t, r, \dot{r}) + F_{ka}^{NSA}(t, r, \dot{r}, \ddot{r}, ...),$$ (I.B.2a)

$$\sum_{a=1,...,N} F_a^{SA} = 0, \quad \sum_{a=1,...,N} r_a \times F_a^{NSA}{}_a = 0, \quad \sum_{a=1,...,N} \dot{r} \cdot F_a^{NSA} = 0, (I.B.2b)$$

where the last set generally constitutes *subsidiary constraints* to the equations of motion (I.B.2a). It is easy to see that under conditions (I.B.2b), Eq.s (I.B.2a) verify the conservation of all conventional, Galilean total quantities [loc. cit.],

$$E = H, \qquad P_a = \sum_a P_{ak},$$ (I.B.36a)

$$M_k = \sum_a \epsilon_{kij} r_{ai} P_{aj}, \qquad G_k = \sum_a (m_a r_{ak} - t P_{ak}).$$ (I.B.3b)

Inspection of conditions (I.B.2b) soon reveals that they can be interpreted as being *seven algebraic conditions* on the 3N components F_{ka}^{NSA}. Unconstrained

II.8 and II.9).It is easy to prove that the granting of a nonnull range to a resistive force implies the existence of a potential because of the action–at–a–distance character. In turn, it is a routine undergraduate exercise to prove that the granting to contact–resistive interactions of a potential *energy* implies trajectories without connection with physical reality. The scientific attitude which is needed for advances is therefore the observation and admission of our Newtonian physical reality which establishes the existence of zero–range instantaneous forces beyond credible doubts. Rather than attempting to adapt physical reality to existing theories, the attitude implemented in these volumes is to seek novel theories capable of representing said physical reality.

details, and examples).

Note that in this approach *strong interactions in closed systems* are *represented by the isounit.*

Since the emphasis of this approach is in total conservation laws under nonhamiltonian internal forces, the underlying methods are of the Lie–isotopic type. In fact, a primary result of ref.s [3] is the following:

Theorem 1.B.1: *The form–invariance of a nonrelativistic (relativistic) system of nonselfadjoint equations of motion under the Galilei–isotopic* $\hat{G}(3.1)$ *(Poincaré–isotopic* $\hat{P}(3.1)$*) symmetry ensures the validity of conventional, total, Galilean (Lorentzian) conservation laws.*

In different terms, the role of the Galilei–isotopic (or Poincaré–isotopic) symmetry is precisely that of imposing subsidiary conditions (1.B.2b). This classical results can be easily anticipated from the Lie–isotopic studies of Vol. 1, e.g., from the property that the basis of a Lie algebra does not change under isotopies. Thus the Galilei and Galilei–isotopic (or Poincaré and Poincaré–isotopic) symmetries have the *same* generators, and only the Lie product is generalized. Theorem 1.B.1 then follows from the fact that the generators of a Lie symmetry are the conserved quantities.

An objective of this Volume II is to show that Theorem 1.B.1 does indeed admit a fully consistent operator image.

II) Classical, open–nonconservative treatment possessing a Lie-admissible structure [4]. In this latter approach, each constituent is studied *per se* by considering the rest as external. In this case the equations of motion are given by Eq.s (1.B.2a) *without* restrictions (1.B.2b)

$$m \, \ddot{r}_{ka} = F_{ka}^{SA}(t, r, \dot{r}) + F_{ka}^{NSA}(t, r, \dot{r}, \ddot{r}, ...) . \tag{1.B.4}$$

Since the emphasis is now in nonconservation, that is, in the *time–rate–of–variation* of the Galilean (or Lorentzian) quantities, the algebraic structure cannot be of Lie–isotopic type, and must be of the covering Lie–admissible type [4].

We can therefore say that *strong interactions in open conditions* are *represented by the genounit.*

To put it differently, while the emphasis of the Lie–isotopic approach to interactions is the preservation of conventional total conservation laws under the broadest possible nonhamiltonian interactions, the emphasis of the more general Lie–admissible approach is to *maximize* (rather than restrict) their possible functional dependence. The above results can be expressed via the following:

Fundamental assumption 1.B.2: *Strong interactions in closed conditions are represented with the isounit and related Lie–isotopic methods, while those in open conditions are represented with the genounits and related*

Lie−admissible methods.

Recall that closed, Hamiltonian, N−body systems are the classical image of the atomic structure. We reach in this way the following:

Fundamental assumption 1.B.3: *Closed nonhamiltonian systems such as Jupiter's structure are the classical image of the structure of nuclei, hadrons and stars.*

We reach in this way the classical origin of an important notion of hadronic mechanics in general, and of the chemical synthesis of particles in particular, that of bound state of particles structurally beyond the representational capabilities of classical Hamiltonian mechanics and therefore beyond the capabilities of quantum mechanics.

In fact, we can say that the Solar system is a classical image of the atomic structure, *but not of the nuclear and hadronic structure.* The latter is given instead by the structure of Jupiter.

We shall study various theoretical and experimental evidence supporting the basic classical assumptions of this appendix. At this point we merely mention a crucial physical difference between closed Hamiltonian and nonhamiltonian systems, the fact that *the former exhibit a central Keplerian nucleus, while the latter do not, and admit instead the "isonucleus" which can be any constituent arbitrarily heavier or ligther than all other constituents* (see Vol. II, App. III.A of ref.s [3]).

Hadronic mechanics can be conceptually and technically derived from this notion alone. Classical experimental evidence establishes the absence of the Keplerian nucleus in the structure of Jupiter. This is sufficient to establish the lack of *exact* validity of Galilei, Lorentz and Poincaré symmetries for the form−invariant description of Jupiter's structure. In fact, the Keplerian nucleus can be absent only under contact nonhamiltonian interactions which break conventional space−time symmetries.

The classical *isogalilean, isolorentz and isopoincaré symmetries* [3] have been constructed by the author for the representing a closed−isolated system *without* the Keplerian nucleus.

In the transition to the operator setting, we can say that the "lack of a nucleus in the nuclear structure" is sufficient evidence, alone,[12] to suggest the lack of exact validity of quantum mechanics in nuclear physics and the need to search for a covering discipline. At any rate, the insistence in the exact applicability of quantum mechanics implies the necessary presence of a central isolated nucleus in the nuclear structure (because of the exact validity of the Galilei and Poincaré symmetries), which is contrary to experimental evidence,

---

[12] As we shall see in Vol. III, there are a number of additional aspects of nuclear physics which also converge toward the same conclusion.

In summary, classical, closed–isolated systems can be studied via the following methods:

1) Hamiltonian mechanics: The verification of the ten conventional (nonrelativistic) conservation laws is ensured by the imposition of form–invariance under the linear, local and canonical Galilei symmetry G(3.1). The only possible closed system then are local–differential and potential–conservative. Global stability is reached via a collection of constituents each in stable orbits. The system as a whole and each of the constituents are invariant under time – reversal.

2) Hamilton-isotopic mechanics: The validity of the ten total conservation laws is now reached via the imposition of the form–invariance under the isogalilean symmetry Ĝ(3.1) or the isopoincaré symmetry P̂(3.1) without subsidiary constraints in a way fully parallel to the corresponding Hamiltonian setting. In this case global stability is reached via a collection of constituents each one in generally nonconservative conditions. The center–of–mass trajectory is time–reversal invariant.

3) Hamilton–admissible mechanics: In this case one studies each individual constituent under the most general possible external interactions caused by the rest of the system. The mechanics then characterizes the time–rate–of–variations of the physical quantities of the constituent considered. The systems are then irreversible, thus requiring the selection of a "time arrow".

APPENDIX 1.C: ACCELERATION–DEPENDENCE OF INTERIOR DYNAMICAL FORCES

In the preceding appendix we have shown that classical experimental evidence disproves the rather general belief that closed–conservative systems exist only under potential internal forces (Fig. II.1.1.3).

In this appendix we point out additional experimental evidence disproving another rather general belief that the forces are acceleration–independent.

To begin, we note that a comparison of the equations of motion for the classical image of the electromagnetic interactions, Eq.s (1.A.1), and of the strong interactions, Eq.s (1.B.1) reveals that the former depend on time, coordinates and velocities, while the latter have the additional dependence on the accelerations as well as any needed additional quantity of interior dynamical problems, such as local density μ, temperature τ, index of refraction n, etc.

$$F^{SA}_{elm} = F^{SA}_{elm}(t, r, \dot{r}), \quad F^{NSA}_{strong} = F^{NSA}_{strong}(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...) . \quad (I.C.1)$$

The issue can be best studied via explicit solutions of systems (1.B.2). As an example, an algebraic solution for the case N = 2 requires a *necessary* dependence of the nonhamiltonian forces on the accelerations (see Vol. II, App.

III.A, ref.s [3] for brevity)

$$F^{NSA}_{ka} = K_a \dot{r}_{ka} \quad (\text{no sum}), \quad k = x, y, z, \quad a = 1, 2, \quad K_a \in R. \quad (\text{I.C.2})$$

The implications of these acceleration–dependence forces are rather deep. In fact, they imply a *classical isorenormalization of the mass* (Vol. II, App. III.A, ref.s [3]) according to which the given closed nonhamiltonian system with masses m_a and forces $F_{ka}{}^{SA}, F_{ka}{}^{NSA}$ can be formally treated as the following closed Hamiltonian system

$$\hat{m}_a \dot{r}_{ka} = F_{ka}{}^{SA}, \quad \hat{m}_a = m_a + M_a. \quad (\text{I.C.3})$$

As we shall see, this mechanism persists in full at the operator level and it is actually extended to all other *intrinsic* characteristics of the constituents.

The net result is that in the transition from closed Hamiltonian to closed nonhamiltonian systems particles change their characteristics. This property, which is apparently verified by current experimental evidence in super-conductivity, nuclear physics and particle physics (Vol. III), is at the foundation of a main application of hadronic mechanics, *the chemical synthesis of all unstable hadrons* from suitably selected, massive, ligther particles. As we shall see in Vol. III, such chemical synthesis is strictly prohibited without isorenormalization, that is, without normalizations originating from *nonhamiltonian or nonlagrangian* interactions at the classical level.

By no means do acceleration–dependent forces appear only in closed nonhamiltonian systems, because they appear in a variety of cases. We here mention, for instance, *Weber's force* [7] among two charges q_1 and q_2

$$F(r, \dot{r}, \ddot{r}) = \frac{q_1 q_2 (r_1 - r_2)}{4 \pi \epsilon_0 r^3} \left(I - \frac{\dot{r}^2}{2 c_0{}^2} + \frac{r \ddot{r}}{c_0{}^2} \right) \quad (\text{I.C.4})$$

where $r = |r_1 - r_2|$, ϵ_0 is the *vacuum permissivity* and c_0 is a constant which turns out to be equal to the speed of light in vacuum even though the treatment is nonrelativistic.

It should be recalled that the above law admits the Coulomb law in first approximation and it is compatible with Newton's laws, as well as with conventional requirements (e.g., derivability from a potential, conservation of the energy, etc.). For extended studies on Weber's electrodynamics we refer the interested reader to Graneau's monograph [8] and literature quoted therein.

It is also intriguing to note that the classical isonormalization of the mass has been reached independently by Assis [9]. Consider a particle of mass m and charge q in the *interior* a hollow metal sphere of radius R at rest and without rotations in the laboratory frame and with charge Q uniformly distributed in its surface. Then, Weber's law predicts the following value of the force on the charge q

$$F_q = \frac{q\,Q}{12\,\pi\,\epsilon_0\,c_0^2\,R}\;\ddot{r}\,,$$ (1.C.5)

where $\ddot{r}$ is now the acceleration of q with respect to the center of the sphere.

When the particle considered is interacting with N other systems (such as Earth, magnetic currents, etc.), Assis [loc. cit.] found that the total force on m is given by

$$\sum_{k\,=\,1,\ldots,N} F_k = \hat{m}\,\ddot{r}\,,$$ (1.C.6a)

$$\hat{m} = m - K,\quad K = q\,Q\,/\,12\,\pi\,\epsilon_0\,c_0^2\,R\,.$$ (1.C.6b)

This yields an alteration of the (inertial) mass of the particle, here called *Assis isorenormalization,* which is fully aligned with the preceding lines of this appendix, because it follows explicitly from the acceleration–dependence of the force.

Assis [9] concluded with the suggestion of an experiment to test isonormalization (1.C.6) which is here recommended because of the implications in elementary particle physics upon isoquantization. In fact, the closed two–body nonhamiltonian systems in particle physics are given by *two charged particles each inside the charge distribution of the other,* thus yielding an operator context whose classical image is precisely given by Eq.s (1.C.6). .Additional acceleration–dependent forces will be studied during the course of our analysis.

To summarize, the chemical synthesis of unstable hadrons (Vol. III) is crucially dependent on the isorenormalization of the intrinsic characteristics of the constituents when totally immersed one inside the other. Particularly important is the isorenormalization of the mass (or rest energy). The classical origin of these isonormalizations is given by a necessary acceleration–dependence of the internal forces compatible with total conservation laws and other conventional requirements. In fact, such acceleration–dependence implies an isorenormalization of the (inertial) mass at the purely classical level. A number of acceleration dependencies have been studied in the literature from different viewpoints, thus confirming the legitimacy of these forces even for conventional Hamiltonian systems[13], the understanding being that acceleration–dependent forces are required for closed nonhamiltonian systems. Finally, the classical isonormalization of the mass here considered is indeed verifiable with experiments, whose conduction recommended.

We finally mention another important result reached by Assis (see the second article in ref.s [9]), that *the resultant force acting on a body is null* under the assumption that the inertial forces are due to gravitational interactions with the rest of the universe (Mach principle). This result is important for the isocosmology introduced in Ch. 11.

---

[13] Note that Assis model [9] is fully derivable from a generalized canonical Hamiltonian.

APPENDIX 1.D: ARINGAZIN'S SUPERSYMMETRIC PROPERTIES OF BIRKHOFFIAN MECHANICS

In this appendix we outline the studies by Aringazin [10] on the supersymmetric properties of Birkhoffian mechanics of Sect. II.1.3. We assume for brevity a knowledge of the supersymmetric properties of conventional Hamiltonian mechanics, BRS, anti–BRS invariant states, supersymmetric charges and all that (see the extensive literature in ref.s [10]).

Following Aringazin [loc. cit.], the *generating functional for Birkhoffian mechanics* has formally the same structure as that for Hamiltonian mechanics,

$$Z = \int Da \, \delta(a^\mu - a^\mu{}_{cl}), \quad \mu = 1, 2, ..., 6N, a = \{r, p\}, \quad (1.D.1)$$

with the differences that the $a^\mu{}_{cl}$ are now solutions of Birkhoff's equations (1.3.5) and

$$\delta(a^\mu - a^\mu{}_{cl}) = \delta[a^\mu - \Omega^{\mu\nu}(a)\,\partial_\nu B]\det\{\partial_t\delta^\mu{}_\nu - \partial_\mu[\Omega^{\mu\nu}(a)\,\partial_\nu B]\}. \quad (1.D.2)$$

where $\Omega^{\mu\nu}(a)$ is the contravariant Birkhoff's tensor.

The *associated Birkhoffian function* can then be written

$$\tilde{B} = q_\mu \,\Omega^{\mu\nu}(a)\,\partial_\nu B + i\,\bar{c}_\mu\,\partial_\nu[\Omega^{\mu\nu}(a)\,\partial_\sigma B]\,c^\sigma, \quad (1.D.3)$$

where B is the Birkhoffian, c^μ ($\bar{c}^\mu$) are anticommuting quantities called *Birkhoffian ghosts (antighosts)* ,and the *Birkhoffian supersymmetric charges* are given by

$$Q = i\,c_\mu\,q_\mu, \quad \bar{Q} = i\,\bar{c}_\mu\,\Omega^{\mu\nu}\,q_\nu - \tfrac{1}{2}\partial_\mu[\Omega^{\rho\sigma}(a)]\,c^\mu\,\bar{c}_\rho\,\bar{c}_\sigma, \quad (1.D.4a)$$

$$C = c^\mu\,\bar{c}_\mu, \quad K = \tfrac{1}{2}\Omega_{\mu\nu}\,c^\mu\,c^\nu, \quad \bar{K} = \tfrac{1}{2}\Omega^{\mu\nu}\,\bar{c}_\mu\,\bar{c}_\nu. \quad (1.D.4b)$$

By introducing the realization as in the conventional Hamiltonian case

$$q_\mu = -i\,\partial/\partial a^\mu, \quad c_\mu = \partial/\partial\bar{c}^\mu, \quad (1.D.5)$$

one reaches the *inhomogeneous* Sp(2) *Lie algebra* as in the conventional Hamiltonian case

$$[Q, Q] = [\bar{Q}, \bar{Q}] = [Q, \bar{Q}] \equiv 0, \quad [C, Q] = Q, \quad [K, Q] = [\bar{K}, Q] = 0, \quad (1.D.6a)$$

$$[K, \bar{Q}] = \bar{Q}, \quad [K, \bar{K}] = C, \quad [C, K] = 2K, \quad [C, \bar{K}] = -2\bar{K}. \quad (1.D.6b)$$

where the commutator is the conventional Lie form.

Aringazin [loc. cit.] further introduces the *Birkhoffian BRS charge*

$$Q_B = e^{\beta B} Q e^{-\beta B} = Q - \beta N, \quad N = c^\mu \partial_\mu B, \tag{1.D.7}$$

where β is a real parameter, and the *Birkhoffian twisted anti–BRS charge*

$$\bar{Q}_B = e^{\beta B} \bar{Q} e^{-\beta B} =$$

$$= i \bar{c}_\mu \Omega^{\mu\nu}(a) q_\nu - \tfrac{1}{2} D_\mu [\Omega^{\rho\sigma}(a)] c^\mu \bar{c}_\rho \bar{c}_\sigma + \beta \bar{c}_\mu \Omega^{\mu\nu}(a) \partial_\nu B :=$$

$$= \bar{c}_\mu D^\mu_+ - \tfrac{1}{2} f^{\mu\nu}_{\ \ \rho} c^\rho \bar{c}_\mu \bar{c}_\nu. \tag{1.D.8}$$

where

$$f^{\mu\nu\rho} = \Omega^{\mu\alpha} \partial_\alpha \Omega^{\nu\rho} - \Omega^{\nu\alpha} \partial_\alpha \Omega^{\mu\rho}, \quad D^\mu_+ = \Omega^{\mu\nu} (\partial_\mu + \beta \partial_\nu B). \tag{1.D.9}$$

It is tedious but easy to prove that the operators D^μ_+ satisfy the closure relations

$$[D^\mu_+, D^\nu_+] = f^{\mu\nu\rho} D^+_\rho, \tag{1.D.10}$$

and therefore characterize a *Birkhoffian symmetry* which is absent in Hamiltonian mechanics because the quantities $f^{\mu\nu\rho}$ are identically null for the latter.

The equations for the BRS states are the same as in the Hamiltonian case. However, those for the anti–BRS states are different owing to the difference of structure (1.D.8) with the corresponding Hamiltonian one. In particular, the supercharges Q and $\bar{Q}$ are nilpotent in the Hamiltonian but non–nilpotent in the Birkhoffian case. However, the *Birkhoffian ergodic density* $\rho(a, c)$ remains BRS and anti–BRS invariant,

$$Q_B \rho(a, c) = 0, \quad \bar{Q}_B \rho(a, c) = 0, \tag{1.D.11}$$

i.e., satisfying the condition of preserving the *Birkhoffian flow*

$$B \rho(a, c) = 0, \tag{1.D. 12}$$

Some of the most interesting states are the following:

1) *Birkhoffian 0–ghost state*, which is anti–BRS invariant;

2) *Birkhoffian 2n–ghost states*, which are BRS invariant;

3) *Birkhoffian 1–ghost state*, which is anti–BRS invariant, and originates form the equivalence of $\bar{Q}_B \rho(a, c) = 0$ with $D^\mu_+ \rho(a, c) = 0$;

3) *Birkhoffian even–ghost sector*. In this case Aringazin [loc. cit.] finds the solution of the *Gibbs form* $\rho(a, c) = k K^n e^{-\beta B}$, $k \in R$;

5) *Birkhoffian odd–ghost sector*. In this case there is no physically relevant solution other than $\rho = 0$.

In summary, the transition from Hamiltonian to Birkhoffian mechanics implies the preservation of the inhomogeneous Sp(2) supersymmetry of the

cotangent bundle. This property is *necessary* for a correct realization of the supersymmetry in Birkhoffian mechanics owing to the *isotopic* character of the lifting and the fact that the conventional symplectic geometry is preserved in full for Birkhoffian mechanics (see Ch. 1.4). The transition is however nontrivial because Birkhoffian mechanics admits additional supersymmetries as well as peculiar properties of the ghost states which are absent in the conventional Hamiltonian case.

The extension of Aringazin's results to the Birkhoffian–isotopic and Birkhoffian–admissible mechanics (which has not been investigated at this writing) is then expected to yield significant additional supersymmetries and properties.

APPENDIX 1.E: ISOTOPIES, GENOTOPIES, MOYAL BRACKETS, q-DEFORMED BRACKETS AND OTHER FORMS

As familiar from Vol. 1, a fundamental aspect of the *isotopies* as proposed by the author back in 1978 [11] is the lifting of a given Lie–admissible product $A \times B$[14] among generic quantities A, B, into a generalized product $A \hat{\times} B$ which, for the lifting to be an isotopy, must preserve the original Lie–admissibility of $A \times B$

$$A \times B \rightarrow A \hat{\times} B. \tag{1.E.1}$$

The above lifting dcfines a corresponding isotopy among the attaches Lie brackets

$$[A, B] = A \times B - B \times A \rightarrow [A \hat{,} B] = A \hat{\times} B - B \hat{\times} A. \tag{1.E.2}$$

In particular, liftings (1.E.1) and (1.E.2) are *directly universal;* that is, they contain as particular cases all possible, axiom–preserving generalization of any given Lie–admissible and, therefore, Lie product (universality), directly in the frame of the observer (direct universality).

The structure of the Poisson brackets is precisely along these lines (App. 1.4.A). In fact, the product

$$A \times B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k}, \tag{1.E.3}$$

is *noncommutative and nonassociative,* yet it is Lie–admissible because the attached brackets

---

[14] Recall from Ch. 1.7 that $A \times B$ is said to be Lie–admissible when it is generally *noncommutative and nonassociative,* but such that the attached product $A \times B - B \times A$ is Lie.

$$[A , B] = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} \frac{\partial A}{\partial p_k} , \qquad (1.E.4)$$

are the conventional Poisson brackets.

The structure of the Hamilton–isotopic brackets (Sect. II.1.4) has been constructed along the lines of the original proposal [11]. In fact, it is based on the following isotopy of structure (1.E.3)

$$A \times B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} \rightarrow \quad A \,\hat{\times}\, B = \frac{\partial A}{\partial r^i} T^i_j(r, p, ...) \frac{\partial B}{\partial p_j} \qquad (1.E.5)$$

under the condition of preserving the original Lie–admissible character; i.e., in such a way that the attached antisymmetric brackets

$$[A \,\hat{,}\, B] = A \times B - B \,\hat{\times}\, A = \frac{\partial A}{\partial r^i} T^i_j(r, p, ..) \frac{\partial B}{\partial p_j} - \frac{\partial B}{\partial r^i} T^i_j(r, p, ...) \frac{\partial A}{\partial p_j} \quad (1.E.6)$$

are Lie–isotopic, where $\hat{1}$ is the isounit of the theory.

The direct universality of isotopic liftings (1.E.1)–(1.E.2) implies the existence of endless additional realizations, such as the Birkhoff–isotopic realization studied in the main text.

One further realization is the *Moyal product* [12]

$$A \,\hat{\times}\, B = \sum_{n=0,...,\infty} \frac{(\frac{1}{2} i\hbar)^n}{n!} \sum_{k=1,..,n} \binom{n}{k} (-1)^k (\partial_r^{n-k} \partial_p^k A)(\partial_r^k \partial_p^{n-k} B) \quad (1.E.7)$$

which is also Lie–admissible, that is, such that the attached product $[A \,\hat{,}\, B] = A\hat{\times}B - B\hat{\times}A$ is Lie, yet it is more general than the Poisson brackets.

As well known, product (1.E.7) recovers the Poisson form (1.E.3) at the limit $\hbar \rightarrow 0$ and verifies the familiar *fundamental commutation rules*

$$[r \,\hat{,}\, p] = r \,\hat{\times}\, p - p \,\hat{\times}\, r = i\hbar , \qquad (1.E.8)$$

for these reasons, it is particular significant for operator realizations as well as for the interconnection between classical and operator formulations. Moreover, product (1.E.7) expresses one among several possible forms of *nonlocality* in the sense that it has a differential structure of polynomial type. This renders the Moyal brackets particularly significant for hadronic mechanics where, as now familiar, a primary emphasis is in nonlocality.

What we learn in this appendix is that *quantization is a form of isotopies*. In fact, a first way of introducing the operator image of Hamiltonian mechanics is precisely via the *Moyal isotopy of the Poisson brackets*. More specific operator mappings are studied in the next chapter.

By no means isotopies exhaust all possible generalizations significant for

these volumes. A second family of generalizations is given by the complementary *genotopies* also induced in the original proposal [11], which are liftings of an original product into a new form,

$$A \times B \quad \rightarrow \quad A \odot B \,, \qquad\qquad (1.E.9)$$

such as to *induce* new covering axioms (the axioms of $A \times B$ are therefore admitted as a particular case of, but are not necessarily verified by $A \odot B$).

The genotopies complete the direct universality of our studies, in the sense that they include all conceivable generalizations-coverings of (classical and quantum) Hamiltonian formulations, i.e., generalizations admitting the original theories as particular case, for which purpose the genotopies were conceived [11].

The effective study of the genotopies requires first the selection of a direction in time and its association to a given ordering, say, $A > B$. The genotopies then result from a differentiation between the liftings of the ordering to the right from that to the left, $A \hat{\times}^{>} B \neq A^{<} \hat{\times} B$. As an example, we have

$$A \times^{>} B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} \rightarrow \quad A \hat{\times}^{>} B = \frac{\partial A}{\partial r^i} T^{>i}{}_j \frac{\partial B}{\partial p_j} \,, \qquad (1.E.10a)$$

$$A^{<} \times B = \frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} \rightarrow \quad A^{<} \hat{\times} B = \frac{\partial A}{\partial r^i} {}^{<}T^i{}_j \frac{\partial B}{\partial p_j} \qquad (1.E.10b)$$

which characterizes the Hamilton-admissible brackets for a suitable choice of the genounits (see Sect. 11.1.5).

A class of deformations of Hamiltonian formulations which is of genotopic, rather than isotopic type is given by the so-called *q-deformations* (see App. 1.7.A and references quoted therein). The study of their *classical* foundations is evidently essential for their proper treatment. In Ch. 1.7 we showed that the *classical origin of operator q-deformations* (of the type $AB - QBA$) is given by the original Hamilton's equations with external terms, although reformulated in their Hamilton-admissible form (1.E.10). The underlying classical formalism is therefore that of the Hamilton-admissible mechanics of Sect. 11.1.5.

As it was the case for the isotopies, there are endless possibilities of formulating the genotopies. Here we mention as an illustration the *q-deformation of Moyal product* introduced by Suzuki [13]

$$A \odot B = \sum_{n=0,\ldots,\infty} \frac{(\tfrac{1}{2}i\hbar)^n}{n!} \sum_{k=1,\ldots,n} \binom{n}{k} (-1)^k [(\partial_r{}^{n-k} \partial_p{}^k \partial_r{}^k \partial_p{}^{n-k} A] X$$

$$X \sum_{m=0,\ldots,\infty} \frac{(-\tfrac{1}{2}\gamma)^n}{m!} \sum_{k=1,\ldots,m} \binom{m}{k} (-1)^k [\partial_\xi{}^{m-k} \partial_\eta{}^k \partial_\xi{}^k \partial_\eta{}^{m-k} A] \,, \quad (1.E.11)$$

where r and p are replaced by $r\exp(i\gamma\xi)$ and $p\exp(i\gamma\eta)$, respectively, which verify

the *q–deformed commutation rules*

$$r \odot p - q \, p \odot r = i\hbar q^{\frac{1}{2}} e^{i\gamma(\xi+\eta)}, \quad q = e^{\gamma^3}. \qquad (1.E.12)$$

For additional aspects one may consult Suzuki [loc. cit.] and references quoted therein.

An aspect important for this appendix is that *operator Lie–admissible formulations are a form of genotopy of the classical Hamiltonian formulations.* In fact, *Suzuki brackets* (1.E.11) *are a genotopy of the Poisson brackets.*

Other generalized classical theories are reported in ref.s [2,3,4]. For recent additional studies one may consult McEwans [14] on a *complex Birkhoffian mechanics* and Jannussis [15] on certain implications of isotopies and genotopies.

A central objective of these volumes is to select, among all infinitely possible isotopies and genotopies of the conventional Hamiltonian formulations, those forms which: A) permit unique and unambiguous operator images; B) possess an axiomatic structure invariant under time evolutions; and C) admit clear experimental verifications.

REFERENCES

1. R. MANN, *The Classical Dynamics of Particles*, Academic Press, New York (1974); E. C. G. SUDARSHAN and N. MUKUNDA, *Classical Dynamics: A Modern Perspective*, Wiley–Interscience, New York (1974)
2. R. M. SANTILLI, *Foundations of Theoretical Mechanics*, Vol. I: *The Inverse Problem in Newtonian Mechanics* (1978), and Vol. II: *Birkhoffian Generalization of Hamiltonian Mechanics* (1983) II, Springer–Verlag, Heidelberg/New York
3. R. M. SANTILLI, *Isotopic Generalization of Galilei and Einstein Relativities*, Vol. I: *Mathematical Foundations* and Vol. II: *Classical Isotopies*, second edition, Academy of Sciences of Ukraine, Kiev (1994)
4. R. M. SANTILLI, *Lie–admissible Approach to the hadronic Structure*, Vol. I: *Nonapplicability of Galilei and Einstein Relativities ?* (1978) and Vol. II: *Coverings of Galilei and Einstein Relativities ?* (1982), Hadronic Press, Tarpon Springs, FL
5. Y. NAMBU, Phys. Rev. **7**, 2405 (1973); see also A. KALNAY Phys. Rev. **D17**, 1552 (1980)
6. A. J. KALNAY and R.M. SANTILLI, Hadronic J. **6**, 1798 (1983)
7. W. WEBER, Ann. Phys. (Germany) **73**, 193 (1848)
8. P. GRANEAU, *Ampere–Neumann Electrodynamics of Metals*, second edition, Hadronic Press, Tarpon Springs (1994)
9. A. K. T. ASSIS, J. Phys. Soc. Japan, Lett., **62**, 1418 (1993) [see also Found. Phys. Lett. 2, 301 (1989)]
10. A. K. ARINGAZIN, Hadronic J. **16**, 385 (1993), and Phys. Lett. **B314**, 333 (1993)
11. R. M. SANTILLI, Hadronic J. **1**, 228 (1978)

12. J. E. MOYAL, Proc. Cambridge Phyl. Soc. **43**, 99 (1948)
13. T. SUZUKI, J. Math. Phys. **34**, 3453 (1993)
14. J. MCEWANS, Found. Phys. **23**, 313 (1993)
15. A. JANNUSSIS and SOTIROPOULOU, in *Frontiers of Fundamental Physics,* F. Selleri and M. Barone, Editors, Plenum, London (1994)
16. T. L. GILL, W. W. ZACHARY, M. F. MAHMOOD and J. LINDESAY, Hadronic J. **17**, 449 (1994)

NOTE ADDED TO THE SECOND EDITION

A new formulation of the isotopies of classical mechanics has been recently formulated in the papers listed below which is based on the *isodifferential calculus* with *isodifferentials* of space and time $\hat{d}r^k = \hat{1}^k{}_i dr^i$, $\hat{d}t = \hat{1}_t dt$ and corresponding isoderivatives $\hat{\partial}/\hat{\partial}r^k = T_k{}^i \partial/\partial r^i$, $\hat{\partial}/\hat{\partial}t = T_t \partial/\partial t$, $\hat{1} = T^{-1}$, $\hat{1}_t = T_t^{-1}$. The main result is that *the isotopies begin with Newton's equations,*

$$\hat{m}\,\hat{d}\hat{v}_k \,/\, \hat{d}t = -\,\hat{\partial}\hat{V} \,/\, \hat{\partial}r^k , \quad \hat{v}_k = \hat{\delta}_{ki}\,\hat{d}r^i \,/\, \hat{d}t$$

by permitting the extension of the conventional equation for point particles with local–differential interactions, to extended, nonspherical and deformable particles under linear and nonlinear, local and nonlocal, potential and nonpotential, as well as Newtonian and non–Newtonian forces. The latter isotopies then extend to all subsequent analytic, geometric and operator levels. The proof of the "direct universality" of the above equations is simple and also applies at all possible levels. In this way, the main interactions studied by hadronic mechanics see their ultimate origin at the primitive level of Newton's equations.

R. M. SANTILLI, Nonlocal–integral isotopies of differential calculus, mechanics and geometries, Rendiconti Circolo Matematico di Palermo, in press

R. M. SANTILLI, Isotopic lifting of Newtonian mechanics, Revista Tecnica, **18**, (1995), in press; Isotopic lifting of analytic and quantum mechanics, Revista Tecnica **19**, (1996), in press; and Isotopic lifting of differential geometries, Revista Tecnica, **19**, (1996), in press.

2: ISOQUANTIZATION AND GENOQUANTIZATION

2.1: STATEMENT OF THE PROBLEM

In the preceding chapter we have identified analytic formulations which are directly universal for all possible nonconservative Newtonian systems.

In this chapter we identify methods for the unambiguous operator formulation of all possible nonconservative systems. These methods are of two complementary types. We first have the map of the classical Lie–isotopic formulations into the Lie–isotopic branch of hadronic mechanics, called *isoquantization,* and then the more general map of classical Lie–admissible formulations into the Lie–admissible branch of hadronic mechanics, called *genoquantization.* The first maps are used for the operator form of closed–isolated nonhamiltonian systems, while the second maps are used for open nonconservative systems.

The term *quantization* shall be kept for the conventional map of Hamiltonian into quantum mechanics (see, e.g., ref. [1]). Different names are used for the map of generalized classical mechanics into hadronic mechanics because, according to the original proposal of hadronic mechanics [2], the very notion of Planck's "quantum" ℏ is lost in favor of integro–differential generalizations.

In particular, for the case of isoquantization we have the lifting of ℏ into the (Hermitean) *isounit* of hadronic mechanics,

$$\hbar \;\;\to\;\; \hat{\hbar} = \hat{\hbar}(t, r, p, \dot{p}, \psi, \partial\psi, ...) = \hbar\hat{1}, \quad \hat{1}^\dagger = \hat{1}, \quad \hbar = 1, \quad (2.1.1)$$

while for the case of genoquantization we have the lifting of ℏ into two (nonhermitean) *genounits* depending on the assumed direction of time

$$\hbar \;\;\to\;\; \begin{cases} \hat{\hbar}^>(t, r, p, \dot{p}, \psi, \partial\psi, ...) = \hbar\hat{1}^>, \\ {}^<\hat{\hbar}(t, r, p, \dot{p}, \psi, \partial\psi, ...) = {}^<\hat{1}\,\hbar, \end{cases} \quad \hat{1}^> = ({}^<\hat{1})^\dagger, \hbar = 1. \quad (2.1.2)$$

with a corresponding dependence on $\bar\psi$ and $\partial\bar\psi$ for antiparticles.

In this chapter we consider *individual* particles or antiparticles. As a result, we avoid a *joint* dependence of the isounits and genounits on both ψ and $\bar\psi$, e.g., a dependence in $(\psi\bar\psi)$, $(\partial\psi\partial\bar\psi)$, etc., because (unlike the case for a ψ–$\bar\psi$ dependence in Lagrangians) it would imply the inability to separate the equations in ψ from

those in $\bar{\psi}$. More general dependencies of the isounits and genounits on both ψ and $\bar{\psi}$ will be considered later on when studying interactions among particles and antiparticles.

We shall also restrict classical systems to be sufficiently smooth and regular over a star–shaped region of their local variables which, under isoquantization, yield hadronic formulations of *Kadeisvili Class I* (see Sect. I.7.9 for a review). The inclusion of impulsive or generally discontinuous forces, which yields hadronic formulations of Classes IV and V, should be considered as a second step. At any rate, after achieving the formulations of Class I, the extension to the remaining classes is rather natural, thus deserving attention.

The first explicit formulation of isoquantization was reached by Animalu and Santilli in a preprint of 1988 (subsequently appeared in 1990) [3] in which they reached a consistent map of the isocanonical Hamilton–Jacobi equation into the isoschrödinger equation

$$\partial_t \hat{A}^\circ + H = 0 \quad \rightarrow \quad i\partial_t \psi = H T \psi, \qquad (2.1.3)$$

where ∂_t is a conventional derivative.

The operator image of the second set of the isocanonical Hamilton–Jacobi equations, i.e., the operator form of the momentum,

$$\nabla_k \hat{A}^\circ = p T \quad \rightarrow \quad -i \hat{1}_k^{\ i} \nabla_i \psi = p_k T \psi, \qquad (2.1.4)$$

as well as the full formulation of map (2.1.3) into the form with isoderivative which is compatible with relativistic theories, was reached by Santilli in ref. [4] of 1989.

The first study of *isosymplectic hadronization* was done by Lin in paper [5] of 1988. The map of Nambu's mechanics for triplets into an operator form was reached by Kalnay and Kalnay and Santilli [6] in 1983.

We should also indicate that the original proposals were made under the name of *hadronization*. This term is now known in high energy physics under different meanings and, therefore, unless no confusion arises, we shall adopt hereon the terms "isoquantization" and "genoquantization".

Besides ref.s [3–6], no additional papers exist in the field at this writing (summer of 1994) to the author's best knowledge. We are here referring to operator maps essentially dependent on *generalizations of the unit*, with the understanding that there are endless contributions on quantizations based on the conventional unit which are not related to these studies. In particular, no paper has appeared until now on the genoquantization which is presented here apparently for the first time.

From now on we shall introduce the important restriction that *all generalized units are independent from the local coordinates*, but dependent on all other variables and quantities, e.g.,

$$1 \;=\; 1(t,\, p,\, \dot p,\, \psi,\, \partial\psi,\, ...)\,, \qquad T = T(t,\, p,\, \dot p,\, \psi,\, \partial\psi,\, ...). \qquad (2.1.5)$$

This restriction is recommendable in order to avoid the formal dominance of gravitational aspects in an arena in which they are inessential In fact, as the reader recalls from Sect. II.1.5.4), isounits with an explicit dependence on the local coordinates do represent curvature, only reformulated via isotopic methods. The extension of the result to a dependence on local coordinates will be done in Ch.s II.8 and II.9.

As we shall see, restrictions (2.1.5) has no impact on hadronic mechanics because drag forces due to motion within physical media have a primary dependence on the *velocities* and are known to be null for a body at rest in the medium. The primary physical relevance of interest in these volumes is therefore in the derivatives of coordinates as well as of the wavefunctions.

2.2: NAIVE QUANTIZATION

For comparative purposes, let us briefly review the simplest possible quantization of Hamiltonian mechanics, known under the name of *naive quantization* [1].

It essentially consists of the following map of the canonical action (II.1.2.2a) computed in the interval [0, t]

$$A^\circ(t,\, r) \quad \rightarrow \quad -\,i\,\hbar\,\mathrm{Ln}\,\psi(t,\, r)\,, \qquad (2.2.1)$$

where ψ is a state in a conventional Hilbert space $\mathcal{H}$.

Under the above map, we have the following behaviour of the canonical Hamilton–Jacobi equations (1.2.10) for one particle in phase space

$$\frac{\partial A}{\partial t} + H = 0 \qquad \rightarrow \qquad -\,i\,\hbar\,\frac{1}{\psi}\,\frac{\partial}{\partial t}\,\psi \;+\; H^{\mathrm{oper}} \;=\; 0\,, \qquad (2.2.2a)$$

$$\frac{\partial A}{\partial r^k} - p_{ak} = 0 \qquad \rightarrow \qquad -\,i\,\hbar\,\frac{1}{\psi}\,\nabla_k\,\psi \;-\; p_k^{\mathrm{oper.}} \;=\; 0\,, \qquad (2.2.2b)$$

where the label "Oper" stands also to indicate proper symmetrization [1]. By dropping such label, Eq.s (2.2.2) can be rewritten in the familiar form of the fundamental *Schrödinger's equations* of quantum mechanics [1]

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(t,\, r) \;=\; H(t,\, r,\, p)\,\psi(t,\, r)\,, \qquad (2.2.3a)$$

$$-\,i\,\hbar\,\nabla_k\,\psi(t,\, r) \;=\; p_k\,\psi(t,\, r)\,. \qquad (2.2.3b)$$

The entire discipline can then be directly or indirectly derived from the above equations, as well known.

2.3: NAIVE ISOQUANTIZATION

The main idea of Animalu and Santilli [3] is the following. The transition from the canonical action (1.2.1) to the isocanonical form (1.4.9), $A \rightarrow \hat{A}°$, is an isotopy. As a consequence, their map into operator forms must also be an isotopy for consistency. The only term in $-i\hbar \, \text{Ln} \, \psi$ which permits an axiom-preserving isotopy is precisely the fundamental hypothesis of hadronic mechanics, the integral generalization of Planck's constant $\hbar \rightarrow \hat{\hbar} = \hbar\hat{1} = \hat{1}$, $\hbar = 1$.

This lead to the *naive isoquantization* [3] as being the following map of isoaction (II.1.3.1) computed in the interval [0, t]

$$\hat{A}° (t, x) \quad \rightarrow \quad -i\,\hat{1}(t, p, \dot{p}, \hat{\psi}, \partial\hat{\psi},...) \, \text{Ln} \, \hat{\psi}(t, x) \,. \tag{2.3.1}$$

where the isounit is hereon assumed to be of Kadeisvili's Class 1 and the symbol $\hat{\psi}$ represents the expected difference of the eigenfunctions from the quantum mechanical form ψ.

At the time of writing paper [3] the first isocanonical Hamilton–Jacobi equation was given by Eq. (1.4.15a) without the isotopic element T_t. The map proposed in ref. [3] is then given by

$$\frac{\partial \hat{A}°}{\partial t} + H = 0 \quad \rightarrow \quad -i \left(\frac{\partial \hat{1}}{\partial t} \right) \text{Ln} \, \hat{\psi} - i\hat{1} \, \frac{1}{\hat{\psi}} \, \frac{\partial}{\partial t} \hat{\psi} + H^{\text{oper.}} = 0 \tag{2.3.2}$$

which can be written

$$i \, \frac{\partial}{\partial t} \hat{\psi}(t, r) \;=\; H^{\text{eff}} * \hat{\psi}(t, r) \; := H^{\text{eff}} \, T(t, p, \dot{p}, ...) \, \hat{\psi}(t, r) \,, \tag{2.3.3a}$$

$$H^{\text{eff}} = H - i \left(\frac{\partial \hat{1}}{\partial t} \right) T \, \text{Ln} \, \hat{\psi}(t, r) \,. \tag{2.3.3b}$$

Thus, naive isoquantization (2.3.1) does indeed yield an eigenvalue equation for the Hamiltonian which has exactly the isotopic structure of quantum mechanics.

The above naive isoquantization was re-examined in ref. [4] after the inclusion of the time isotopic element T_t in Eq.s (II.1.4.15a) on the isospace

$$\hat{E}(t,\hat{R}_t) \times \hat{E}(r,\hat{\delta},\hat{R}) \colon \quad \hat{R}_t \approx R\,\hat{1}_t, \hat{1}_t = T_t^{-1} > 0, \quad \hat{1} = T^{-1} > 0, \quad \hat{1}_t \neq \hat{1}\,, \tag{2.3.4}$$

by reaching the *Schrödinger–isotopic* (or *isoschrödinger*) *equation for the*

energy

$$i \frac{\partial}{\partial t} \hat{\psi}(t, r) = i \hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) = H^{eff} * \hat{\psi}(t, r) = H^{eff} T(t, p,...) \hat{\psi}(t, r), \qquad (2.3.5)$$

where we have used the notion of isoderivative (Sect. 1.6.7), which is the final form of the equation for an overall compatibility of hadronic mechanics, including its relativistic form, classical Hamilton–isotopic mechanics, functional isoanalysis, and other aspects.

Note that the above naive isoquantization is also applicable to the Birkhoffian Hamilton–Jacobi equations (1.3.10a), as outlined in App. II.2.C.

The achievement of a correct isotopic form of the momentum was the single most laborious problem for the entire study of hadronic mechanics which delayed the developments and applications of the theory for years and forced the construction of yet an new generalization of Hamiltonian and Birkhoffian mechanics, as indicated in the preceding chapter.

It may be of interest for the reader to identify the technical difficulties for the operator form of Birkhoffian mechanics. All difficulties rest in the second set of the Hamilton–Jacobi equations (1.3.10b). In fact, when written in the disjoint form (II.1.3.11), one can identify at least two primary difficulties. The first is due to the fact that the Birkhoffian action now depends also on the linear momenta, $A = A(t, r, p)$. Any map of such action requires a mechanics beyond quantum mechanical axioms, e.g., because the "wavefunctions" must have a necessary dependence on the momenta, i.e.,

$$A_{Birkh} = A(t, r, p) \rightarrow \psi_{Birkh} = \hat{\psi}(t, r, p). \qquad (2.3.6)$$

A "wavemechanics" possessing such a general dependence, even though conceivable (see Appendix II.2.C), is unknown at this writing. At any rate, the emphasis of hadronic mechanics is to achieve the minimal possible generalization of quantum mechanics preserving its basic axioms. This requires the necessary preservation of the functional dependence of conventional wavefunctions $\hat{\psi} = \hat{\psi}(t, r)$.

To state the occurrence in different terms, the dependence $\psi = \psi(t, r)$ of the conventional wavefunctions of quantum mechanics is due to the fact that the canonical action has the dependence on (t, r) only,

$$A_{Ham} = A^°(t, r) \rightarrow \psi_{Ham} = \psi(t, r). \qquad (2.3.7)$$

This property is generally lost in Birkhoffian mechanics, thus yielding map (2.3.6).

The second difficulty is that, even assuming the subclass of Birkhoff's mechanics for which $A = A(t, r)$, which occurs for $Q(r, p) \equiv 0$ in Eq.s (II.1.3.2), the naive isoquantization of the second set of Birkhoffian Hamilton–Jacobi equation

(11.1.3.11a) yields the expression

$$\frac{\partial A}{\partial r^k} - P_k(r, p) = 0 \rightarrow -i1 \frac{1}{\hat{\psi}} \nabla_k \hat{\psi} - P_k{}^{oper}(r, p) = 0, \qquad (2.3.8)$$

where P^{oper} is the operator image of the functions $P(r, p)$ of Eq.s (11.1.3.2), which can be rewritten

$$-i \nabla_k \hat{\psi}(t, x) = P_k{}^{oper}(r, p) T(t, p, \dot{p}, ...) \hat{\psi}(t, r). \qquad (2.3.9)$$

As one can see, an isotopic structure is indeed reached, but not in a form which is generally usable for practical applications. In fact, the operator form for the momentum p, if and when obtained by solving operator equations (2.3.9), is of little practical value.

Also, recall from Sect. 1.7.8 that, from an axiomatic viewpoints, the acceptable solution for the momentum must be such to have the *fundamental isocommutation rule*

$$[r \hat{,} p] = r T p - p T r = i1, \quad 1 = T^{-1}. \qquad (2.3.10)$$

In fact, as recalled in the Preface, the above rules emerge under the most general possible *nonunitary* transformation of the conventional commutation rules (see also Sect. 11.2.5)

$$\hbar U U^\dagger = \hat{\hbar} := \hbar 1 = 1^\dagger \neq 1, \quad T = (U U^\dagger)^{-1} = T^\dagger, \quad 1 = T^{-1}, \quad \hbar \qquad (2.3.11a)$$

$$U [r, p] U^\dagger = U r U^\dagger (U U^\dagger) U p U^\dagger - U p U^\dagger (U U^\dagger)^{-1} U r U^\dagger =$$

$$= r' T p' - p' T r' = i \hbar U 0 = i \hat{\hbar} = 1 (\hbar = 1), \quad r' = U r U^\dagger, \quad p' = U p U^\dagger. \qquad (2.3.11b)$$

After achieving isotopic structure (2.3.10), the isocommutators remain invariant under the most general possible transformations, as we shall have ample opportunity to study later on.

Structure (2.3.11) excludes other forms, e.g., in which the r.h.s. is not i multiplied by the inverse of the isotopic element T. In fact, it is easy to show that such alternative forms are not invariant under their own time evolution and, therefore, not acceptable for hadronic mechanics.[15]

It is evident that Birkhoffian form (2.3.9), even after solving the operator equations in p, does not generally yield a form of the momentum operator verifying in general Eq. (2.3.10).

Despite the knowledge of the isoeigenvalue equation for the energy since the time of the original proposal of hadronic mechanics [2] of 1978, the lack of a

---

[15] As we shall see, all generalized commutators of the generic type $rTp - pTr = if(r, p, ...)$, $f \neq T^{-1}$, are axiomatically inconsistent because non invariant under the time evolution.

general expression of the momentum p verifying axiom (2.3.10) put the entire new discipline in a state of "suspended animation", evidently because of the impossibility to construct the operator form of the isosymmetries, inability to apply the new discipline to concrete problems, etc.

These are the problematic aspects that forced this author to go back again to the foundation of our contemporary knowledge, *classical mechanics*, and build a second, step–by–step generalization of Hamilton's and Birkhoff's mechanics specifically suited for the *operator* needs here considered.

The Hamilton–isotopic mechanics (Sect. 1.4) resolved the above (as well as other) problems. In fact, the use of naive isoquantization (2.3.2) for the isocanonical Hamilton–Jacobi equations (II.1.4.16a) yields the map

$$\frac{\partial \hat{A}^\circ}{\partial r^k} - p_i T^i{}_k = 0 \;\rightarrow\; -i\hat{1}\frac{1}{\hat{\psi}}\nabla_k \hat{\psi} - p_i{}^{Oper} T^i{}_k = 0, \qquad (2.3.12)$$

which can be rewritten in the final form of the *isoschrödinger equation for the momentum* [4]

$$p_k * \hat{\psi}(t, r) = p_k T(t, p, \dot{p}, ...) \hat{\psi}(t, r) = -i\hat{\nabla}_k \hat{\psi}(t, r) - i\hat{1}_k{}^i(t, p, \dot{p}, ...)\nabla_i \hat{\psi}(t, r), \quad (2.3.13)$$

where we have again used the isoderivative $\hat{\nabla}_k = \hat{1}_k{}^i\nabla_i$.

As one can see, the above expression, first of all, is universal for all possible Hamilton–isotopic systems. Second, it does indeed verify axiomatic rules (2.3.10)

$$[r^i \,\hat{,}\, p_j] = r^i T p_j - p_j T r^i = -i\hat{1}^i{}_j, \qquad (2.3.14)$$

yielding the *fundamental isocommutation rules* of hadronic mechanics

$$([a^\mu \,\hat{,}\, a^\nu]) = (i\,\omega^{\mu\alpha}\hat{1}_{2\alpha}{}^\nu) = i \begin{pmatrix} 0_{3N\times3N} & \hat{1}_{3N\times3N} \\ -\hat{1}_{3N\times3N} & 0_{3N\times3N} \end{pmatrix}. \qquad (2.4.15)$$

By recalling Eq.s (II.1.14), we see that the fundamental classical and operator commutation rules of hadronic mechanics have exactly the same geometric structure, thus confirming the achievement of a consistent operator map.

As we shall see, after the laborious achievement of expression (2.3.13), the developments, applications and verifications of hadronic mechanics reached a rapid pace which appears to be just at the beginning at this writing.

A comment to indicate the *necessity* of *functional isoanalysis* (Ch. 1.6) is recommendable. Consider a conventional Hamiltonian $H = K + V = -\Delta/2m + V$ ($\hbar = 1$). We have indicated in the preface the derivability of the isoheisenberg equation via nonunitary transformations of the conventional form. One could therefore expect a similar derivability of the isoschrödinger equation via nonunitary transforms of the conventional expression. It is important to understand that

nonunitary transforms *alone* are insufficient to build hadronic mechanics.

Consider a time–independent nonunitary transform of type (2.3.11a). Then the image of the Schrödinger equation does indeed yield an isotopic structure in the r.h.s.,

$$i\, U\, \partial_t\, \psi \;=\; i\, \partial_t\, \hat{\psi} \;=\; \hbar^{-1} U H \psi \;=\; \hbar^{-1}\, U\, H\, U^\dagger\, (U\, U^\dagger)^{-1}\, U\, \psi \;=$$

$$=\; H'\, (\hbar\, U\, U^\dagger)^{-1}\, \hat{\psi} \;=\; H'T\hat{\psi} \;=\; H * \hat{\psi}, \quad \hat{\psi} \;=\; U\psi, \quad H' \;=\; U H U^\dagger. \quad (2.3.16)$$

However, the above lifting has no clue on the appropriate definition of the D'Alembertian $\hat{\Delta}$ in the Hamiltonian $H = -\hat{\Delta}/2m + V$ and the time derivative ∂_t. The need of the functional isoanalysis is then consequential.

We close this section by indicating that, according to Eq.s (2.3.16), *quantum mechanics already possesses an isotopic structure, although realized by the simplest possible space isotopic element* $T = \hbar^{-1} = constant$ *and time isotopic element* $T_t = 1$. In fact, the entire quantum mechanics can be reformulated into the isotopic formalism of hadronic mechanics[16]. After such isotopic reformulation, the operator mapping can be unified into one, single, abstract, geometric form for both quantum and hadronic mechanics.

2.4: NAIVE GENOQUANTIZATION

Isoquantization (II.2.3.2) holds under the condition that isoaction (II.1.4.9) is time–reversal invariant, in which case the isounit $\hat{1}$ is Hermitean.

We now present, apparently for the first time, the operator form of the *two* genoactions (II.1.5.10) which, by conception, are not time–reversal invariant and describe irreversible processes.

In this case the operator units must be *nonhermitean*, and interconnected by a conjugation representing time–reversal at the operator level, here assumed to be the Hermitean conjugation,

$$\hat{1}^> \;=\; R(t, p, \dot{p}, ...) \;=\; (^<\hat{1})^\dagger \;=\; [\, S(t, p, \dot{p} ...)\,]^\dagger. \quad (2.4.1)$$

We reach in this way the *naive genoquantization*

$$\hat{A}^>(t, r) \;\to\; -\, i\, \hat{1}^>\, \mathrm{Ln}\, \hat{\psi}^>, \quad {}^<\hat{A}(t, x) \;\to\; +\, i\, {}^<\hat{1}\, \mathrm{Ln}\, {}^<\hat{\psi}, \quad (2.4.2a)$$

$$\hat{\psi}^> \;=\; (^<\hat{\psi})^\dagger, \quad (2.4.2b)$$

---

[16] As we shall see in the symplectic quantization of Sect. II.2.5 the isotopic character of $\hbar^{-1}$ emerges rather forcefully, e.g., from Eq. (II.2.5.8).

By using the first set of the Eq.s (II.1.5.15a), we reach *Schrödinger-admissible* (or *genoschrödinger*) *equations for the energy* for motion forward and backward in time, respectively,

$$i \hat{1}_t^{>} \frac{\partial}{\partial t} \hat{\psi}^{>}(t, r) = H^{eff} > \hat{\psi}^{>}(t, r) := H^{eff} R(t, p, \dot{p}, ...) \hat{\psi}^{>}(t, r), \qquad (2.4.3a)$$

$$- i {}^{<}\hat{\psi}(t, r) \frac{\partial}{\partial t} {}^{<}\hat{1}_t = {}^{<}\psi(t, r) < H^{eff} := {}^{<}\hat{\psi}(t, r) S(t, p, \dot{p}, ...) H^{eff}, \qquad (2.4.3b)$$

$$H^{eff} = H - i (\partial_t {}^{<}\hat{1}^{>}) {}^{<}T^{>} Ln {}^{<}\hat{\psi}^{>}, \qquad (2.4.3c)$$

where H^{eff} is conventionally Hermitean (and symmetrized), thus *observable under nonconservative conditions.*[17]

By using the second set of Eq.s (II.1.5.15b) we reach the *Schrödinger-admissible equations for the momentum*

$$-i \hat{1}^{>}{}_k{}^{i}(t, p,...) \nabla_i \hat{\psi}^{>}(t, r) = p_k > \hat{\psi}^{>}(t, r) = p_k R(t, p, ...) \hat{\psi}^{>}(t, r) \qquad (2.4.4a)$$

$$+ i {}^{<}\hat{\psi}(t, r) \nabla_i {}^{<}\hat{1}_k{}^{i}(t, p, ...) = {}^{<}\hat{\psi}(t, x) < {}^{<}p_k = {}^{<}\hat{\psi}(t, x) R(t, p, ...) p_k \qquad (2.4.4b)$$

The rest of the Lie–admissible branch of hadronic mechanics can be constructed accordingly, as we shall see.

The understanding of hadronic mechanics requires the knowledge that *both the naive isoquantization and genoquantization coincide with the conventional quantization at the abstract level.*

This is first due to the fact that the underlying classical spaces, the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{R})$ and genoeuclidean spaces ${}^{<}\hat{E}^{>}(r,{}^{\partial}\hat{\delta}^{>}{}^{<}\hat{R}^{>})$ (Ch. I.3) coincide at the abstract level with the conventional Euclidean space by construction. In fact, in both cases the deformation of the original metric δ is equal to the *inverse* of the deformation of the unit

$$\delta \rightarrow \hat{\delta} = \delta T, \qquad 1 \rightarrow \hat{1} = T^{-1}, \qquad (2.4.5a)$$

$$\delta \rightarrow {}^{<}\hat{\delta}^{>} = \delta {}^{<}T^{>}, \qquad 1 \rightarrow {}^{<}\hat{1}^{>} = ({}^{<}T^{>})^{-1}. \qquad (2.4.5a)$$

$$E(r,\delta,R) \approx \hat{E}(r,\hat{\delta},\hat{R}) \approx {}^{<}\hat{E}^{>}(r,{}^{\partial}\hat{\delta}^{>}{}^{<}\hat{R}^{>}), \qquad (2.3.5c)$$

resulting in an evident equivalence to such an extent to imply the identity of trajectories in all spaces, despite profound physical differences of the respective systems.[18]

At the level of operators we similarly have that the underlying *isohilbert*

---

[17] As we shall see in the next chapter, this requires a *genohilbert space* (Sect. 1.6.2) characterized by the genotopic element ${}^{<}T^{>}$ and corresponding genounits ${}^{<}T^{>}$ as a necessary condition to preserve hermiticity

[18] Recall from Ch. 1.3 that the trajectories are different when projected in our physical space.

spaces $\mathcal{H}$ and *genohilbert spaces* $^<\mathcal{H}^>$ (Ch. I.6) coincide at the abstract level with the original Hilbert space $\mathcal{H}$, also by construction. In fact, the original inner product over the field of complex numbers $<\psi, \phi> \in C(c,+,\times)$ is also deformed in an amount equal to the inverse of the deformation of $C(c,+,\times)$,

$$<\psi, \phi> \in C(c,+,\times) \quad \rightarrow \quad <\hat{\psi}, \phi> \; = <\psi, T\phi> \hat{1} \in C(c,+,\times), \tag{2.4.6a}$$

$$<\psi, \phi> \in C(c,+,\times) \quad \rightarrow \quad <\psi \overset{<>}{,} \phi> \; = <\psi, {}^<T^> \phi> {}^<\hat{1}^>, \tag{2.4.6a}$$

$$\mathcal{H} \approx \mathcal{H} \approx {}^<\mathcal{H}^>, \tag{2.4.6a}$$

thus resulting in the evident equivalence of all three Hilbert spaces despite the nonhermiticity of $^<T^>$. In fact when T is a real constant and $^<T^>$ a complex constant (or just the imaginary number i) all inner products (2.3.6) coincide with the conventional ones and the same happens for all the separations on the carrier spaces.

2.5: ISOSYMPLECTIC AND GENOSYMPLECTIC HADRONIZATION

2.5.A: Statement of the problem. We now reinspect the above results from a more technical and, therefore, more abstract viewpoint.

Despite aspects which continue to be debated, *symplectic quantization* (see, e.g., Sniatycki in ref.s [1] and references quoted therein) is a rigorous approach for the map of Hamiltonian into quantum mechanics.

The objective of the *isosymplectic hadronization* is to reach a corresponding map from the Hamilton-isotopic mechanics into the Lie-isotopic branch of hadronic mechanics according to studies initiated by Lin [5].

The main idea is so simple to appear trivial. Recall (Fig. 2.5.1) that the basic disciplines involved in symplectic quantization, the symplectic geometry, Hamiltonian mechanics and quantum mechanics, admit unique isotopic images.

symplectic geometry	ISOTOPIES	isosymplectic geometry
Hamiltonian mechanics		Hamiltonian-isotopic mechanics
quantum mechanics		Lie-isotopic branch of Hadronic Mechanics
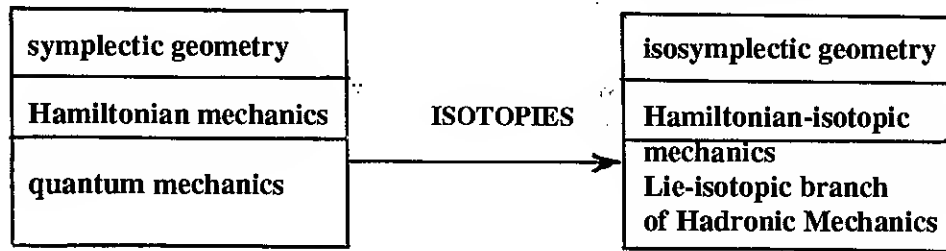
FIGURE 2.5.1: A schematic view of the main disciplines involved in isosymplectic hadronization and their inter-relations characterized by isotopies.

The main objective of isosymplectic hadronization is therefore that of identifying the isotopic image of the conventional symplectic quantization.

At this point it is important to identify the mechanism in each isotopy because the same mechanism will then yield the desired isosymplectic hadronization.

For this purpose we recall from the preceding chapter that all aspects of Birkhoffian mechanics (such as Birkhoff's tensor $\Omega_{\mu\nu}$, Birkhoff's equations, fundamental commutation rules, etc.) can be constructed from the corresponding aspects of Hamiltonian mechanics (canonical tensor $\omega_{\mu\nu}$, Hamilton's equations, fundamental commutation rules, etc.) via *noncanonical transformations* in the cotangent bundle (phase space) with chart $a = (r, p) \rightarrow a' = a'(a) = (r'(r, p), p'(r, p))$ (see ref. [7], Sect. 5.3, for detailed study).

The transition from the Birkhoffian to the Hamilton–isotopic mechanics then requires the additional factorization of the original canonical structure. For instance, the map of the canonical tensor $\omega_{\mu\nu}$ into the Hamilton–isotopic tensor $\omega_{\mu\rho} T^{\rho}{}_{\nu}$ via the intermediate passage through the Birkhoff's tensor $\Omega_{\mu\nu}$ is given by

$$\omega_{\mu\nu} \;\rightarrow\; \frac{\partial a^{\rho}}{\partial a'^{\mu}} \, \omega_{\rho\sigma} \, \frac{\partial a^{\sigma}}{\partial a'^{\nu}} \;=\; \Omega_{\mu\nu}(a') \;=\; \omega_{\mu\rho} \, T^{\rho}{}_{\nu}(a') \tag{2.5.1}$$

Note that the transformations $a \rightarrow a'$ must be local–differential for the correct map to the Birkhoffian mechanics, while they can be nonlocal–integral when mapping into the Hamilton–isotopic mechanics. Note also that the factorization of Birkhoff's tensor into the Hamilton–isotopic form *always* exists (Sect. II.1.4).

The map from quantum to hadronic mechanics is then given by a corresponding *nonunitary* transformation. Consider any given quantum structure on a Hilbert space $\mathcal{H}$ and a transformation which, by assumption, is such that $UU^{\dagger} \neq 1$. It is then easy to see that, under such transforms, all main aspects of quantum mechanics (such as Heisenberg's equations, Schrödinger's equation, fundamental commutation rules, etc.) are mapped into the corresponding aspects of hadronic mechanics.

As an example, the nonunitary transformation of the fundamental commutation rules, where now $a = (r, p)$ represents operators on $\mathcal{H}$, is given by

$$U\,U^{\dagger} = \hat{1} = \hat{1}^{\dagger} \neq I, \quad T = (U\,U^{\dagger})^{-1} = T^{\dagger}, \quad \hat{1} = T^{-1}, \tag{2.5.2a}$$

$$U\,[a^{\mu}, a^{\nu}]\,U^{\dagger} = U\,a^{\mu}\,a^{\nu}\,U^{\dagger} - U\,a^{\nu}\,a^{\mu}\,U^{\dagger} =$$

$$= U\,a^{\mu}\,U^{\dagger}(U\,U^{\dagger})^{-1}U\,a^{\nu}\,U - U\,a^{\nu}\,U^{\dagger}(U\,U^{\dagger})U\,a^{\mu}\,U^{\dagger} =$$

$$= a'^{\mu}\,T\,a'^{\nu} - a'^{\nu}\,T\,a'^{\mu} = [a'^{\mu}\,\hat{,}\,a'^{\nu}] = i\,U\,\omega_{\mu\nu}\,U^{\dagger} = i\,\omega_{\mu\nu}\hat{1}, \quad a' = U\,a\,U^{\dagger}, \tag{2.5.2b}$$

thus yielding precisely the *fundamental isocommutation rules* of hadronic mechanics (Sect. II.2.7). The correct formulation of rules (2.5.2) is then completed by lifting the original Hilbert space $\mathcal{H}$ with inner product $< \psi \mid \phi >$ into the *isohilbert space over the isofield of isocomplex numbers* (Sect. I.6.2)

$$\mathcal{H}_T: \quad <\hat{\psi}\,\hat\uparrow\,\hat\phi> \;=\; <\hat{\psi}\,|\,T\,|\,\phi>\hat{1} \;\in\; \hat{C}(\hat{c},+,*), \tag{2.5.3}$$

where $\mathcal{H}_T$ and $\hat{C}(\hat{c},+,*)$ are characterized by the same isotopic element T and isounit $\hat{1} = T^{-1}$ of rules (2.5.2), and the original associative product of arbitrary quantities ab is replaced by the isoassociative product $a*b = aTb$.

Similarly, the nonunitary transform of the quantum mechanical momentum can be written (for U independent of r)

$$U\,p\,\psi \;=\; U\,p\,U^\dagger\,T\,U\,\psi \;=\;=\; p'\,T\,\hat{\psi}\; :=\; p'*\hat{\psi} \;=$$

$$-\,i\,U\,\nabla\,\psi \;=\; -\,i\,\hat{\nabla}\,U\,\psi \;=\; -\,i\hat{1}\,\nabla\,\hat{\psi}, \qquad p' \;=\; U\,p\,U^\dagger, \qquad \hat{\psi} \;=\; U\,\psi. \tag{2.5.4}$$

Thus, *nonunitary transformations preserve the linearity in the momentum when expressed in the isolinear form*

$$p*(\hat{m}*\hat{\psi} + \hat{n}*\hat{\phi}) \;=\; \hat{m}*(p*\hat{\psi}) + \hat{n}*(p*\hat{\phi}), \qquad \hat{n},\hat{m}\,\in\,\hat{C}. \tag{2.5.5}$$

We can therefore say that *the isosymplectic hadronization is the isotopy of conventional symplectic quantization characterized, in the classical part, by noncanonical transforms factorized into the isocanonical structure $\hat{\omega} = \omega T$ over isofields with isounits $\hat{1} = T^{-1}$ and, in the operator part, by corresponding nonunitary transforms reinterpreted as isounitary on isohilbert spaces over the isofield of isocomplex numbers with the same isounit $\hat{1}$.*

The *genosymplectic hadronization* is the further generalization permitting the map of the Hamilton−admissible mechanics into the Lie−admissible branch of hadronic mechanics.

2.5.B: Symplectic quantization. The conventional *symplectic quantization* (see Sniatycki in ref.s [1] and literature quoted therein for a detailed presentation) can be summarized as follows. Considered a symplectic manifold M(a, ω, R) with local chart a and *canonical* structure ω over the reals R(n,+,×). Classical observables are the set $C^\infty(M)$ of real−valued and smooth functions on M(a,ω,R). In correspondence of any function $f \in C^\infty(M)$ there is a *Hamiltonian vector−field* X_f characterized by

$$X_f \,\lrcorner\, \omega \;=\; -\,df. \tag{2.5.6}$$

This permits the mapping of a Lie algebra g of functions $f_k \in C^\infty(M)$ characterized by the Poisson brackets into the Lie algebra $g' \approx g$ of vector−fields characterized by the commutators

$$\{f_i, f_j\} \;=\; \omega(\,[X_i, X_j]\,). \tag{2.5.7}$$

To quantize the time evolution on $M(a,\omega,R)$ we consider the Hamiltonian function $H \in C^{\infty}(M)$ and related vector field X_H. Then we have the symplectic quantization when X_H is lifted into an operator H on the *prequantization bundle* (also called *quantum line bundle*) L over a complex Hilbert space $\mathcal{K}(C)$ in such a way to preserve the connection one-form, where L is referred to as a complex line bundle over $C(c,+,\times)$ with connection ∇ such that

$$\text{curvature } \nabla = -\omega \, \hbar^{-1}. \tag{2.5.8}$$

where $\hbar$ is Planck's constant.

More generally, we can say that the symplectic quantization of a system of classical observables $f_i \in C^{\infty}(M)$ forming a Lie algebra g consists in constructing an irreducible representation of the isomorphic algebra g′ characterized by linear selfadjoint operators on a complex Hilbert space $\mathcal{K}(C)$.

This requires a prequantization bundle L over $M(a,\omega,R)$ indicated above plus a *polarization* P of $M(a,\omega,R)$ and a *metaplectic structure* on $M(a,\omega,R)$. A polarization is a sub-bundle of the complexification $T^{C*}M$ of the tangent bundle T^*M such that a fibre in the chart a, $P_a \in T^C_a{}^*M$, is a Lagrangian subspace for all possible charts a and the space of sections of the bundle P is closed under the Poisson brackets. A metaplectic structure is that of the *metaplectic group* $Mp(2n, R)$ which is the connected double covering of the *symplectic group* $Sp(2n, R)$, the latter being the natural symmetry of a symplectic manifold.

The above structures permit the introduction of a Hilbert space $\mathcal{K}_P$ with states ψ on the space $L^2(M, L)$ defined as the completion of the space of smooth sections of L over $M(a,\omega,R)$ with compact support with respect to the Hermitean inner product $< ..., ... >$ of $\mathcal{K}_P$. This allows the map of the algebra g of classical observables on $M(a,\omega,R)$ into the isomorphic linear algebra g′ of selfadjoint operators on $\mathcal{K}_P$.

2.5.C: Isosymplectic and genosymplectic hadronization. Lin [5] has been the first to show that the above symplectic quantization does indeed admit an isotopic lifting, thus permitting the isosymplectic hadronization of the Hamilton-isotopic mechanics into the Lie-isotopic branch of hadronic mechanics. In particular, Lin follows precisely the two steps indicated in Sect. 2.5.A, first the lifting of a Hamiltonian into a Birkhoffian formulation and then its Hamilton-isotopic reformulation.

The Birkhoffian lifting of the first part of the symplectic quantization (excluding the introduction of a Hilbert space) is straightforward and formally coincides with the conventional part. One starts from a symplectic manifold $M(a,\Omega,R)$ where Ω is now a Birkhoffian structure (the most general possible exact, nowhere-degenerate, thus symplectic two-form). In correspondence of a function $f \in C^{\infty}(M)$ one can define a *Birkhoffian vector-field* X_f which is such that (App. I.5.A)

$$X_f \lrcorner \ \Omega \ = \ - \, df \ . \tag{2.5.9}$$

This permits the lifting of the *Lie-isotopic algebra* $\tilde{g}$ of functions on $C^{\infty}(M)$ with isotopic product (II.1.3.8) into the Lie-isotopic algebra of vector-fields and related *Lie-isotopic second theorem* (Ch. I.4)

$$\{f_i \, , \, f_j\}^* \ = \ \Omega \, (\, [X_{f_i} \, , X_{f_j} \,]^* \,). \tag{2.5.10}$$

The mapping of the above structure into operator forms would however imply the loss of linearity, thus preventing a successful map. This originates from the fact that the classical algebra $[X_{f_i} \, , \, X_{f_j}]^*$ itself is generally nonlinear. The reformulation of the Birkhoffian structure into an *identical* Hamilton-isotopic form resolves the problem, thus permitting the achievement of an operator image which is *isolinear* (Sect.s I.2.4, and I.4.4).

The above reformulation is also straightforward. Introduce the isotopic decomposition of the Birkhoff form

$$\Omega \ = \ \omega \, T \ = \hat{\omega} \, , \tag{2.5.11}$$

where T is *symmetric*. Assume T as the isotopic element of the theory and reformulate the Birkhoffian manifold $M(a,\Omega,R)$ into the isosymplectic manifold $\hat{M}(a,\hat{\omega}, \hat{R})$ which is now defined over the isoreals $\hat{R}(\hat{n},+,*)$ with isounit $\hat{1} = T^{-1}$.

In correspondence of every function $f \in C^{\infty}(M)$ we have a *Hamilton-isotopic vector-field* X_f which is such that

$$X_f \lrcorner \ \hat{\omega} \ = \ - \, \hat{d}f \, , \tag{2.5.12}$$

where now $\hat{d}$ is, of course, the *isodifferential* (Sect. I.6.7). A set of observable $f_k \in C^{\infty}(M)$ is then equipped with the Lie-isotopic product of type (II.1.4.13) and can be mapped into the Lie-isotopic product

$$\{f_i \, \hat{,} \, f_j\} \ = \ \hat{\omega} \, (\, [X_i \, \hat{,} \, X_j] \,) \ = \ \omega \, T \, (\, X_i \, T \, X_j - X_j \, T \, X_i \,) \tag{2.5.13}$$

The crucial point is that the algebra of vector-fields when defined on an isofield $\hat{R}(\hat{n},+,*)$ with isounit $\hat{1} = T^{-1}$ is now isolinear. All steps of the symplectic quantization then admit an isotopic image into an *isolinear* algebra of isoselfadjoint operators on a isohilbert space.

In fact, we can introduce the *isoprequantization bundle* L over a isocomplex isohilbert space $\hat{\mathcal{H}}(\hat{C})$ in such a way to preserve the connection one-isoform (Ch. I.5), where $\hat{L}$ is referred to as a complex line bundle over $\hat{C}(\hat{c},+,*)$ with isoconnection $\hat{\nabla}$ such that

$$\text{isocurvature } \hat{\nabla} = -\omega \hat{\hbar}^{-1} = -\hat{\omega} \qquad (2.5.14)$$

The above result can be simply read off rule (2.5.8) because the fundamental assumption of hadronic mechanics is the replacement of $\hbar$ with $\hat{1} = T^{-1}$. We can therefore say that structure (2.5.8) *already has* isotopic structure, although of the simple type with $T = \hbar^{-1} = \text{const.}$

The *isopolarization* can also be defined via a step–by–step isotopy of the conventional polarization, with the understanding that we now have the conditions of the existence of an *isolagrangian*[19] for all charts and the entire formalism is based on isofields.

The isosymplectic manifold $\hat{M}(a,\hat{\omega},\hat{R})$ admits a well defined *isosymplectic symmetry* $\hat{Sp}(2n,\hat{R})$ with connected double covering given by the *isometaplectic symmetry* $\hat{Mp}(2n,\hat{R})$.

The above structures then imply a mapping of the isoalgebra $\hat{g}$ of vector fields X_k into an isolinear isoalgebra $\hat{g}' \approx \hat{g}$ of operators on an isohilbert space $\hat{\mathcal{H}}(\hat{C})$.

The reader familiar with the content of Volume 1 can readily see all the above results from a simple abstract viewpoint. The transition from the symplectic to the isosymplectic geometry is based on the *dual* lifting of the canonical structure ω *and* of the unit I of the theory in such a way that the deformation of the former is the *inverse* of the deformation of the latter,

$$\omega \rightarrow \hat{\omega} = \omega T, \qquad I \rightarrow \hat{1} = T^{-1}. \qquad (2.5.15)$$

This implies the complete abstract identity of the two manifolds $M(a,\omega,R)$ and $\hat{M}(a,\hat{\omega},\hat{R})$ to such an extent that the trajectories in the two manifolds coincide even though the represented systems are inequivalent. Moreover, the inclusion of nonlocal–integral systems, which is prohibited by the symplectic geometry, is permitted by its isosymplectic covering precisely because of dual lifting (2.5.15). The existence of a step–by–step isotopy of the symplectic quantization is then consequential.

We close this section with the indication without treatment that the above procedures appears to be extendable to the *genosymplectic hadronization*, that is, the mapping of *genosymplectic manifolds* $^<\hat{M}^>(a,^<\hat{\omega}^>,^<\hat{R}^>)$ (Sect. I.4.4) into the Lie–admissible branch of hadronic mechanics. In fact, basic rule (2.5.15) reads

$$\omega \rightarrow {}^<\hat{\omega}^> = \omega {}^<T^>, \qquad 1 \rightarrow {}^<\hat{1}^> = (^<T^>)^{-1}. \qquad (2.5.16)$$

where the selection of one given direction of time is understood, and the genotopic element is now *nonsymmetric.*

---

[19] As the reader recall, this is an ordinary first–order Lagrangian $\hat{L}(t, r, p)$ defined on an isospace and, thus with all products, operations etc. , being isotopic.

Rule (2.5.16) essentially states that the totally antisymmetric form ω is mapped into a generalized form $<\hat{\omega}>$ *which is no longer totally antisymmetric.* However, the basic unit I of the theory is jointly lifted into the inverse of the deformation of ω. The net result is that, again, $M(a,\omega,R)$ and $<\hat{M}>(a,<\hat{\omega}>,<\hat{R}>)$ coincide by construction at the abstract level, thus implying the consequential expectation of the genosymplectic hadronization which it is hoped will be investigated by interested mathematicians.

2.6: OPERATOR IMAGE OF NAMBU'S MECHANICS

We now outline the operator image of Nambu's mechanics for triplets studied by Kalnay and Kalnay and Santilli [6].

Recall that in this case naive quantization, symplectic quantization and their isotopic coverings are not applicable because of the loss of the underlying symplectic and isosymplectic geometries at the classical level.

We therefore search for (trilinear) *triple brackets* [..., ..., ...] which are the operator image of Nambu's classical triple brackets (..., ..., ...) (Sect. II.1.6) which permit the operator image of the basic time evolution

$$\dot{F} = (F, H_1, H_2) \quad \rightarrow \quad i\dot{F} = [F, H_1, H_2], \qquad (2.6.1)$$

(where the quantities in the r.h.s are evidently operators in a properly symmetrized form hereon implied).

To achieve the above objective, the triple brackets must verify the following conditions:

1) Let F_k, $k = 1, 2, 3$, be a set of operators forming a *triple system*. Then the operator $[F_1, F_2, F_3]$ must also belong to the same set, i.e., the triple system with the brackets considered must be a closed set.

2) The triple brackets must be alternatively antisymmetric, i.e.,

$$[F_1, F_2, F_3] = - [F_2, F_1, F_3], \text{ etc.} \qquad (2.6.2)$$

3) The triple brackets must be defined on an enveloping algebra of operators with product $A \odot B$ which permits the *differential rules* (also called *derivations*)

$$[A \odot B, C, D] = A \odot [B, C, D] + [A, C, D] \odot B, \text{ etc} \qquad (2.6.3)$$

To identify the appropriate realization of the triple brackets, consider an associative enveloping algebra ξ with conventional associative product AB and (right and left) unit I, $IA = AI = A$, $\forall A \in \xi$, defined over a field $C(c,+,\times)$ of complex

numbers

We now consider the *fundamental realization of a Lie-admissible algebra* U of operators A, B, C, ... over C introduced by this author in 1978 [2] (see Sect. 1.7.3) with bilinear product

$$(A , B) = A P B - B Q A , \qquad (2.6.4)$$

with attached Lie-isotopic algebra

$$[A , B]_U = (A , B) - (B , A) = A T B - B T A , \quad T = P + Q , \qquad (2.6.5)$$

and associator

$$(A , B , C) = ((A , B), C) - (A , (B , C)) =$$

$$= A P C Q B - B Q A P C + B P C Q A - C Q A P B . \qquad (2.6.6)$$

The operator image [... , ... , ...] of Nambu's classical brackets { ..., ..., ... } is then given by a triple system whose enveloping algebra is a Lie-admissible algebra, i.e.,

$$A \odot B = (A , B) = A P B - B Q A , \qquad (2.6.7)$$

and, in its most general possible form, is given by (see Kalnay in ref.s [6])

$$[A , B , C] := \qquad (2.6.8a)$$

$$= \alpha \left[((A, B), C) + ((B, C), A) + ((C, A), B) - ((B, A), C) - ((A, C), B) - ((C, B), A) \right] +$$

$$+ \beta \left[(A, (B, C)) + (B, (C, A)) + (C, (A, B)) - (B, (A, C)) - (A, (C, B)) - (C, (B, A)) \right] .$$

$$\alpha, \beta \in C , \qquad | \alpha | + | \beta | \neq 0 \qquad (2.6.8b)$$

Tedious but simple calculations prove that the terms within square brackets multiplied by α and β are equal. Thus, we can assume for operator realization of the triple brackets the reduced form, say, with $\alpha = 1$ and $\beta = 0$,

$$[A , B , C] = \qquad (2.6.9)$$

$$= ((A, B), C) + ((B, C), A) + ((C, A), B) - ((B, A), C) - ((A, C), B) - ((C, B), A) .$$

It is an instructive exercise to prove that the above realization does indeed verify conditions 1, 2, 3 above. Kalnay (loc. cit.) now assumes

$$P = H_1^{-1} , \qquad Q = H_2^{-1} , \qquad (2.6.10)$$

in which case the fundamental form (2.6.7) of the Lie-admissible envelope

assumes the realization

$$A \odot B = (A, B) = A H_1^{-1} B - B H_2^{-1} A, \qquad (2.6.11)$$

and the triple brackets assume the explicit form

$$[A, B, C] :=$$

$$= (A H_1^{-1} B H_1^{-1} C + B H_1^{-1} C H_1^{-1} A + C H_1^{-1} A H_1^{-1} B) -$$

$$- (B H_1^{-1} A H_1^{-1} C + A H_1^{-1} C H_1^{-1} B + C H_1^{-1} B H_1^{-1} A) -$$

$$- (A H_2^{-1} B H_2^{-1} C + B H_2^{-1} C H_2^{-1} A + C H_2^{-1} A H_2^{-1} B) +$$

$$+ (B H_2^{-1} A H_2^{-1} C + A H_2^{-1} C H_2^{-1} B + C H_2^{-1} B H_2^{-1} A). \qquad (2.6.12)$$

which also verifies the basic conditions 1, 2, 3 above.

Simple algebra via the use of triple brackets (2.6.12) then yields the following

Theorem 2.6.1 (Kalnay and Santilli [6]): *The operator image of Nambu's time evolution of an operator F on a (conventional) Hilbert space $\mathcal{H}$ expressed in terms of two Hamiltonians H_1 and H_2*

$$i \dot{F} = [F, H_1, H_2], \qquad (2.6.13)$$

is given by the time evolution of the Lie-isotopic branch of hadronic mechanics with realization

$$i \dot{F} = [F, H_1, H_2] \equiv [\hat{F, H}] = F T H - H T F, \qquad (2.6.14a)$$

$$H = H_1 + H_2, \qquad (2.6.14b)$$

$$T = H_1^{-1} + H_2^{-1}. \qquad (2.6.14c)$$

The identity (rather than the equivalence) $[F, H_1, H_2] \equiv [\hat{F, H}]$ then permits the activation for the operator Nambu's triplets of the entire isotopic methods outlined in Vol. 1.

As we shall see in Vol. III, the above results are nontrivial for quark theories. In fact, they permit the construction of isotopies of quark theories, called *isoquark theories*, verifying the following basic properties:

1) The isotopic $S\hat{U}(3)$ symmetry constructed within the context of Theorem 2.6.1 (i.e., for isotopic element $T = H_1^{-1} + H_2^{-1}$) is locally isomorphic to the conventional SU(3), thus permitting the preservation of all conventional quantum numbers as currently known.

2) The isohilbert space $\mathcal{H}_T$ for $T = H_1^{-1} + H_2^{-1}$ of the interior structural

Hilbert space $\mathcal{H}$ of the exterior problem in vacuum, thus permitting a *strict confinement*, that is, a theory with an identically null transition probability for free quark even for a null potential barrier.

3) The isoquark theory constructed via Theorem 2.6.1 admits convergent perturbative expansions for Hamiltonians $H = H_1 + H_2 > 1$, and thus for the isotopic element $T = H_1^{-1} + H_2^{-1} < 1$, by therefore offering realistic possibility of achieving a *convergent isoperturbative theory for strong interactions* beginning at the level of first isoquantization and prior to any field theoretical extension.

But, in the opinion of this author, the most important implications of Theorem 2.6.1 are of conceptual character. As we shall see, the theorem implies that isoquarks are composite, thus permitting the identification of their constituents with ordinary massive particles freely produced in the spontaneous decays, of course, under the validity of the covering hadronic mechanics for the interior structural problem.

To see the implications, recall that quark theories in their current formulation via conventional quantum mechanics have no conceivable practical application of any nature, precisely because the hadronic constituents are permanently confined in the interior of hadrons. By contract, the new *hadronic technology* recently appeared in the scientific horizon with a number of novel practical applications is due precisely to the assumption that the hadronic (that is, isoquark) constituents are ordinary massive particles which can be produced free either in the spontaneous or in the stimulated decays.

The study of these aspects is a primary task of these volumes.

2.7: MADELUNG–MROWKA–SCHUCH METHOD FOR OPEN–IRREVERSIBLE SYSTEMS

By no means do the methods considered so far in this chapter exhaust all possible mappings from classical to operator[20] formulations. Among a variety of additional methods developed in this century (which we cannot possibly review for brevity), we present in this section an additional method which focuses attention on the classical origin as well as universality of the logarithmic term in Eq.s (11.2.3.4b).

A simple yet effective method for the *quantum* mechanical image of classical, *closed–reversible systems* was identified by Madelung and Mrowka [8] in 1950. The same method was then extended by Schuch et al. [9] in 1983 to the *operator* image of *open–irreversible systems*. In this section we shall outline the

---

[20] We continue to use the word "quantum" for the operator image of *conservative-potential* systems, but avoid it for *nonconservative-nonpotential* interactions because, as established in these volumes, the latter imply the loss of the very notion of quantum of energy.

approach here called *Madelung–Mrowka–Schuch method.*

The method is based on three assumptions taken from experimental evidence:

1) Uncertainty principle;

2) Interference phenomena of material systems such as particles; and

3) Correspondence principle (Ehrenfest theorem).

Let us first review the conservative profile. Assumption 1 implies the characterization of all physical quantities, such as the position r, velocities $\dot{r}$, etc., via a distribution function $\rho(t, r)$ and a continuity equation with familiar expressions

$$< r > = \int \rho\, r\, d\tau, \qquad \int \rho\, d\tau = 1, \quad d\tau = dr^3, \qquad (2.7.1a)$$

$$\dot{\rho} + \mathrm{div}\, j = \dot{\rho} \equiv \mathrm{div}\,(\rho\, v) = 0 \qquad (2.7.1b)$$

and $\dot{\rho} = \partial\rho/\partial t$.

In analogy with optics where the intensity is a quadratic function of the amplitude, Assumption 2 requires a bilinear character of ρ and j in terms of two complex field amplitudes $\alpha(t, r)$ and $\beta(t, r)$ and we can write

$$\rho(t, r) = \alpha(t, r)\, \beta(t, r) \geq 0, \qquad (2.7.2a)$$

$$j(t, r) = C\,(\beta \nabla \alpha - \alpha \nabla \beta), \qquad (2.7.2b)$$

where C is a constant, from which we obtain

$$v = C\left(\frac{\nabla\alpha}{\alpha} - \frac{\nabla\beta}{\beta} \right) = C \nabla \mathrm{Ln}\, \frac{\alpha}{\beta}. \qquad (2.7.3)$$

In order to reach a complete solution of Eq. (2.7.1b), the phase of α (which enter in j) has to be determined. This can be best done by separating Eq. (2.7.1b) into two equations in α and β via a "separation function" $f(t, r)$ independent from α and β under which we have

$$\dot{\alpha} + C \Delta \alpha = -f \alpha \qquad (2.7.4a)$$

$$\dot{\beta} - C \Delta \beta = f \beta \qquad (2.7.4b)$$

Assumption 3 then permits the identification of the physical meaning of f. In fact, under the Ehrenfest theorem, we can write

$$< F > = m \frac{d}{dt} < v > = m \frac{d}{dt} \int \rho\, v\, d\tau =$$

$$= m \int 2\,C(\beta \nabla \alpha - \dot{\alpha} \nabla \beta)\, d\tau = m \int [-\nabla (2 C f)]\, d\tau \qquad (2.7.5)$$

It is then obvious that the separation function is proportional to the potential,

$$f = \frac{1}{2\,m\,C}\,V(r)\,.$$
(2.7.6)

By putting

$$\alpha(t,r) = \psi(t,r)\,, \quad C = \hbar/2\,m\,i\,,$$
(2.7.7)

Eq. (2.7.4a) yields the *conventional Schrödinger equation* identically,

$$i\,\hbar\,\frac{\partial}{\partial t}\,\psi(t,r) = [-\frac{\hbar^2}{2\,m}\,\Delta + V(r)]\,\psi(t,r)\,.$$
(2.7.8)

This completes our review of the *conservative–reversible case*.

Schuch's generalization for open–irreversible systems can be summarized as follows. In order to represent nonconservation[21] and irreversible time evolutions, it is necessary to extend the continuity equation (2.7.1b) to include a *diffusion current density* j_D,

$$j_D(t,r) = -D\,\nabla\,\rho(t,r) = -D\,(\beta\,\nabla\,\alpha + \alpha\,\nabla\,\beta)\,,$$
(2.7.9)

where D is the *diffusion coefficient*. The breaking of time–reversal invariance is then evident. The *fundamental classical equation* of the *Madelung–Mrowka–Schuch method* is therefore of the Fokker–Planck type

$$\dot\rho + \mathrm{div}\,(\rho\,v) - D\,\Delta\,\rho = 0\,,$$
(2.7.10)

from which we obtain

$$-D\,\Delta\,\rho = -D\,[\alpha\,\Delta\,\beta - \beta\,\Delta\,\alpha + 2\,(\nabla\,\alpha)\,(\nabla\,\beta)]\,.$$
(2.7.11)

Schuch [loc. cit.] shows that, among the various possibilities for j_D, the realization permitting the separability of the α and β components, that is, permitting two separate equations for ψ and its conjugate[22], is given by

$$D\,\Delta\,\rho\,/\,\rho = \gamma\,(\mathrm{Ln}\,\rho + \tilde{Z})\,,$$
(2.7.12)

---

[21]With the term *dissipation* we refer to systems with energy monotonically *decreasing* in time, while *nonconservation* refers to systems in which the energy can monotonically *decrease or increase*.

[22] We should recall that, if ψ represents a given particle, $\bar\psi$ represents its antiparticle. The separability we are referring to is, therefore, the achievement of two separate equations, one for particles and the complementary one for antiparticles. Equations in the mixed states $(\psi\bar\psi)$ are avoided at this stage because, as indicated at the beginning of this section, they would represent particle–antiparticle interactions.

where γ is constant and $\tilde{Z} = Z + \bar{Z} = 2\,\mathrm{Re}(Z)$ is a function independent of α and β. Ansatz (2.7.12) implies that the normalization is possible iff

$$\tilde{Z} = -<\mathrm{Ln}\,\rho>. \tag{2.7.13}$$

Schuch's complete condition of separability [loc. cit.] then reads

$$-D\,\Delta\rho/\rho = \gamma\,(\mathrm{Ln}\,\rho - <\mathrm{Ln}\,\rho>). \tag{2.7.14}$$

Substitution into Eq. (2.7.10) yields the following generalization of Schrödinger's equation

$$i\hbar\,\frac{\partial}{\partial t}\psi(t,r) = [-\frac{\hbar^2}{2m}\Delta + V(r) - i\gamma\hbar(\mathrm{Ln}\,\psi - <\mathrm{Ln}\,\psi>)]\,\psi(t,r). \tag{2.7.15}$$

here called *logarithmic Schrödinger equation for open–irreversible systems*, in order to differentiate it from other types of nonlinear forms.

Moreover, according to Ehrenfest theorem, the total force is given by the sum of a conservative term and a nonconservative one,

$$<F> = \int \rho\,(-\nabla V - m\gamma v)\,d\tau = <-\nabla V> - m\gamma<v>. \tag{2.7.16}$$

Also, the nonconservative contribution can be represented by the term

$$\tilde{W}_S := W_S - <W_S> = -i\gamma\hbar(\mathrm{Ln}\,\psi - <\mathrm{Ln}\,\psi>) = \tag{2.7.17a}$$

$$= \frac{\gamma\hbar}{2i}(\mathrm{Ln}\,\frac{\psi}{\bar{\psi}} - <\mathrm{Ln}\,\frac{\psi}{\bar{\psi}}>) + \frac{\gamma\hbar}{2i}(\mathrm{Ln}\,(\psi\bar{\psi}) - <\mathrm{Ln}\,(\psi\bar{\psi})>). \tag{2.7.17b}$$

As one can see, $\tilde{W}_S$ is complex, but $<\tilde{W}_S> = 0$. Assuming that at the value $t = 0$ the system is conservative with energy $<H> = E_O =$ constant, them at a later time t we have

$$<H> = <K> + <V> = E(t) = \text{nonconserved energy}. \tag{2.7.18}$$

This confirms that Eq. (2.7.15) represents the interaction of a particle in a resistive medium considered as external, which is the operator image of the classical projectile moving in our atmosphere. For additional intriguing properties, we refer the reader to Schuch's papers [9].

As one can see, Eq. (2.7.15) confirms the logarithmic term in Eq.s (2.3.4) and identifies its alternative interpretation as originating from a classical diffusion term in the continuity equation. Moreover, by putting

$$\psi = R\psi^>, \qquad -i\tfrac{1}{2}\hbar\gamma = -i(\partial^>_t 1)^> T^>, \tag{2.7.19a}$$

$$- i \frac{1}{2} \hbar \gamma < \mathrm{Ln} \, \psi' > = - i \, (\, \partial^>_t \, 1^>) \, T^> \, \mathrm{Ln} \, 1^>, \tag{2.7.19b}$$

the r.h.s. of Eq. (2.4.3a) assumes precisely the structure of Eq. (2.7.15), i.e.,

$$[- \hbar^2 \Delta / 2m + V(r) - i \gamma \hbar (\mathrm{Ln} \, \psi' - < \mathrm{Ln} \, \psi' >)] \, \psi'(t, r) . \tag{2.7.20}$$

Recalling that Eq.s (2.4.3) are "directly universal" for all possible nonconservative systems, one therefore sees the corresponding "direct universality" of the logarithmic term for all possible open-irreversible systems.

The primary difference between eq.s (2.4.3) and (2.7.15) is that the former are defined on genofields, genospaces, etc. beginning with the classical formulations, while the latter are defined on conventional fields, carrier spaces, etc. also beginning with classical settings.

The reformulation of Eq.s (2.7.15) in the formalism of hadronic mechanics (left here to the interested reader for brevity) is not a mere mathematical curiosity because it permits: the treatment of open-irreversible systems under *genolinearity,* that is, in a form axiomatically equivalent to the conventional linearity for conservative-reversible conditions; the achievement of Hermiticity for nonconserved Hamiltonians, that is, the first rigorous formulation of the experimental evidence according to which the energy is indeed observable when nonconserved; and other aspects.

APPENDIX 2.A: FUNDAMENTAL ROLE OF LIE-ADMISSIBLE ALGEBRAS IN OPERATOR MAPPINGS

In App. II.1.E we have shown that Lie-admissible algebras are at the foundations of classical representations of systems with nonpotential-nonhamiltonian interactions, the Lie-isotopic approach being a *particular case* of the Lie-admissible formulations.

It is important to review these algebraic aspects from the viewpoint of their mappings into operator forms and illustrate the reason why the original proposal [2] suggested the Lie-admissible algebras as the algebraic foundations of hadronic mechanics.

Recall from App. II.1.E that the *nonassociative* Lie-admissible algebras originate at the primitive level of the Poisson brackets

$$(A, B) = \frac{\partial A}{\partial r_k} \frac{\partial B}{\partial r_k} = \textbf{Nonass. Lie-admis. algebra} \tag{2.A.1a}$$

$$[A, B] = (A, B) - (B, A) =$$

$$\frac{\partial A}{\partial r^k} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r_k} \frac{\partial A}{\partial p_k} = \frac{\partial A}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \textbf{Lie algebra} \tag{2.A.1b}$$

where: a = (r, p), ω is the canonical Lie tensor, and the *nonassociativity* of the product (A , B) is trivially proved by the property $((A , B), C) \neq (A, (B, C))$.

The covering Birkhoffian mechanics (Sect. 11.1.3) has an algebraically similar structure, but of the more general isotopic type

$$(A \,\hat{,}\, B) = \frac{\partial A}{\partial a^{\mu}} \, S^{\mu\nu} \frac{\partial B}{\partial a^{\nu}} = \text{\textbf{Isotopic Lie-adm. Algebra}} , \qquad (2.A.2a)$$

$$[A \,\hat{,}\, B] = (A \,\hat{,}\, B) - (B \,\hat{,}\, A) =$$

$$= \frac{\partial A}{\partial a^{\mu}} \, \Omega^{\mu\nu}(a) \frac{\partial B}{\partial a^{\nu}} = \text{\textbf{Lie-isotopic algebra}} , \qquad (2.A.2b)$$

where $S^{\mu\nu} \neq - \hat{S}^{\nu\mu}$ is the Birkhoff–admissible tensor of Sect. 11.1.5, i.e., such that $\Omega^{\mu\nu} = S^{\mu\nu} - S^{\nu\mu}$, where $\Omega^{\mu\nu}$ is Birkhoff's tensor. On the basis of these results (which were fully known in 1978), ref. [2] proposed the construction of a generalization of quantum mechanics according to the operator structure

$$(A , B) = A \, R \, B - B \, S \, A = \text{\textbf{Gen. Nonass. Lie-adm. Algebra}} \qquad (2.A.3a)$$

$$[A \,\hat{,}\, B] = (A , B) - (B , A) \equiv ATB - BTA = \text{\textbf{Lie-isotopic algebra}} \qquad (2.A.3b)$$

where T = R + S, whose equivalence to the classical structures (2.A.2) and (2.A.3) is manifest.

We now examine the above algebraic structures from the viewpoint of their mappings into operator forms. The conventional (naive or symplectic) quantization essentially deals with the mapping of a Lie algebra of functions equipped with the *Poisson brackets* to an algebra of operators equipped with the *commutator*

$$[A, B]_{Cl \, M} = (A, B) - (B, A) = \text{\textbf{Lie}} \; \Rightarrow \; [A, B]_{QM} = A \, B - B \, A = \text{\textbf{Lie}} . \quad (2.A.4)$$

But at the level of enveloping algebras, this implies the mapping of a *nonassociative* into an *associative* algebra

$$(A , B) = \frac{\partial A}{\partial r_k} \frac{\partial B}{\partial r_k} = \text{\textbf{Nonass. Lie-adm.}} \; \Rightarrow \; AB = \text{\textbf{Ass. Lie-adm.}} \quad (2.A.5)$$

which evidently does not exist, with a consequential algebraic inequivalence between classical and quantum formalisms which may well result to be the origin of a number of still unsettled aspects in conventional quantization.[23]

---

[23] Recall from Sect. 11.2.5 that the problem is circumvented in symplectic quantization via the lifting of the algebra of *functions equipped with the Poisson brackets,* into the algebra of *vector–fields equipped with the conventional commutator,* and then passage to operator settings. Despite that, inequivalence (2.A.5) clearly persists.

On the contrary, the covering isotopic theories are based on the following mapping which was conceived [2] to resolve the above inequivalence

$$[A \hat{,} B]_{Cl. M.} = (A \hat{,} B) - (B \hat{,} A) = \textbf{Lie-isotopic} \quad \Rightarrow$$

$$\Rightarrow \quad [A \hat{,} B]_{H M} = (A \hat{,} B) - (B \hat{,} A) = \textbf{Lie-isotopic} , \qquad (2.A.6)$$

under the crucial condition of compatibility of classical and operator envelopes

$$(A \hat{,} B) = \frac{\partial A}{\partial a^\nu} S^{\mu\nu} \frac{\partial B}{\partial a^\nu} = \textbf{Nonass. Lie-adm.} \Rightarrow$$

$$\Rightarrow \quad (A \hat{,} B) = A R B - B S A = \textbf{Nonass. Lie-adm.} \qquad (2.A.7)$$

This implies an evident algebraic equivalence, first, in the classical and operator envelopes and, then, in the attached Lie products.

The second foundational aspect (also fully known in 1978) can be expressed via the following

Lemma 2.A.1 [2]: *Any Lie–isotopic theory with* a *nonassociative Lie–admissible* envelope *characterized by the product*

$$(A, B) = A R B - B S A, \quad \text{Det. R, S, R} \pm \text{S} \neq 0, \qquad (2.A.8)$$

can *always* be *identically formulated in terms of* an *isoassociative* envelope *with product* ATB, T = R + S *under the rule*

$$(A, B) - (B, A) = A (R+S) B - B (R+S) A \equiv A T B - B T A \equiv [A \hat{,} B] . \quad (2.A.9)$$

This illustrates the reasons why the nonassociative character of the envelope of the Lie–isotopic formulations is de–emphasized, and replaced with the isoassociative envelope.

Alternatively, we can say that *the isoassociative character of the envelope of hadronic mechanics permits a reinterpretation of the operator brackets with a nonassociative envelope which achieves full algebraic equivalence with the corresponding classical formulations.*

Rather than being mere mathematical curiosities, the emergence of nonassociative Lie–admissible algebras at the foundations of classical and operator mechanics has deep physical implications studied in this volume and in the next. It is sufficient at this moment to recall the lack of achievement of an exact confinement of quarks even under infinite potential barriers (because of the tunnel effects originating from Heisenberg's uncertainties), while the theory of isoquarks does indeed achieve an exact confinement (because of the incoherence of the internal and external Hilbert spaces) even in the absence of a

potential barrier (see Vol. III).

The ultimate origin of the above disparity lies in the content of this section. In fact, quark theories are based on quantization (2.A.4), while isoquark theories are based on the more general isoquantization (2.A.7) and rule (2.A.9).

Numerous additional properties of our treatment of the strong interactions, such as the lack of convergent perturbation theory for conventional quantization (2.A.4) as compared to a convergent isoperturbation theory for isoquantization (2.A.7), also see their origin in the algebraic properties identified in ref. [2].

APPENDIX 2.B: OPERATOR IMAGE OF BIRKHOFF, BIRKHOFF-ISOTOPIC AND BIRKHOFF-ADMISSIBLE MECHANICS

In this appendix we present, apparently for the first time, the operator image of Birkhoff's equations (II.1.3.5). As discussed in this chapter, a main property is that the Pfaffian action (II.1.3.1) depends also on the velocities, $A = A(t, r, p) = A(t, a)$, a $= (r, p)$. Therefore, the emerging "wavefunctions" depend also on momenta, $\hat{\psi} = \hat{\psi}(t, r, p) = \psi(t, a)$. It then follows that *the operator image of the Birkhoffian mechanics is outside the axioms of hadronic mechanics because the wavefunctions of the latter have the dependence* $\hat{\psi}(t, r)$. Nevertheless, such operator image is intriguing both per se, as well as a possible generalization of hadronic mechanics itself.

Recall that the Pfaffian action $A(t, a)$ is noncanonical. We therefore introduce the following *naive generalized operator map* for the simpler case in which the isounit is independent of a

$$A(t, a) \rightarrow -i\hat{T}(t, \dot{a}, ...) \, \text{Ln} \, \hat{\psi}(t, a), \quad \hat{T} = \hat{T}^\dagger > 0, \qquad (2.B.1)$$

which maps the Birkhoffian Hamilton–Jacobi equations (II.1.3.10) into the forms

$$\frac{\partial A(t, a)}{\partial t} + B(t, a) = 0 \quad \rightarrow \quad -i\frac{\partial \hat{T}}{\partial t} \, \text{Ln} \, \hat{\psi} - i\hat{T}\frac{\hat{T}}{\hat{\psi}}\frac{\partial}{\partial t}\hat{\psi} = 0, \qquad (2.B.2a)$$

$$\frac{\partial A(t, a)}{\partial a^\mu} - R_\mu = 0 \quad \rightarrow \quad -i\hat{T}\frac{\hat{T}}{\hat{\psi}}\frac{\partial}{\partial a^\mu}\hat{\psi} - R_\mu = 0. \qquad (2.B.2b)$$

The latter then yield the *Birkhoffian–Schödinger equations*

$$i\frac{\partial}{\partial t}\hat{\psi}(t, r) = B^{eff} * \hat{\psi} = B^{eff}(t, a)\,\hat{T}(t, \dot{a}, ...)\,\hat{\psi}(t, a), \qquad (2.B.3a)$$

$$R_\mu * \hat{\psi}(t, R) = R_\mu(t, a)\,T(t, \dot{R}, ...)\,\hat{\psi}(t, R) = -i\frac{\partial}{\partial a^\mu}\hat{\psi}(t, R). \qquad (2.B.3b)$$

$$B^{eff} = B(t, a) - i(\partial_t \hat{T})\,\hat{\psi}(t, a)\hat{T}. \qquad (2.B.3c)$$

As one can see, the latter equations do have an isotopic structure, nevertheless, *the derivative are conventional.* We leave it as an exercise for the interested reader to prove that the derivative themselves become isotopic when one considers the Birkhoff-isotopic equations. The operator form of the Birkhoff-admissible equations can be constructed by relaxing the Hermiticity of the isounit and the method of Sect. 11.2.4.

As one can see, in the Hamilton-isotopic and hadronic mechanics the conjugate quantities are the three-dimensional coordinates r and momenta p, as established from the primitive one-isoform $\hat{\theta} = p*dr$. In the more general formulations of Birkhoffian type the conjugate quantities are the six-dimensional variables a and R, as it can be see from the corresponding primitive one-isoform $\tilde{\theta} = R*da$ (or, equivalently, the structure, the Pfaffian (11.1.4.34). In fact, Birkhoff's tensor $\Omega_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu$, $\partial_\mu = \partial/\partial a^\mu$, is nowhere degenerate, det $\Omega \neq 0$. As a result, the change of coordinates $a \to a'(a) \equiv R(a)$ is always possible. The Pfaffian action A(t, a), the Birkhoffian function B(t, a) and Birkhoffs equations can therefore be transformed from the coordinates {t, a} to new coordinates {t, R} and a corresponding transformation is possible at the operator level.

REFERENCES

1. P. ROMAN, *Advanced Quantum Theory* , Addison Wesley, Reading MA (1965); J. SNIATYCKI, *Geometric Quantization and Quantum Mechanics* , Springer-Verlag, New York (1979)
2. R. M. SANTILLI, Hadronic J. 1, 574 (1978)
3. A. O. E. ANIMALU and R. M. SANTILLI, in *Hadronic Mechanics and Nonpotential Interactions* , M. Mijatovic Editor, Nova Science, New York (1990).
4. R. M. SANTILLI, Hadronic J. Suppl. 4B, issue no. 1 (1989)
5. E. B. LIN, Hadronic J. 11, 81 (1988)
6. A. J. KALNAY, Hadronic J. 6, 1 (1983); A. J. KALNAY and R. M. SANTILLI, Hadronic J. 6, 1798 (1983)
7. R. M. SANTILLI, *Foundations of Theoretical Mechanics,* Vol. 11, Springer-Verlag, Heidelberg (1983)
8. E. MADELUNG, *Die Mathematischen Hilfsmittel des Physikers,* Springer-Verlag, Berlin (1950; B. MROWKA, Z. Phys. 130, 164 (1951)
9. D. SCHUCH, K.M. CHUNG and H. HARTMANN, J. Math. Phys. 24, 1652 (1983); D. SCHUCH and K.-M. CHUNG, Intern. J. Quantum Chem. 19, 1561 (1986); D. SCHUCH 23, 59 (1989)

3: BASIC AXIOMS

3.1: FUNDAMENTAL EQUATIONS OF HADRONIC MECHANICS

In Ch. II.1 we have studied the direct universality of generalized classical mechanics of isotopic and genotopic types for nonlinear, nonlocal and nonhamiltonian systems. In Ch. II.2 we have studied the map of such generalized classical theories into hadronic mechanics by therefore confirming the direct universality for all possible operator *systems*[24] previously identified in Vol. I on mathematical grounds. In this chapter we shall identify the fundamental axioms of the operator theory.

It is recommendable to begin with a review of the basic equations of hadronic mechanics with the inclusion of the isooperator form of the momentum which was unavailable in the mathematical studies of Vol. I.

A) Isotopic branch of hadronic mechanics for the description of systems of particles with conserved, positive-definite, total energy and nonlinear-nonlocal-nonhamiltonian internal forces.

Physical systems are represented by *two* different quantities, the conventional local–differential Hamiltonian $H = K + V$ representing the kinetic energy K and the potential energy V, and a Hermitean integro–differential generalization $\hat{h} = h\hat{1}$, of Planck's constant $h = 1$ representing all nonlinear,[25] nonlocal–nonhamiltonian interactions.

We shall hereon assume that $\hat{h}$ has an arbitrary functional dependence on time t, momenta p, accelerations $\dot{p}$, density of the medium considered μ, local temperature τ, and other interior physical quantities

$$\hat{h} = h\hat{1} = \hat{1}(t, p, \dot{p}, \mu, \tau, ...) = \hat{1}^{\dagger} = T^{-1} > 0, \quad h = 1, \qquad (3.1.1)$$

---

[24] The reader should keep in mind the comments in the Preface according to which such direct universality refers to the *equations of motion* and, by no means, to the various operator *theories* existing in the literature for their treatment.

[25] Nonlinearity is here referred to the velocities and, later on, to the wavefunctions and their derivatives.

but it is independent from the the coordinates r. The latter dependence will be studied in Ch.s II.8 and II.9. At any rate, measures done from the outside require the averaging of $\hat{h}$ to a constant, in which case the internal dependence can be unrestricted.

The new quantity $\hat{h}$ is assumed as the basic *space isounit* of the theory, with its inverse $T = \hat{h}^{-1}$ being the *space isotopic element*. The theory is then characterized by the same isoproduct to the right and to the left among generic quantities A, B, $A*B = A\hat{h}^{-1}B = ATB$, where T is fixed.

Hadronic mechanics require an additional complementary quantity, the *time isounit* $\hat{1}_t$ and related *time isotopic element* $T_t = I_t^{-1}$, which are independent from the corresponding space quantities 1 and T (in fact, the latter generally are matrices while the former are generally functions).

From these basic assumptions the following isostructures can be uniquely and unambiguously derived (see Vol. 1 for details):

A-1) *Space isofields* $\hat{F}_T(\hat{a},+,*)$ of *isoreal* or *isocomplex numbers* $\hat{a} = a\hat{1}$, *isomultiplication* $\hat{a}*\hat{\beta} = (a\beta)\hat{1}$, *isonorm*

$$| \hat{a} | = | a |\hat{1} > 0, \tag{3.1.2}$$

where $| a |$ is the conventional norm, and other properties studied in Ch. I.2 with corresponding time isofields $\hat{R}_t(t,+,*)$.

A-2) *Isoassociative enveloping operator algebras* $\hat{\xi}_T(L)$ of an n-dimensional Lie algebra L with generators X_k, i = 1, 2, ..., n, with infinite-dimensional basis

$$\hat{\xi}_T(L): \quad \hat{1}, \ X_k, \ X_i * X_j \ (i \leq j), \qquad X_i * X_j * X_i \ (i \leq j \leq k), \quad .... \tag{3.1.3}$$

and related *isoexponentiation*

$$\hat{e}^{i\,\hat{w}*\,X} = \{e^{i\,X\,T\,w}\}\hat{1} = \hat{1}\{e^{i\,w\,T\,X}\}, \quad \hat{w} \in \hat{F}_T, \quad w \in F. \tag{3.1.4}$$

A-3) *Isohilbert spaces* $\mathcal{H}_G$, with *isoinner product*

$$<\hat{\psi}|\hat{\phi}> \ = \ <\hat{\psi}|G|\hat{\phi}>\hat{1} \ \in \ \hat{F}_T(\hat{a},+,*) \tag{3.1.5}$$

and *isonormalization* $<\hat{\psi}|\hat{\phi}> = \hat{1}$, where $<\hat{\psi} | \hat{\phi}>$ is the ordinary inner product and $G = G^\dagger \neq T$.

A-4) *Heisenberg-isotopic* (or *isoheisenberg*) *equation* *in the infinitesimal form* first submitted in 1978 by Santilli (ref. [1], p. 752, Eq.s (4.15.49)) with an ordinary time derivative, and then finalized in ref. [2] with the *isoderivative* (Sect. I.6.7 and Sect. II.3.2 for details)

$$i\,\frac{\partial Q}{\partial t} = i\hat{1}_t\,\frac{dQ}{dt} = [Q\hat{,}H] := Q*H - H*Q = Q\,T\,H - H\,T\,Q, \tag{3.1..6}$$

characterized by the *fundamental Lie–isotopic brackets* $[A, \hat{} B] = ATB - BTA$, where the *time isoderivative* is given by $\hat{\partial}/\hat{\partial t} = \hat{1}_t d/dt, \hat{1}_t = T_t^{-1} \neq \hat{1}$.

A–5) *Isoheisenberg equations in their finite form* first proposed by Santilli in 1978 (ref. [1], Sect. 4.18),

$$Q(t) = \hat{U} * Q(0) * \hat{U}^{\hat{\dagger}} = \{\hat{e}^{iX*\hat{t}}\} * Q(0) * \{\hat{e}^{-i\hat{t}*X}\} =$$

$$= \{e^{iXTt}\} Q(0) \{e^{-itTX}\}, \tag{3.1.7}$$

characterized by the *Lie–isotopic group* of isounitary transformations on $\mathfrak{K}_G$.

A–6) *Schrödinger–isotopic (or isoschrödinger) equation for the energy* on isospace $\hat{E}(t,\hat{R}_t) \times \hat{E}(r,\hat{\delta},\hat{R})$ (see Ch. 1.3) over $\mathfrak{K}_G$ first proposed by Myung and Santilli [3] and, independently, by Mignani [4] with the conventional derivative, and finalized in 1989 by Santilli [2] with the isoderivative

$$i \frac{\hat{\partial}}{\hat{\partial t}} \hat{\phi}(t, r) = i \hat{1}_t \frac{\partial}{\partial t} \hat{\phi}(t, r) = H * \hat{\phi}(t, r) := H T \hat{\phi}(t, r), \tag{3.1.8a}$$

$$-i \hat{\phi}^{\dagger}(t, r) \frac{\hat{\partial}}{\hat{\partial t}} = -i \hat{\phi}^{\dagger}(t, r) \frac{\partial}{\partial t} \hat{1}_t = \hat{\phi}^{\dagger}(t, r) * H = \hat{\phi}^{\dagger}(t, r) T H, \tag{3.1.8b}$$

where H is assumed to be conventionally Hermitean on $\mathfrak{K}$ and isotopically Hermitean on $\mathfrak{K}_G$ [26]

$$H \equiv H^{\dagger} \equiv H^{\hat{\dagger}}. \tag{3.1.9}$$

A–7) *isoschrödinger equations for the momentum components* first identified by Santilli [2] in 1989

$$p_k * \hat{\phi}(t, r) = p_k T \hat{\phi}(t, r) = -i \hat{\nabla}_k \hat{\phi}(t, r) = i \hat{1}_k^i \nabla_i \hat{\phi}(t, r), \tag{3.1.10a}$$

$$\hat{\phi}^{\dagger}(t, r) * p_k = \hat{\phi}^{\dagger}(t, r) T p_k = +i \hat{\phi}^{\dagger}(t, r) \hat{\nabla}_k = +i \hat{\phi}^{\dagger}(t, r) \nabla_i \hat{1}_k^i, \tag{3.1.10b}$$

where the operator p is also assumed to be conventionally Hermitean on $\mathfrak{K}$ and isotopically Hermitean on $\mathfrak{K}_G$, $p \equiv p\dagger \equiv p\hat{\dagger}$, with *fundamental isocommutation rules* identified by Santilli [1,2]

$$[a^\mu \hat{} a^\nu] = \begin{pmatrix} [r^i \hat{} r^j] & [r^i \hat{} p_j] \\ [p_i \hat{} r^j] & [p_i, p_j] \end{pmatrix} = \begin{pmatrix} 0 & i\hat{1} \\ -i\hat{1} & 0 \end{pmatrix}. \tag{3.1.11}$$

---

[26] As shown in Sect. I.6.3, Hermiticity on $\mathfrak{K}$ automatically implies that on $\mathfrak{K}_G$ when G = T, otherwise it implies restrictions on the isotopic element G (see also next chapters).

B) Isodual isotopic branch for the description of systems of antiparticles[27] **with conserved, negative-definite total energy and nonlinear-nonlocal-nonhamiltonian internal forces.**

These latter formulations are the image of the preceding ones under isoduality (see Ch. 1.2 and Sect. II.3.2)

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1}, \tag{3.1.12}$$

with consequential isodual isonumbers, isodual isofields, isodual isospaces, etc. (see Vol. I for details).

The isotopic branch of hadronic mechanics is classified into the following *Kadeisvili's five classes* (Ch. I.1):

Class I, for isounits which are sufficiently smooth, bounded, nowhere-degenerate, Hermitean and positive–definite, characterizing *isotopies* properly speaking,

Class II, for isounits which are the same as in Class I, but negative-definite characterizing *isodualities,*

Class III, the union of Classes I and II,

Class IV, for singular isounits, and

Class V, for arbitrary isounits, e.g., given by discontinuous functions, distributions, lattices, etc.

C) Genotopic branch of hadronic mechanics for the description of particles with positive-definite nonconserved energy and nonlinear-nonlocal-nonhamiltonian external forces.

Systems are now assumed to be irreversible and are represented by the same Hamiltonian H = K + V of the isotopic branch plus *two* different integro-differential generalizations $\hat{\hbar}^>$ of Planck's constant ħ, one for motion forward to future time > and one for motion forward from past time < (see Sect. II.3.3 for details), which represent all nonlinear–nonlocal–nonhamiltonian interactions in each direction of time.

The new quantities $\hat{\hbar}^>$ are assumed as the basic *space genounits* of the theory, they are interconnected by Hermitean conjugation (or some other conjugation dependent on the problem at hand)

$$\hat{\hbar}^> \; := \; \hbar\,\hat{1}^> \; = \hat{1}^>(t, p, \dot{p}, \mu, \tau, \, ...) \; = \; R^{-1}, \; \hbar = 1, \tag{3.1.13a}$$

$$^<\hat{\hbar} \; := \; \hbar\, ^<\hat{1} \; = \; ^<\hat{1}(t, p, \dot{p}, \mu, \tau, ...) = S^{-1}, \tag{3.1.13b}$$

$$\hat{1}^> = (\,^<\hat{1})^\dagger, \tag{3.1.13c}$$

---

[27] See Ch. II.10 on relativistic field equations for the characterization of antiparticles via isodual methods.

and characterize two different ordered products among generic quantities A, B, one for multiplication to the right $A>B = ARB$, and one to the left $A<B = ASB$, with R and S fixed, $A>B \neq A<B$. The theory further require a complex generalization of the unit of time $\hat{1}_t$ with related genofields, according to which time assumed the two forward directions $\hat{t}> = t\hat{1}_t$ and $<\hat{t} = \hat{1}_t\, t$, where t is the ordinary real time. The above basic assumptions imply in a unique and unambiguous way the following genostructures (see Ch. 1.7 for all details):

C-1) *Genofields* $\langle\hat{F}\rangle(\langle\hat{a}\rangle,+,<>)$ of *genoreal* or *genocomplex numbers* $\langle\hat{a}\rangle := \alpha\langle\hat{1}\rangle$ with genomultiplications $\langle\hat{a}\rangle<>\langle\hat{\beta}\rangle = (\alpha\beta)\langle\hat{1}\rangle$ and *genonorms*

$$|\langle\hat{a}\rangle| = |\alpha|\langle\hat{1}\rangle\ ,$$
<div align="right">(3.1.14)</div>

C-2) *Genoassociative enveloping operator algebras* $\langle\hat{\xi}\rangle(L)$ with infinite-dimensional basis

$$\langle\hat{\xi}\rangle_T(L):\quad \langle\hat{1}\rangle\ ,\quad X_k,\ X_i<>X_j\ (i\leq j),\quad X_i<>X_j<>X_k\ (i\leq j\leq k),\dots$$
<div align="right">(3.1.15)</div>

and related *genoexponentiation*

$$\langle\hat{e}\rangle^{\,i\langle\hat{w}\rangle<>X} = \{e^{\,iX<>w}\}\langle\hat{1}\rangle = \langle\hat{1}\rangle\{e^{\,iw<>X}\}\ .$$
<div align="right">(3.1.16)</div>

C-3) *Genohilbert spaces* $\langle\hat{\mathcal{H}}\rangle$ with *genoinner product*

$$<\hat{\psi}\,\langle\hat{P}\rangle\,\hat{\phi}> := <\hat{\psi}|\,\langle\hat{G}\rangle\,|\hat{\phi}><\hat{1}> \in \langle\hat{F}\rangle(\langle\hat{a}\rangle,+,<>)\ ,$$
<div align="right">(3.1.17)</div>

and *genonormalization* $<\hat{\psi}\,\langle\hat{P}\rangle\,\hat{\phi}> = \langle\hat{1}\rangle$.

C-4) *Heisenberg-admissible* (or *genoheisenberg*) *equation in the infinitesimal form* first submitted by Santilli in 1978 (ref. [1], p. 746, Eq.s (4.15.34)) with the conventional derivative, and then finalized in ref. [2] with the *genoderivative* (Sect. 1.7.8).

$$i\,\frac{\langle\hat{d}\rangle Q}{\langle\hat{d}\rangle t} = i\,\langle\hat{1}\rangle_t\,\frac{dQ}{dt} = (Q,H) = Q<H - H>Q = QRH - HSQ,\quad (3.1.18)$$

characterized by the *fundamental Lie-admissible brackets* $(A,B) = ARB - BSA$.

C-5) *Genoheisenberg equations in the finite form* first submitted by Santilli in ref. [1], p. 783, Eq.s (4.18.16),

$$Q(t) = \hat{U}> > Q(0) < <\hat{U}^\dagger = \{\hat{e}>^{\,iXt}\} > Q(0) < \{<\hat{e}^{\,-itX}\} =$$

$$= \{e^{\,iXRt}\}Q(0)\{e^{\,-itSX}\}\ ,$$
<div align="right">(3.1.19)</div>

characterized by a *Lie-admissible group* of *genounitary transformations*

C-6) *Schrödinger-admissible* (or *genoschrödinger*) equation *for the energy for motion forward in time* on genospaces $<\hat{E}>(t,<\hat{R}>_t)\times<\hat{E}>(r,<\hat{S}>,<\hat{R}>)$ over $<\mathcal{H}>$, first submitted by Myung and Santilli [3] and, independently, by Mignani [4] with the conventional derivative, and then finalized by Santilli [2] with the genoderivative

$$i \; \frac{\partial^>}{\partial^> t} \hat{\psi}^>(t,r) \; = \; i \, 1_t^> \frac{\partial}{\partial t} \hat{\psi}^>(t,r) = \; H > \hat{\psi}^>(t,r) \; = \; H \, R \, \hat{\psi}^>(t,r) , \qquad (3.1.20a)$$

$$-i \; {}^<\hat{\psi}(t,r) \frac{{}^<\partial}{\partial^< t} = \; - \, i \, {}^<\hat{\psi}(t,r) \frac{\partial}{\partial t} \, {}^<1_t = {}^<\hat{\psi}(t,r) < H \; = \; {}^<\hat{\psi}(t,r) \, S \, H , \qquad (3.1.20b)$$

$$R^\dagger \, = \, S , \qquad 1_t^> \, = \, (\, {}^<1_t)^\dagger , \qquad (3.1.20c)$$

where the operator H is also assumed to be conventionally Hermitean on $\mathcal{H}$ and *genohermitean* on $<\mathcal{H}>$ although *nonconserved* (because of the lack of antisymmetry of the Lie-admissible product $(A, B) = ARB - BSA$).

C-7) *Schrödinger-admissible* (or *genoschrödinger*) equations *for the momentum components* first formulated by Santilli [2] in 1989

$$p_k > \hat{\psi}^>(t,r) \; = \; p_k \, R \, \hat{\psi}^>(t,r) \; = \; - \, i \, \hat{\nabla}_k^> \, \hat{\psi}^>(t,r) \; = \; - \, i \, 1^>{}_k{}^i \, \nabla_i \, \hat{\psi}^>(t,r) , \qquad (3.1.21a)$$

$$ {}^<\hat{\psi}(t,r) < p_k \; = \; {}^<\hat{\psi}(t,r) \, S \, p_k \; = \; + \, i \, {}^<\hat{\psi}(t,r) {}^<\hat{\nabla}_k \; = \; + \, i \, {}^<\hat{\psi}(t,r) \, \nabla_i \, {}^<1_k{}^i , \qquad (3.1.21b)$$

where the operator p is also conventionally Hermitean and genohermitean, with *fundamental Lie-admissible rules* first formulated by Santilli in 1978 (ref. [2], p. 746, Eq.s (4.15.34b))

$$(a^\mu , a^\nu) \; = \; a^\mu < a^\nu \; - \; a^\nu > a^\mu \; = \; a^\mu \, R \, a^\nu \; - \; a^\nu \, S \, a^\mu \; = \; i \, S^{\mu\nu} , \qquad (3.1.22a)$$

$$(S^{\mu\nu}) \; = \; \begin{pmatrix} r(R - S) r & + i \, {}^<1> \\ - i \, {}^<1> & p (R - S) p \end{pmatrix} \qquad (3.1.22b)$$

where $S^{\mu\nu}$ is a suitably symmetrized) operator image of the corresponding classical, fundamental, Lie-admissible tensor (II.1.5.8).

D) Isodual genotopic branch of hadronic mechanics for the description of antiparticles with negative-definite nonconserved energy and nonlinear-nonlocal-nonhamiltonian external forces.

This branch is the image of the preceding one under the isodualities

$$({}^<1>)^d = - {}^<1> , \qquad ({}^d1_t^>)^d \; = \; - {}^<1_t> , \qquad (3.1.23)$$

with consequential isodual images of genofields, genospaces, etc. (see Vol. 1 for

details). In particular, this latter formulation characterizes the remaining two directions in time, $\hat{t}^{>d} = -\hat{t}^> = (\hat{1}_t^{>d}) t$ for *motion backward from future time* and $^{d<}\hat{t} = -^<\hat{t} = (^<\hat{1}_t^{>d}) t$ for *motion backward to past time*. As we shall have ample opportunity to study, these features permit hadronic mechanics to achieve an axiomatic characterization of irreversibility in each of the four possible directions of time, as well as the reduction of macroscopic irreversibility to the ultimate structure of matter, that at the level of elementary constituents in open–nonconservative interior conditions.

The genotopic branch of hadronic mechanics can also be classified into Kadeisvili's five classes, although referred now to the Hermitean part of the genounits.

E) Isotopic branch of hadronic mechanics and its isodual for the characterization of quarks and antiquarks, respectively.

Quarks are represented by *two* operators, the conventional Hamiltonian $H = K + V$ (representing all conventional potential interactions) decomposed into the form $H = H_1 + H_2$, plus the isounits and related isotopic element

$$\hat{1} = \hat{1}^\dagger = T^{-1} > 0, \qquad T = H_1^{-1} + H_2^{-1}, \tag{3.1.24}$$

representing the internal nonlinear–nonlocal–nonhamiltonian interactions due to mutual penetration of the quarks wavepackets. Only positive–definite (negative–definite) isounits are admitted for quarks (antiquarks) because of the stability of their orbits (thus excluding Lie–admissible formulations for nonconserved quark states). [28]

The time evolution is given by the following particular form of the general isoheisenberg equation identified by Kalnay [5] and Kalnay and Santilli [6] in 1983

$$i\dot{F} = [F, H_1, H_2] \equiv [F\hat{,}H] = F T H - H T F, \tag{3.1.25a}$$

$$H = H_1 + H_2, \qquad T = H_1^{-1} + H_2^{-1}. \tag{3.1.25b}$$

The theory is then completed by corresponding isofields, isoenveloping operator algebras, isohilbert spaces and remaining methods of the Lie–isotopic branch of hadronic mechanic. Antiquarks are represented by the isodual image of the preceding formulations resulting negative–definite energies referred to negative–definite isounits.

In summary, as anticipated in Sect. 1.1.5, *hadronic mechanics is composed of a total of twelve different branches each pone in a variety of Kadeisvili Classes,* and this illustrates the diversification of the theory.

---

[28] The reader should recall that it takes at least *four* constituents to have *unstable* orbits in a closed nonhamiltonian systems. The quark orbits therefore remain stable in hadronic mechanics.

In this chapter we shall identify the basic postulates of hadronic mechanics via the appropriate isotopies of quantum mechanical postulates. The isotopic postulates were indicated by Santilli [1] in the original proposal of 1978 to build hadronic mechanics; they were first studied in technical details by Myung and Santilli [3] in 1982 and by Mignani, Myung and Santilli [7] in 1983; they were finalized by Santilli in ref. [2] of 1989; and reviewed by Lopez [8] in 1993. No additional reference is on record at this writing (early 1994), specifically, on the basic postulates of hadronic mechanics to the author's best knowledge.

For simplicity, we shall first study the isotopic axioms of hadronic mechanics (Class I) and their isoduals (Class II) characterized by one single isotopic element $T = G$. The broader genotopic axioms and their isoduals for will be studies subsequently.

Since there is no risk of ambiguity, we shall drop the subscript T in the structures $\hat{F}_T$, $\hat{\xi}_T$, and $\hat{\mathcal{H}}_T$. The reader should remember that, unless otherwise stated, the space isotopic element T is assumed not to be explicitly dependent on the local coordinates r and the time isotopic element T_t not to depend explicitly on time (but dependent in a generally nonlinear and nonlocal way on all other quantities and their derivatives).

Moreover, isotopic and genotopic generalizations of Planck's constants are assumed to recover the conventional value for mutual distances D bigger than one fermi

$$\hat{\hbar}_{D>1fm} \equiv \hbar I, \qquad \langle\hat{\hbar}\rangle_{D>1fm} \equiv \hbar I . \tag{3.1.26}$$

All isotopic and genotopic generalizations therefore recover quantum mechanical axioms by construction at large mutual distances.

Equivalently, we can say that *hadronic mechanics recovers quantum mechanics by construction for mutual distances bigger than the range of the strong interactions,* and this explains the reason for the name "hadronic mechanics" suggested for the discipline [1].

The isodual isotopic and genotopic axioms are restricted by the conditions

$$\hat{\hbar}^d_{D>1fm} \equiv -\hbar I, \qquad \langle\hat{\hbar}\rangle^d_{D>1fm} \equiv -\hbar I . \tag{3.1.27}$$

and recover the axioms of a hitherto unknown antiautomorphic image of quantum mechanics called *isodual quantum mechanics,* as we shall see in this chapter.

The basic postulates of this chapter will be implemented into physical laws in the next chapter, and then developed in the remaining chapters of this volume into the various aspects of the theory as in conventional quantum mechanics. The basic postulates will be applied to specific cases and subjected to confrontation with experimental evidence in Vol. III.

As we shall see, even though preliminary, available experimental and

phenomenological evidence supports quite clearly the isotopic postulates, provided that they are applied in the arena of their conception, the *interior* particle problem at mutual distances of the order of one Fermi.

To avoid major misconceptions, it is therefore essential to abandon the familiar concept of point particle in vacuum for which quantum mechanics is *exactly* valid, and consider instead fundamentally different physical conditions, such as an *extended* proton within the hyperdense medium in the core of a star called *hadronic medium* [1]. Along similar conceptual lines, we shall study hadrons whose constituents have extended wavepackets of the same dimension as that of the hadrons themselves ($\approx$ 1 fm), thus implying the conditions of total mutual penetration at the foundations of hadronic mechanics.[29]

3.2: BASIC AXIOMS OF THE ISOTOPIC BRANCH OF HADRONIC MECHANICS

We shall now identify the basic *isotopic axioms,* characterizing a system of particles considered as isolated from the rest of the Universe, and therefore verifying conventional total conservation laws, yet admitting nonlinear-nonlocal-nonhamiltonian internal effects.

Such axioms will be constructed under the condition of coinciding with the conventional axioms of quantum mechanics (see, e.g., ref.s [9]) at the abstract realization-free level, beside recovering them identically for mutual distances bigger than the range of the strong interactions.

The above conditions essentially allow the construction of both, quantum and hadronic mechanics, in terms of one single set of abstract axioms, and their differentiations into exterior problems in vacuum and interior problems within physical media under suitable corresponding realizations of the isounits.

Unless otherwise stated, all formulations of this section are assumed to be of Class 1 and all isounits to be diagonalized for simplicity[30]

---

[29] Recall from Ch. 1.1 that, as usually proved in undergraduate courses in quantum mechanics, all massive particles have a wavepacket of the order of one fermi which is of the same order of magnitude of the radius of the charge distributions of all hadrons as well as of the range of the strong interactions themselves.

[30] The diagonalization of the isounit is always possible because it is positive-definite, although it is not necessary. In fact, we shall encounter in Ch. II.10 an isotopy of the Dirac equation originally due to Dirac himself with an truly intriguing *nondiagonal* isotopic element. A number of applications with nondiagonal isounits (deformations of charge distributions in nuclear physics, Bose-Einstein correlation in particle physics, Cooper pairs in superconductivity, and others) are presented in Vol. III. The extension of the basic axioms of hadronic mechanics to nondiagonal isounits is simple and will be tacitly implied later on.

3.2.A: Isostates and isoobservables. Recall that a quantum mechanical state is an element $|\psi>$ of a conventionally modular Hilbert space $\mathcal{H}$, that is, a state under the *associative modular action* of a Hamiltonian H

$$H|\psi> = E_0|\psi>. \tag{3.2.1}$$

This structure, however, is linear, local and Hamiltonian and, as such, cannot represent the nonlinear, nonlocal and nonhamiltonian interactions under study in these volumes.

The representation of the latter interactions is permitted by the isotopy of structure (3.2.1) which can be expressed via the following axioms.

AXIOM I: The states $|\hat{\psi}>$ of the isotopic branch of hadronic mechanics, called <isostates>, are elements of a right isomodular Hilbert space $\hat{\mathcal{H}}$ verifying the isoschroedinger equation

$$i\frac{\partial}{\partial t}|\hat{\psi}> = H*|\hat{\psi}> = HT|\hat{\psi}> = \hat{E}*|\hat{\psi}>, \tag{3.2.2a}$$

$$p_k*|\hat{\psi}> = p_k T|\hat{\psi}> = -i\hbar \partial/\partial r^k|\hat{\psi}> = -ik\hat{1}_k{}^i \partial/\partial r^i|\hat{\psi}>, \tag{3.2.2b}$$

$$\partial/\partial t = \hat{1}_t \partial/\partial t, \quad \hat{1}_t = T_t{}^{-1}, \quad \hat{t} = tT_t \,(\hat{1}_t = T_t{}^{-1}), \quad \hat{E} = E\hat{1} \in \hat{C}(\hat{c},+,*), \quad E \in C(c,+,*), \tag{3.2.2c}$$

where, as known from the preceding section, $\partial/\partial t$ is the *time isoderivative,* $\hat{t} = \hat{t}\hat{1}_t$ is the *isotime,* $\hat{1}_t = T_t{}^{-1}$ is the *time isounit,* T_t the *time isotopic element,* $\hat{1} = T^{-1}$ is the *space isounit* and T is the *space isotopic element.*

AXIOM II: Measurable physical quantities called <isoobservables>, are characterized by isohermitean operators on an isohilbert space $\hat{\mathcal{H}}$ which therefore possesses isoreal isoeigenvalues (Sect. I.6.2). Measured quantities are then characterized by ordinary real numbers,

$$H^{\hat{\dagger}} \equiv H^{\dagger} \equiv H^{\hat{\dagger}}, \qquad H*|\hat{\psi}> = \hat{E}*|\hat{\psi}> = E|\hat{\psi}>, \quad E \in R(n,+,\times). \tag{3.2.3}$$

By recalling also from Sect. I.6.3 that the isohermiticity of an operator H coincides with the conventional Hermiticity for the case considered, we can say that *all conventional observables of quantum mechanics remain observable for the isotopic branch of hadronic mechanics.*

Moreover, note from Eq.s (3.2.3) that $\hat{E}*|\psi> = E\hat{1}T|\hat{\psi}> = E|\psi>$. Thus, *the numbers actually measured in quantum and hadronic mechanics are elements of the same conventional field of real numbers.*

Comparison of Eq.s (3.2.1) and (3.2.3) confirms a result of Sect. I.6.3 to the effect that *the eigenvalues of the <same> Hermitean operator H are <different> in quantum and hadronic mechanics,* $E_0 \neq E$. This is a first property

of hadronic mechanics with fundamental implications for applications and verifications of the theory studied in Vol. III.

As an example, the above result indicates that *the conventional Hamiltonian for the hydrogen atom* H_{Coul} *has <two> different sets of eigenvalues, one given by the conventional form* E_0 *at large mutual distances and a new, hitherto unknown eigenvalue* E *when the proton and electrons are in conditions of total mutual penetration/overlapping at mutual distances of the order of one fermi.*

Note the fundamental role of the *nonhamiltonian* character of the interactions for the achievement of the above result. In fact, for conventional potential interactions one adds a new potential V' to the old Hamiltonian. This implies two *different* eigenvalues E_0 and E for two *different* Hamiltonians H_{Coul} and H = H_{Coul} + V', which is a trivial occurrence.

Equivalently, the above result is fundamentally dependent on the admission of novel interactions beyond the representational capabilities of quantum mechanics, those of *zero-range*[31] due to the mutual *contact* of the wavepackets. If the interactions are of action-at-a-distance type, they are of conventional potential character and no novelty emerges.

As it is well known in quantum mechanics, different observables cannot be measured simultaneously unless they commute. We then have the following isotopic image (see ref. [3], pp. 1321-1322 for a proof):

Lemma 3.2.1: *A necessary and sufficient condition for different hadronic observables* A_k, k = 1, 2, ..., n, *to be measurable simultaneously is that they isocommute,*

$$[A_i \,\hat{,} A_j] = A_i T A_j - A_j T A_i \equiv 0, \quad i, j = 1, 2, ..., n. \quad (3.2.4)$$

But the basis of a vector space remains unchanged under isotopy (Ch.s 1.3 and 1.5). As a result, *the generators of quantum mechanics and those of the isotopic branch of hadronic mechanics coincide,* as it must be the case since they represent physical quantities such as coordinates, momenta, angular momenta, energy, etc. Hadronic mechanics merely generalizes the *operations* among them. Also, operators which commute conventionally do not necessarily isocommute. We then have the following

Corollary 3.2.1A: *Observables which can be simultaneously measured in quantum mechanics (at large mutual distances) do not necessarily remain simultaneously measurable in hadronic mechanics (at small distances) and viceversa.*

The identification of the *maximal set of isocommuting isoobservables* is

---

[31] This aspect can be better studied at the isorelativistic level of Ch. II.9.

essentially the same as that for the conventional case [9] and it will be assumed as known.

Conventional quantum mechanical quantities, such as the Hamiltonian operator H = H(t, r, p), are constructed from time t and the pair of canonically conjugated quantities r and p with familiar *fundamental commutation rules*

$$[\, r_i \,,\, r_j \,] \;=\; [\, p_i \,,\, p_j \,] \;=\; 0, \qquad [\, r_i \,,\, p_j \,] \;=\; i\, \delta_{ij}, \quad (\hbar = 1). \tag{3.2.5}$$

For the case of n particles in three-dimensional Euclidean space, the above rules can be written in the now familiar unified form

$$[a^\mu \hat{,} a^\nu] \;=\; i\, \omega^{\mu\nu}, \qquad a = \{a^\mu\} = \{r, p\}, \qquad \mu, \nu = 1, 2, \dots 6n, \tag{3.2.6a}$$

$$(\omega^{\mu\nu}) \;=\; \begin{pmatrix} 0_{3n\times 3n} & I_{3n\times 3n} \\ -I_{3n\times 3n} & 0_{3n\times 3n} \end{pmatrix} \tag{3.2.6b}$$

which carry important algebraic and geometric meanings.

In fact, $\omega^{\mu\nu}$ is the Lie tensor in canonical realization as the contravariant version of the symplectic tensor $\omega_{\mu\nu}$, also in canonical realization. In turn, the appearance of the *classical* quantity $\omega^{\mu\nu}$ in the above *quantum mechanical* expressions is important to establish quantum mechanics as the correct and unique operator image of classical Hamiltonian mechanics via the symplectic quantization and other means.

The results of Sect. II.1.4 then allow the formulation of the following:

AXIOM III: The operators of the isotopic branch of hadronic mechanics are constructed from time t and the isoconjugated coordinates r^i and momenta p_j, with <fundamental isocommutation rules> in disjoint r and p formulation for diagonal isounits

$$[r^i \hat{,} r^j] \;=\; [p_i \hat{,} p_j] \;=\; 0, \quad [r^i \hat{,} p_j] \;=\; i\hat{1}^i_j \quad i, j = 1, 2, \dots, 3n, \tag{3.2.7}$$

or in the unified notation a = (r,p)

$$[a^\mu \hat{,} a^\nu] \;=\; i\, \hat{\omega}^{\mu\nu} \;=\; i\hat{1}\, \omega^{\mu\nu}, \quad \mu, \nu = 1, 2, \dots, 6n. \tag{3.2.8}$$

We assume the reader is familiar with the following properties studied earlier:

1) The isotopic image of the exact Hamiltonian two-form $\omega = \omega_{\mu\nu} da^\mu \wedge da^\nu = d\theta$ is the isoexact Hamilton-isotopic form $\hat{\omega} = \hat{\omega}_{\mu\nu} \hat{d}a^\mu \wedge \hat{d}a^\nu = \omega_{\mu\alpha} T^\alpha_\nu da^\mu \wedge da^\nu = \hat{d}\hat{\theta}$;

2) The isotopic image of the Hamiltonian Lie tensor $\omega^{\mu\nu}$ is the isotopic form

$$\hat{\omega}^{\mu\nu} = \hat{1}\omega^{\mu\nu}, \ (\omega^{\mu\nu}) = (\omega_{\mu\nu})^{-1}, \ \hat{1} = T^{-1};$$

3) The integrability conditions for the Hamiltonian tensor $\omega^{\mu\nu}$ to be Lie are given by the closure property of exact two-forms $d\omega = d(d\theta) = 0$ and, in a fully similar way, the integrability conditions for the isocanonical tensor $\hat{\omega}^{\mu\nu}$ to be Lie-isotopic are given by the isoclosure property of isoexact two-isoforms $\hat{d}\omega = \hat{d}(\hat{d}\theta) = 0$.

Therefore, *the fundamental isocommutation rules* (3.2.8) *confirm that the isotopic branch of hadronic mechanics is the unique and unambiguous operator image of the classical Hamilton–isotopic mechanics of Sect. 1.1.4 in exactly the same way as it occurs for classical and quantum Hamiltonian mechanics* .

The above result also confirms the validity of the naive isoquantization of the preceding chapter and provides rigorous grounds for the isosymplectic quantization and other approaches.

3.2.B: Time evolution of isostates and isoobservables. By using the content of Sect.s 1.6.2 and 1.6.3, the isotopy of the conventional evolution of a quantum mechanical state can be formulated via the following

AXIOM IV: The time evolution of the hadronic states is characterized by isounitary transformations with the isohermitean Hamiltonian as the generator, and the time evolution of hadronic observables is characterized by an isoequivalent, one-dimensional, Lie-isotopic group also with the Hamiltonian as the generator.

We consider now the *isotime*

$$\hat{t} = t\hat{1}_t, \tag{3.2.9}$$

and hadronic states $|\hat{\psi}(\hat{t})>$ at a given isotime $\hat{t}$. According to Axiom IV, the same state at a subsequent isotime $\hat{t}'$, $|\hat{\psi}(\hat{t}')>$ is characterized by the isotransform

$$|\hat{\psi}(\hat{t}')> = \hat{U}(\hat{t},\hat{t}') * |\hat{\psi}(\hat{t})>, \tag{3.2.10}$$

where $\hat{U}(\hat{t}, \hat{t}')$ is an isounitary operator verifying the condition for a Lie–isotopic group with the Hamiltonian H as a generator,

$$\hat{U}(\hat{t},\hat{t}') = \hat{e}^{-\hat{i}(\hat{t}'-\hat{t})H} = \hat{1}\{e^{-i(\hat{t}'-\hat{t})T_t T H}\} = \{e^{-iHTT_t (\hat{t}'-\hat{t})}\}\hat{1}, \tag{3.2.11}$$

and we have used isoexponentiation (3.1.4). The condition for the isotransform to constitute a Lie–isotopic group then ensures the composition law

$$\hat{U}(\hat{t},\hat{t} + \delta\hat{t}) * \hat{U}(\hat{t} + \delta\hat{t},\hat{t}') = \hat{U}(\hat{t},\hat{t}'), \tag{3.2.12}$$

and the infinitesimal form (see Sect. I.4.5 for details)

$$\text{Lim}_{\delta t \to 0} \frac{\hat{U}(t, \hat{t}) - \hat{U}(t - \delta t, \hat{t})}{\delta t} = \frac{\partial}{\partial t} \hat{U}(t, \hat{t}) = \hat{1}_t \frac{\partial}{\partial t} \hat{U}(t, \hat{t}) =$$

$$= - i H * \hat{U}(t, \hat{t}) = - i H T \hat{U}(t, \hat{t}), \quad T \neq T_t. \tag{3.2.13}$$

The time evolution implied by Axiom IV is therefore of the form

$$i \hat{1}_t \frac{\partial}{\partial t} | \hat{\psi}(t) > = H T | \hat{\psi}(t) >, \tag{3.2.14}$$

which is precisely the isoschrödinger's form of Axiom I. Note the presence of the time isounit $\hat{1}_t$ in the l. h. s. which was absent in the earlier formulations of the equation, as recalled in Sect. II.2.3.

Axiom IV also implies that *the isonormalization of a hadronic state* (Sect. 1.6.2) *does not change in time (or in isotime)*, i.e.,

$$< \hat{\psi}(t) | \hat{\psi}(t) > = < \hat{\psi} | T | \hat{\psi} > \hat{1} = \hat{1} \equiv$$

$$\equiv < \hat{\psi}(t) | * \hat{U}^\dagger * \hat{U} * | \hat{\psi}(t) > \hat{1} = < \hat{\psi}(t) | T | \hat{\psi}(t) > \hat{1} \equiv \hat{1}. \tag{3.2.15}$$

We learn in this way that *the consistent formulation of the isotopic branch of hadronic mechanics requires a redefinition of time into isotime* (3.2.9). As we shall see in our studied at the relativistic level, this occurrence has seemingly fundamental implications because it implies that, in additional to the Einsteinian relativity of time, we have a *relativity of the unit of time*, with far reaching implications, such as: different times in different regions of the Universe with the same speed relative to an inertial system but different gravitational fields (e.g., time on Earth and that in Jupiter are predicted to be different); an experimentally verifiable "space−time machine" and others.

In the following, whenever no confusion arises, we may continue to use the symbol "t" rather than "$\hat{t}$", with the understanding that it refers to the isotime.

The transition to the isoheisenberg representation is straightforward. In the latter case Axiom IV implies that the time evolution of an operator A is given by the (one−dimensional) Lie−isotopic group

$$A' = \hat{U}^\dagger * A * \hat{U}, \tag{3.2.16}$$

or, more explicitly, by

$$A(t) = \hat{U}^\dagger(t, \hat{t}) * A(0) * \hat{U}(t, \hat{t}) = \hat{e}^{- i H (t' - t)} * A(0) * e_{\hat{f}}^{i (t' - t) H} \tag{3.2.17}$$

thus implying the now familiar infinitesimal form

$$A(\delta t) = (1 - i \, H \, T \, \delta t) \, A(0) \, (i \, \delta t \, T \, H - I),$$ (3.2.18)

from which the fundamental isoheisenberg equation of the theory follows,

$$i \, \hat{1}_t \frac{dA}{dt} = A \, T \, H - H \, T \, A = [A \, \hat{,} \, H].$$ (3.2.19)

In particular, we have the following important properties:

Lemma 3.2.2: *The isounitary time evolution of hadronic mechanics leaves invariant the basic isounit of the theory,*

$$\hat{1}' = \hat{U}^\dagger * \hat{1} * \hat{U} \equiv \hat{U}^\dagger * \hat{U} \equiv \hat{U} * \hat{U}^\dagger \equiv \hat{1}.$$ (3.2.20)

Lemma 3.2.3: *Isounitary transformations leave invariant the Lie-isotopic brackets*

$$\hat{U}^\dagger * [A \, \hat{,} \, B] * \hat{U} = \hat{U}^\dagger * (A * B - B * A) * \hat{U} =$$

$$= A'T \, B' - B'T \, A' = A' * B' - B' * A' \equiv [A' \, \hat{,} \, B'].$$ (3.2.21)

by therefore leaving invariant the isoheisenberg time evolution

$$i \, \dot{A} = [A \, \hat{,} \, H] \quad \Rightarrow \quad i \, \hat{U}^\dagger * \dot{A} * \hat{U} = \hat{U}^\dagger * [A \, \hat{,} \, H] * \hat{U} \equiv i \, \dot{A}' = [A' \hat{,} \, H'],$$ (3.2.22)

as well as the fundamental isocommutator rules (3.2.8)

$$[a^\mu \, \hat{,} \, a^\nu] = i \, \hat{\omega}^{\mu\nu} \quad \Rightarrow \quad [a'^\mu \, \hat{,} \, a'^\nu] = i \, \hat{\omega}^{\mu\nu}.$$ (3.2.23)

The importance of the preceding results should be indicated. First, note the invariant character of the fundamental isounit $\hat{1}$. The lifting $\hbar \Rightarrow \hat{\hbar}$ is an isotopy, not only in the preservation of the original characteristics, but, more deeply, in the preservation of the invariant character as the basic unit of the theory.

Secondly, recall that quantum mechanics is invariant under a rather restricted class of transformations, the unitary ones. By comparison, hadronic mechanics preserves conventional unitary transformations, evidently because it is a covering of quantum mechanics, but admits in addition the invariance under a class of transformations considerably broader than the unitary ones.

To understand how broad the latter invariance is, one should keep in mind that hadronic mechanics admits an infinite variety of isotopic elements T for each given Hamiltonian H with corresponding infinite variety of form–invariance

under isounitary transformations.

Finally, note the invariance of the Lie–isotopic tensor $\hat{\omega}^{\mu\nu}$ in the fundamental isocommutation rules (3.2.23). This invariance also has a fundamental character for the theory because it establishes that the branch of hadronic mechanics under study is indeed a complete theory, without evidently precluding the existence of yet more general theories (see next section).

In different terms, the important aspect for any given theory in a chain of generalizations is that the particular theory considered is axiomatically self–consistent, i.e., derivable from primitive axioms and left invariant by its own transformations, while admitting the old theory as a particular case.

Quantum mechanics is indeed self–consistent because its own transformations, the conventional unitary transformations, preserve the Lie tensor $\omega^{\mu\nu}$ of its fundamental commutation rules. If unitary transformations had mapped $\omega^{\mu\nu}$ into a different tensor, quantum mechanics would not have been a self–consistent theory.

We learn from Lemma 3.2.3 that an equivalent situation exists at the covering level of hadronic mechanics in which the quantity now preserved is the Lie–isotopic tensor $\hat{\omega}^{\mu\nu}$, while admitting the quantum mechanical structure $\omega^{\mu\nu}$ as a simple subcase for $\hat{1} = 1$.

The isoequivalence of the isoschrödinger and isoheisenberg representations is studied in App. III.A.

In Sect. II.I.3 we recalled that the (classical) Birkhoffian mechanics can be constructed via <noncanonical> transformations of the conventional Hamiltonian mechanics. In proposal [1] we suggested the following corresponding operator form for hadronic mechanics.

Lemma 3.2.4: *The various structures of hadronic mechanics can be built via <nonunitary> transformations of the corresponding structures of quantum mechanics, e.g., under a conventionally modular (time-independent) nonunitary transformation Heisenberg's equations transform into their isotopic covering*

$$U\,U^\dagger = \hat{1} \neq I, \qquad U^\dagger \neq U^{-1}, \tag{3.2.24a}$$

$$i\,\dot{A} = [A, H] = A\,H - H\,A \;\Rightarrow\; i\,U\,\dot{A}\,U^\dagger = U\,[A, H]\,U^\dagger =$$

$$= i\,\dot{A}' = [A'\,\hat{,}\,B'] = A'\,T\,H' - B'\,T\,H', \quad A' = \hat{U}\,A\,U^\dagger, \quad B' = U\,B\,U^\dagger, \tag{3.2.24b}$$

with the isotopic element T *given precisely by the hermitean inverse of the isounit*

$$T = U^{\dagger-1}\,U^{-1} = (U\,\hat{0})^{-1} = \hat{1}^{-1}. \tag{3.2.25}$$

All additional nonunitary time evolution can then be written in the unique

isotopic form

$$W\,W^\dagger \;\equiv\; \hat{U}*\hat{U}^\dagger = \hat{1} \neq 1\,, \qquad W = \hat{U}\,T^{1/2}, \tag{3.2.25}$$

under which the basic dynamical equations remain invariant.

Note that the isotopic element T represents the deviation from I. This is desired because the isotopic element represents noncanonical interactions by conception.

The following result has important implications for other generalizations of quantum mechanics.

Corollary 3.2.4.A: *The fundamental commutation rules of quantum mechanics are mapped by nonunitary transformations into the corresponding rules of hadronic mechanics*

$$U(r\,p - p\,r)\,U^\dagger = r'\,T\,p' - p'\,T\,r' = i\,U1\,U^\dagger = i\hat{1}\,, \tag{3.2.27a}$$

$$U(r\,r - r\,r)\,U^\dagger = r'\,T\,r' - r'\,T\,r' = U\,0\,U^\dagger = 0\,, \tag{3.2.27b}$$

$$U(p\,p - p\,p)\,U^\dagger = p'\,T\,p' - p'\,T\,p' = i\,U\,0\,U^\dagger = 0\,. \tag{3.2.27c}$$

As one can see, among all infinitely possible generalizations of the canonical commutation rules, a generalization which is axiomatic, that is, derivable from first principles, and invariant under its own time evolution is given by the structure

$$r\,T\,p - p\,T\,r = i\hat{1}, \tag{3.2.28}$$

where the condition $\hat{1} = T^{-1}$ is crucial for consistency, as it has been the case throughout all aspects of the isotopic theories, including isonumbers, isofields, isospaces, isoalgebras, etc.

This result implies that the rather large literature on the so-called noncanonical generalizations of the fundamental commutation rules of the type

$$r\,p - p\,r = i\,f(t, r, p)\,, \tag{3.2.29}$$

do not possess a consistent axiomatic structure. In fact, the above generalizations are manifestly outside the axiomatic structure of quantum mechanics, yet they do not verify the fundamental rule that the r.h.s. function f(t, r, p) is the inverse of the isotopic element T. A similar lack of axiomatic structure exists for other deformations of quantum mechanical formalisms, as studied in Appendix II.3.C.

Lemma 3.2.5: *The isoeigenvalues of hadronic observables are invariant under isounitary transformations,*

$$H * | \hat{\psi} > = \hat{E} * | \hat{\psi} > = E | \hat{\psi} > \Rightarrow \hat{U} * H * | \hat{\psi} > = H' * | \hat{\psi}' > =$$

$$= \hat{U} * \hat{E} * | \hat{\psi} > = \hat{E} * | \hat{\psi}' > \equiv E | \hat{\psi}' > . \qquad (3.2.30)$$

Note the abstract identity between the isotopic properties and the corresponding ones of quantum mechanics. This implies that the quantum mechanical properties considered in this section are genuine axioms of the theory because invariant under isotopies. On the contrary, we shall see later on that there exist quantum mechanical structures (such as Bell's inequality) which are not true axioms because not invariant under isotopies.

3.2.C: Isoexpectation values. Recall that the conventional expectation value of a quantum mechanical observable is given by [9]

$$< A > = \frac{< \psi | A | \psi >}{< \psi | \psi >} \in R(n, +, \times), \qquad (3.2.31)$$

The isotopy of the preceding structure then yields the *isoexpectation value of a hadronic observable*

$$\hat{<} A \hat{>} = \frac{< \psi | * A * | \psi >}{< \psi | * | \psi >} \in R(n, +, \times), \quad A = A^\dagger . \qquad (3.2.32)$$

AXIOM V: The values expected in measurements of isoobservables are given by the isoexpectation values and they are conserved when representing total Galilean (or Lorentzian) quantities of a system assumed as isolated from the rest of the Universe.

Note that the isoinner product in $\hat{\mathcal{H}}$ coincides with the conventional product in $\mathcal{H}$ when the isotopic element T can be factorized (e.g., when it does not depend on the integration variables or it is a scalar)

$$< \psi \hat{|} \psi > = < \psi | \psi > T\hat{1} \equiv < \psi | \psi >. \qquad (3.2.33)$$

Under the same assumption, the conventional and isotopic expectation values also coincide, $\hat{<} A \hat{>} = < A > T\hat{1} T\hat{1} \equiv < A >$. Note in particular that

$$\hat{<} \hbar \hat{>} = \hat{<} \hbar \hat{1} \hat{>} = \hbar, \qquad \hat{<} \hat{1} \hat{>} = 1, \qquad \hat{<} 1 \hat{>} \neq 1. \qquad (3.2.34)$$

The above property therefore implies that

Lemma 3.2.6: *The experimental measure of a given expectation value cannot distinguish whether the underlying theory is quantum or hadronic mechanics.*

In order to reach experimental measures on such a distinction, we must identify <novel> predictions of hadronic mechanics, that is, predictions beyond the technical capabilities of quantum mechanics, as we shall study in detail in Volume III.

The proof of the following property is elementary.

Lemma 3.2.7: *Isoexpectation values are invariant under isounitary transformations.*

It is easy to see that *the isoexpectation values coincide with the isoeigenvalues*. Let $c \in C(c,+,\times)$ be the isoeigenvalues of an operator A with eigenstate $| \hat{\psi} >$, $A * | \hat{\psi} > = \hat{c} * | \hat{\psi} > = c | \hat{\psi} >$. Then we have the evident expression

$$\hat{<} A \hat{>} = \frac{< \hat{\psi} | * A * | \hat{\psi} >}{< \hat{\psi} | * | \hat{\psi} >} = c \frac{< \hat{\psi} | * | \hat{\psi} >}{< \hat{\psi} | * | \hat{\psi} >} = c \in C(c,+,\times) . \qquad (3.2.35)$$

Note that the identity between isoeigenvalues and isoexpectation values persists also under the differentiation between the isotopic element G of the isohilbert space and the isotopic element T of the operator algebra.

Next, we introduce the notion of *iso-wave-function* which can be expressed via the structure

$$\hat{\psi}(t, r) = < t, r | T | \hat{\psi} > . \qquad (3.2.36)$$

A simple example, particularly significant for practical applications, is the *iso-plane-wave* , say, in one space-dimension

$$\hat{\psi}(t, r) = \hat{N} * \hat{e}^{i p r} * \hat{e}_t^{-i E t} = N e^{i(p T r - E T_t t)} =$$

$$= N e^{i(p \hat{r} - E \hat{t})} = \psi(\hat{t}, \hat{r}) , \ \hat{r} = Tr, \hat{t} = T_t t , \qquad (3.2.37)$$

where N is a suitable isonormalization constant (see below) and $\hat{e}_t$ is the isoexponentiation for the isotopic element T_t. Note that structure (3.2.37) can be equivalently interpreted either as a new function $\hat{\psi}$ in the old variables t, r or as the old function ψ on the new isovariables $\hat{t}$ and $\hat{r}$.

The primary function of isoplanewaves studied in detail in Ch. II.8 is the *geometrization of the inhomogeneity and anisotropy of the medium in which*

the wave propagates in a form suitable for experimental verifications, such as an electromagnetic wave propagating throughout Earth's atmosphere or, along conceptually similar lines, an electron moving within the hadronic medium of the interior of a star. The scope of the isoplane waves is then to provide a quantitative representation of the *deviation* from motion in empty space suitable for verification of the theory.

Structure (3.2.37) verifies the basic eigenvalues equations for the position operator R, momentum operator P and energy operator $H = i\hat{1}_t \, \partial/\partial t$

$$r_{opr} * \hat{\psi}(t, r) = r_{eigenv} \, \hat{\psi}(t, r), \tag{3.2.38a}$$

$$P_{oper} * \hat{\psi}(t, r) = -i\hat{1} \frac{\partial}{\partial r} \hat{\psi}(t, r) = p_{eigenv} \, \hat{\psi}(t, r), \tag{3.2.38b}$$

$$i\hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) = E \, \hat{\psi}(t, r). \tag{3.2.38c}$$

Isoplanewaves therefore verify isoschrödinger equation (3.2.2).

As we shall see later on, the "symmetrization" of the space isotopy pTr with the time one $ET_t t$ is *necessary* for the mutual compatibility of nonrelativistic and relativistic hadronic formulations. In fact, as indicated earlier, such compatibility forces the lifting of the derivative of time into the isoderivative form, that is, it requires the generalization of the Galilean or Einstein time into the isotime.

3.2.D: Isoprobabilities. We then introduce the notion of *isoprobability density*

$$\mathscr{P} = |\hat{\psi}(t, r)|^2 = |\hat{\psi}^2(t, r) * \hat{\psi}(t, r)|\hat{1} = |\hat{\psi}^\dagger(t, r) \, T \, \hat{\psi}(t, r)| \, T^{-1}, \tag{3.2.39}$$

with normalization

$$\int dV \, \hat{\psi}^\dagger * \hat{\psi} = 1, \tag{3.2.40}$$

or, equivalently, isonormalization to $\hat{1}$. Note that, when T is a scalar, isoprobabilities and conventional probabilities coincide, thus illustrating again the abstract identity of quantum and hadronic mechanics of Class 1.

Consider now a hadronic observable A and suppose that it possesses a finite number of isoeigenvalues $\hat{c}_k$, $k = 1, 2, ..., n$. Let $|\hat{\psi}>$ be the hadronic state vector of A, with $|\hat{\psi}_k>$ the (isoorthogonal) states corresponding to the individual isoeigenvalues $\hat{c}_k$. It is then easy to see that the expressions

$$\hat{w}_k = w_k \hat{1} \in \hat{R}(\hat{n},+,*), \qquad w_k = |<\hat{\psi}_k | T | \hat{\psi}>|^2 \in R(n,+,\times), \tag{3.2.41}$$

represent the *isoprobability for obtaining the isoeigenvalue* $\hat{c}_k$ *in a given measure.*

The assumed isonormalization implies the recovering of the familiar unit value for the total probability,

$$\sum_k \hat{w}_k = 1, \qquad \sum_k w_k = \sum_k |<\psi_k|\psi>|^2 = 1. \qquad (3.2.42)$$

The above results are sufficient to show the existence of a fully consistent isotopy of the quantum mechanical theory of probability. The isotopies of other aspects are left to the interested reader.

A simple generalization of the proof of the corresponding quantum mechanical property shows that *isostates are unique except for possible phase isofactors.* The conservation of total physical quantities will be studied in subsequent chapters. The interested reader can easily work out additional isotopies.

We can therefore conclude by saying that *all basic axioms of quantum mechanics admit consistent, nontrivial, axiom–preserving isotopic formulation.* This confirms the identity of quantum mechanics and hadronic mechanics of Class I at the abstract, realization free level .

3.2.E: Basic axioms of isodual isotopic hadronic mechanics.

According to hadronic mechanics, all preceding axioms apply solely for particles, because antiparticles are described by isodual formulations (see Ch. 1I.10).

The basic axioms for the isodual isotopic branch of hadronic mechanics were studied for the first time by Santilli in 1989 [2] and are given by the simple isodual image of Axioms I–V. First, we should recall that the "numbers" are now the elements of the isodual isofields $\hat{R}^d(\hat{n}^d,+,*^d)$ and $\hat{C}^d(\hat{c}^d,+,*^d)$ with *negative-definite norms*

$$|\hat{n}^d| = |n|\hat{1}^d < 0, \qquad |\hat{c}^d| = |c\,\bar{c}|\hat{1}^d < 0. \qquad (3.2.43)$$

The *isodual isoeuclidean space* of the theory is the expression

$$\hat{E}^d(t,\hat{R}_t{}^d) \times \hat{E}^d(r,\hat{\delta}^d,\hat{R}^d): \hat{1}_t{}^d = (T_t{}^d)^{-1} = -\hat{1}_t, \hat{\delta}^d = \hat{\delta}T^d = -\hat{\delta}, \hat{1}^d = (T^d)^{-1} = -\hat{1}, \qquad (3.2.44)$$

with *isodual isotime* $\hat{t}^d = -\hat{t} = t\,\hat{1}_t{}^d$, *isodual time isounit* $\hat{1}_t{}^d = -\hat{1}_t$ and *isodual space isounit* $\hat{1}^d = -\hat{1}$

The *isodual isohilbert space* $\mathfrak{H}^d$ is characterized by the *isodual states* $|\hat{\psi}>^d = -(|\hat{\psi}>)^\dagger = -<\hat{\psi}|$, with *isodual isoinner product* and related *isodual isonormalization*

$$\mathfrak{H}^d: \quad <\hat{\psi}|\hat{\psi}>^d = <\hat{\psi}|^d T^d|\hat{\psi}>^d \hat{1}^d \in \hat{C}^d(\hat{c}^d,+,*^d), \qquad (3.2.45a)$$

$$< \hat{\Psi} | \hat{\phi} >^d \; = \; 1^d \; = \; -1, \quad < \hat{\phi} | T^d | \hat{\psi} > \; = \; -1 . \qquad (3.2.45b)$$

As one can see, *the isoinner products of* $\mathcal{H}$ *and* $\mathcal{H}^d$ *coincide*. However, by no means this implies that these spaces are equivalent, mathematically *or* physically. On mathematical grounds the spaces $\mathcal{H}$ and $\mathcal{H}^d$ are defined over antiautomorphic fields F and F^d, respectively, with different elements, the isostates $| \hat{\phi} >$ for $\mathcal{H}$ and their isoduals $| \hat{\psi} >^d = -< \hat{\psi} |$ for $\mathcal{H}^d$ (see Sect. 1.6.2). On physical grounds the isodual isospaces ca*nnot* admit positive eigenvalues. In particular, as we shall see when studying relativistic equations, *the isodual map* $\mathcal{H} \rightarrow \mathcal{H}^d$ *is equivalent to charge conjugation.*

The *isodual universal enveloping algebra* is the image ξ^d of ξ with the isodual isounit, element and product

$$\xi \rightarrow \xi^d : \quad 1 \rightarrow 1^d = -1 , \quad A \rightarrow A^d = -A, \qquad (3.2.46a)$$

$$A * B = A \, T \, B \; \rightarrow \; A^d *^d B^d = A^d \, T^d \, B^d \; = \; -A * B . \qquad (3.2.46b)$$

The notion of isohermiticity of an operator $H = H^{\hat{\dagger}} = H^\dagger$ is mapped in a simple way into the *isodual isohermiticity* $H = H^{\hat{\dagger}d} = H^{\dagger d}$. The action of an isohermitean operator $H \in \xi^d$ on $\mathcal{H}^d$ is now *isodual isomodular* with *isodual isoeigenvalue equation*

$$< \hat{\phi} |^d *^d H \; = \; -< \hat{\phi} | *^d E^d \; = \; -< \hat{\phi} | T^d 1^d E = -< \hat{\phi} | E, \qquad (3.2.47)$$

where E is the positive–definite eigenvalue of H.

The *isodual isoprobability density* is given by

$$\mathcal{P}^d \; = \; | \hat{\psi}(t, r) |^{2d} \; = \; | \hat{\psi}\dagger(t, r) \, T^d \, \hat{\psi}(t, r) | 1^d \; = \; -| \hat{\psi}\dagger(t, r) \, T \, \hat{\psi}(t, r) | 1 = -\mathcal{P} , \quad (3.2.48)$$

with isodual normalization

$$\int dV \, \hat{\psi}^\dagger T^d \, \hat{\psi} \; = -1 , \qquad (3.2.49)$$

and *isodual isoprobabilities for obtaining the isodual isoeigenvalues* $\hat{c}^d_k$

$$\hat{w}_k{}^d = w_k \, 1^d \in \hat{R}^d(\hat{n}^d, +, *^d), \qquad w_k{}^d \; = \; | < \hat{\psi}_k | T^d | \hat{\psi} > |^2 \in R(n, +, \times) . \qquad (3.2.50)$$

The sum of all such probabilities then yields again the underlying unit, the isodual isounit,

$$\sum_k \hat{w}_k{}^d = 1^d. \qquad (3.2.51)$$

The *isodual isopostulates* can then be expressed as follows.

AXIOM I^d: The states of the isodual isotopic hadronic mechanics,

called <isodual isostates>, are elements $< \hat{\psi} \,|$ of a left, isodual, isomodular, Hilbert space $\mathcal{H}^d$ with isodual isoschroedinger equation

$$- i < \hat{\psi} \,| \frac{\partial^d}{\partial t^d} \; = \; < \hat{\psi} \,|\ast^d H \; = \; < \hat{\psi} \,|\, T^d \, H \; = \; - < \hat{\psi} \,|\ast^d \, \hat{E} \;, \qquad (3.2.52)$$

$$< \hat{\psi} \,|\ast^d p_k{}^d \; = \; = \; < \hat{\psi} \,|\, T^d \, pk^d \; = \; i\,\hbar < \hat{\psi} \,|\, \partial^d / \partial r^{kd} \; = \; i\,\hbar < \hat{\psi} \,|\, \partial/\partial r^i \, \hat{1}_k{}^i \;, \quad (3.2.52b)$$

$$\partial^d/\partial t^d = \hat{1}^d{}_t \partial/\partial t, \; \hat{E}^d \; = \; E\hat{1}^d \in \hat{C}^d(\hat{c}^d, +, \ast^d), \; -E \in C^d(c^d, +, \times^d) \;. \qquad (3.2.52c).$$

AXIOM II^d: Measurable physical quantities of the isodual isotopic hadronic mechanics, called <isodual isoobservables>, are characterized by isodual isohermitean operators on an isodual isohilbert space $\mathcal{H}^d$ whose isoeigenvalues are isoreal (Sect. I.6.3). Measured quantities are then characterized by real, negative-definite numbers

$$H \; \equiv \; H^{\hat{\dagger}} \; \equiv H^{\dagger}, \;\; < \hat{\psi} \,|^d \ast^d H \; = \; < \hat{\psi} \,|^d \ast^d \, \hat{E}^d \; = \; - < \hat{\psi} \,|\, E \;. \qquad (3.2.53)$$

AXIOM III^d: The operators of isodual isotopic hadronic mechanics are constructed from the isodual time $t^d = - t$, the isodual isoconjugated coordinates $r^{id} = - r_i$ and momenta $p_j{}^d = - p_j$ with <fundamental isodual isocommutation rules> in disjoint r, and p formulation

$$[\, r^i \,\hat{,}\, p_j \,]^d \; = \; r^{id} \, T^d \, p_j{}^d - p_j{}^d \, T^d \, r^{id} \; = \; -[\, r^i \,\hat{,}\, p_j \,] \; = \; i\,\delta^{di}{}_j \; = -i\hat{1}\,\delta_j{}^i, \quad (3.2.54a)$$

$$[\, r^i \,\hat{,}\, r^j \,]^d \; = \; [\, p_i \,\hat{,}\, p_j \,]^d \; = \; 0, \quad i, j = 1, 2, ..., 3n, \qquad (3.2.54b)$$

or in the unified formulation $a^d = (r^d, p^d)$

$$[\, a^{\mu} \,\hat{,}\, a^{\nu} \,]^d \; = \; i\,\hat{\omega}^{\mu\nu d} \; = \; i\hat{1}^d \, \omega^{\mu\nu} \; = \; -\hat{\omega}^{\mu\nu}, \quad \mu, \nu = 1, 2, ..., 6n \;. \quad (3.2.55)$$

AXIOM IV^d: The time evolution of the isodual isotopic hadronic states is characterized by isodual isounitary transformations $\hat{U}^d$ with the isodual isohermitean Hamiltonian as the generator, while the time evolution of isodual isoobservables is characterized by an isoequivalent, one-dimensional, isodual Lie-isotopic group also with the Hamiltonian as the generator,

$$\hat{U}^d \ast^d \hat{U}^{d\hat{\dagger}} \; = \; \hat{U}^{d\hat{\dagger}} \ast^d \hat{U}^d \; = \; \hat{1}^d \; = \; -\hat{1} \;, \qquad (3.2.56a)$$

$$< \hat{\psi}(t^d) \,| \; = \; < \hat{\psi}(0) \,|\ast^d \hat{U}^d \,(t^d, 0) \; = \; < \hat{\psi}(0) \,|\ast^d \hat{e}^d {}^{-i\hat{t}^d H} \; =$$

$$= \langle \hat{\psi}(\hat{0}^d) | \ e^{-i \, t \, T_t^d \, T \, H} \ , \quad 0 \equiv \hat{0} \equiv \hat{0}^d \ .$$

(3.2.56b)

AXIOM V^d: The values expected in the measurements of the isodual isoobservables are given by negative-definite isodual isoexpectation values and they are conserved when representing total Galilean (or Lorentzian) quantities of a system of antiparticles assumed as isolated from the rest of the Universe.

$$\hat{\langle} A^d \hat{\rangle} \ = \ \frac{\langle \hat{\psi} |^{*d} A^d {}_{*d} | \hat{\psi} \rangle}{\langle \hat{\psi} |^{*d} | \hat{\psi} \rangle} \ = \ -\hat{\langle} A \hat{\rangle} \quad \in \ R^d(n^d, +, \times^d), \quad A = A^{\dagger d}.$$

(3.2.57)

The interested reader can work out the isodualities of other aspects. Note that the above formulation implies the existence of a hitherto unknown antiautomorphic image of quantum mechanics, the *isodual quantum mechanics* which evidently holds for the particular case when $1^d = 1^d$.

3.3: BASIC AXIOMS OF THE GENOTOPIC BRANCH OF HADRONIC MECHANICS

Hadronic mechanics was originally suggested [1]in its most general possible form, as a *genotopic* (or *Lie-admissible*) generalization of quantum mechanics, because in such a form it admits the *isotopic* (or *Lie-isotopic*) generalization and the conventional quantum mechanics as particular cases.

In these volumes we have elected to present the isotopic and genotopic generalizations in a separate, progressive way primarily for clarity of presentation, but with the understanding that, on formal grounds, our presentation could have been restricted to the genotopic branch.

A further reason for the selected presentation is that a primary objective of these volumes is the study of closed-isolated systems of strongly interacting particles with nonlocal-nonpotential internal forces, in which case the isotopic branch of hadronic mechanics becomes predominant.

Before outlining the basic genotopic axioms it may be recommendable to review certain foundational results of Vol. I. Unless otherwise states, all genounits are assumed to have Hermitean parts of Kadeisvili's Class 1 which are assumed to be diagonalized for simplicity.

3.3.A: Genonumbers and their isoduals. The fundamental notion of hadronic mechanics, mathematically and physically, is the most general known

formulation of numbers, the *genonumbers* with related *genofields,* and the isodual *genonumbers* with related *isodual genofields* studied in Ch. 1.2. We are here referring to:

CHAIN OF COVERING THEORIES STUDIED IN THESE VOLUMES

SYSTEMS	ALGEBRAS	MECHANICS
Closed systems witb potential internal forces	Lie algebras	Quantum mecbanics
Closed systems witb potent. and nonpotent. internal forces	Lie-isotopic algebras	Isotopic branch of hadronic mecbanics
Open systems witb potent. and nonpotent. external forces	Lie-admissible algebras	Genotopic brancb of badronic mecbanics

FIGURE 3.3.1: A schematic view of the chain of generalizations under study and related enclosure properties.

1) The ordering of the multiplication of generic numbers a, b to the right, indicated with the script a>b, and to the left, indicated with a<b with their isoduals $a>^d b = - a>b \neq a<b$ and $a<^d b = - a<b \neq a>b$;

2) The differentiation of the ordering to the right from that to the left, a>b $\neq$ a<b, $a>^d b \neq a<^d b$; and

3) The characterization of *motion forward to future time* with the product a>b, *motion forward from past time* with the product <, *motion backward from future time* with the product $a>^d b$ and *motion backward to past time* with $a<^d b$ (see Fig. 3.3.2 for more details).

Mathematical consistency of the theory then requires the following four generalizations of the basic unit of quantum mechanics, Planck's constant ħ = 1,

Forward genounit	$\hat{h}^> = \hbar \hat{1}^>$,	$\hat{1} = S^{-1}$,	
Conjugated forward genounit	$^<\hat{h} = {}^<\hat{1}\,\hbar$,	$^<\hat{1} = R^{-1} = (\hat{1}^>)^\dagger$	
Backward genounit	$\hat{h}^{>d} = -\hat{h}^>$,	$\hat{1}^{>d} = -\hat{1}^>$,	(3.3.1)
Conjugated backward genounit	$^<\hat{h}^d = -{}^<\hat{h}$,	$^<\hat{1}^d = -{}^<\hat{1}$.	

The quantities hereon written in unified notation $^<\hat{h}^> = \{^<\hat{h}, \hat{h}^>\}$ and $^<\hat{h}^{>d} = \{^<\hat{h}^d, \hat{h}^{>d}\}$ are assumed as the *space genounits.* Their inverses $<> = \{R, S\}$ and $<>^d$

$= \{R^d, S^d\} = - <>$ are the *space genotopic elements,* with corresponding *genomultiplications*

$$a <> b = \{ a<b = a \, R \, b, \quad a > b := a \, S \, b \}, \quad R = S^\dagger, \qquad (3.3.2a)$$

$$a <>^d b = \{ a >^d b, \quad a >^d b \quad \cdot \} = a <> b. \qquad (3.3.2b)$$

The above assumptions imply for mathematical consistency four different notions of time

Forward genotime	$\hat{1}^> = tS_t,$	$\hat{1}_t^> = S_t^{-1},$
Conjugated forward genotime	$^<\hat{1} = t\,R_t,$	$^<\hat{1}_t = R_t^{-1} = (\hat{1}_t^>)^\dagger,$
Backward genotime	$\hat{1}^{>d} = -\hat{1}^>,$	$\hat{1}^{>d} = -\hat{1}^>,$
Conjugated backward genotime	$^<\hat{1}^d = -\,^<\hat{1},$	$^<\hat{1}^d = -\,^<\hat{1},$

$$(3.3.3)$$

where the quantities $^<\hat{1}_t^>$ and $^<\hat{1}_t^>$ are the *time genounits* with corresponding time products written in the unified form $^<\hat{1}_1^> <>_t {}^<\hat{1}_2^>$ and $^<\hat{1}_1^{>d} <>_t^d {}^<\hat{1}_2^{>d} = - {}^<\hat{1}_1^> <>_t {}^<\hat{1}_2^>$. Note that the space and time genounits are structurally different because the former are three–dimensional matrices while the latter are scalars.

The time genounits can therefore be ordinary complex functions, such as $f_1(r, ..) + i f_2(r, ...)$, where f_1 and f_2 are real. The form possible "time arrows" are then given by $\hat{1}^> = t(f_1 + 1f_2)$, $^<\hat{1} = t(f_1 + if_2)$, $\hat{1}^{>d} = t(-f_1 + -if_2)$, $^{d<}\hat{1} = t(-f_1 + if_2)$. Note that *the four possible "time arrows" are represented by the genounits, while time remains the ordinary real quantity* . As we shall see in Vol. III, rather than being a mere mathematical curiosity, trhe above characterizations of time raises intriguing novel *experimental* issues.

THE NOTIONS OF TIME OF HADRONIC MECHANICS

Motion forward from past time t > 0   Motion forward to future time t > 0

$$t_o$$

———>

current time
Motion backward to past time $t^d < 0$ Motion backward from future time $t^d < 0$
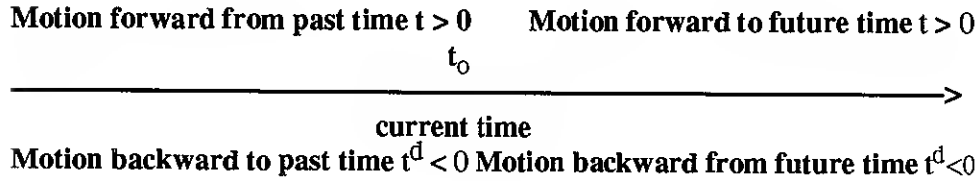
FIGURE 3.3.2: A schematic view of the four possible "time arrows" which are characterized by hadronic mechanics via four possible genounits interconnected by complex conjugation and isoduality. Note that the isotime and its isodual are a particular case when the forward and backward genounits coincide. As shown in Ch.s II.8 and II.9, the generalization of the *unit* of time is necessary for interior relativistic and gravitational isotopic formulations and have rather deep implications, e.g., the possible locality of

time.

We proved in Ch. 1.7 that each ordered multiplication and related genounit implies the preservation of all axioms of a field. This permitted the construction of the genofields $\langle \hat{F} \rangle (\langle \hat{a} \rangle, +, \langle \rangle)$ of real numbers, complex numbers and quaternions, with a corresponding form for the octonions which is important for the isotopies and genotopies of Dirac's equation studied later on in this volume.

As a simple example, consider the genotopy of the field of real numbers $R(n, +, \times)$ characterized by elements R and S belonging to the original field (i.e, R and S are ordinary numbers n). In this case (only) the new elements $\langle \hat{n} \rangle$ can be assumed to coincide with the original elements, the conjugation $R = S^\dagger$ must be relaxed and replaced with a different conjugation $R = S^C$ such that S^C remains an element of the original field (e.g., $S^C = 1/S$). Such genofields can then be written $\langle \hat{R} \rangle (n, +, \langle \rangle)$.

The reader is now encouraged to work out specific realizations. Assume for instance the genounit $\hat{1}^{>} = 3$. Then the multiplication of the number 2 time the number 3 to the right is given by $2 > 3 = 2$. Suppose that $\langle \hat{1} = (\hat{1}^{>})^C = 1/\hat{1}^{>}$. Then the multiplication of the number 3 time the number 2 to the left is $2 < 3 = 18$.

The important point is that all original axioms of $R(n, +, \times)$ are preserved by each ordering of the genofields $\langle \hat{R} \rangle (n, +, \langle \rangle)$. For instance, the original multiplication $\times$ (the ordinary multiplication of numbers) is commutative, $2 \times 3 \equiv 3 \times 2$. Exactly the same property exists for each of the two ordered multiplications, $2 > 3 \equiv 3 > 2 = 2$ and $2 < 3 \equiv 3 < 2 = 18$ while being compatible with the basic condition $a > b \neq a < b$.

The same situation occurs for all other properties as studied in detail in Ch. 1.7. As a result, *fields and their corresponding genofields coincide at the abstract level.*

One should note that the emerging revision of the theory of numbers is rather deep, including a covering formulation of prime numbers, factorization, etc. For instance, statements such as "the number four is not prime" become incomplete because lacking the identification of the underlying unit. In fact, in the above example with genounit $\langle \hat{1} = 1/3$, *the number four becomes a prime number.* Regrettably, these generalizations of the theory of numbers do not appear to have propagated to mathematicians in the field at this writing (early 1994).

Note that the theory of isonumbers is a simple particular case of that of genonumbers when the difference in the ordering of the multiplication is relaxed, therefore resulting in the single isomultiplication $a > b \equiv a < b \equiv a * b$.

It should be indicated that genonumbers and their isoduals are the broadest known, *axiom-preserving,* generalization of the conventional numbers of classical and quantum mechanics. In fact, any further generalization requires the modification of the addition $a + b$ which violates the distributive law and, as such, is no longer axiom-preserving (Sect. I.2.3).

3.3.B: Genospaces and their isoduals. The assumption of the four basic genounits $^<\hbar$, $\hbar^>$ and $^{d<}\hbar$, $\hbar^{>d}$ implies the consequential, unique and unambiguous characterization of all aspects of the genotopic branch of hadronic mechanics.

For instance, the conventional Euclidean space $E(r,\delta,R)$ of classical and quantum mechanics is uniquely lifted into the following four different genospaces

$$^<E^>(r,^<\delta^>,^<R^>): \quad r^{<2>} = (r^i \, ^<\delta^>_{ij} \, r^j) \, ^<1^> \; \in \; ^<R^>(^<\hat{n}^>,+,<>) \,, \tag{3.3.4a}$$

$$^{<}E^{>d}(r,^<\delta^{>d},^<R^{>d}): \quad r^{<2>d} = (r^i \, ^<\delta^{>d}_{ij} \, r^j) \, ^<1^{>d} \; \in \; ^<R^{>d}(^<\hat{n}^{>d},+,<>^d) \,. \tag{3.3.4b}$$

In turn, this implies four different generalized notions of straight lines, spheres, angles, trigonometric functions, etc. (see Ch. I.3 and I.5).

The understanding of the genotopic branch of hadronic mechanics requires the knowledge that *even though the deformed metrics are no longer Hermitean, genospaces coincide at the abstract level with the conventional Euclidean space , while the isodual genospaces coincide at the abstract level with the isodual Euclidean spaces,*

$$^<E^>(r,^<\delta^>,^<R^>) \; \approx \; E(r,\delta,R), \qquad ^<E^{>d}(r,^<\delta^{>d},^<R^{>d}) \; \approx \; E^d(r,\delta^d,R^d) \,. \tag{3.3.5}$$

This is due to the fact that each generalized space is constructed via a deformation of the original metric say, $\delta \to R(t, r, \dot{r}, \ddot{r}, ...)\delta$, $R \neq R^\dagger$, while jointly the unit I of the original space is deformed of the inverse amount, $I \to R^{-1}$, thus preserving the original geometric axioms unchanged.

As an illustration, consider an extended hadron in irreversible conditions (e.g., a proton in the core of a star). In this case the genounits $^<1^> = <>^{-1}$ can be decomposed into a nonhermitean scalar part $^<1_0^> \; <>_0^{-1}$ multiplied by a three-dimensional, Hermitean and diagonal part D^{-1}, $^<1^> = (<>_0 D)^{-1}$. The nonhermitean part is then useful to represent the irreversible conditions, while the diagonal part permits a direct representation of the *nonspherical* shape of the charge distribution of hadron considered and all its infinitely possible *deformations*, via the following different *genospheres* and *isodual genospheres*

$$r^{<2>} = (x \, ^<b^>_1{}^2 \, x + y \, ^<b^>_2{}^2 \, y + z \, ^<b^>_3{}^2 \, z) D^{-1} \,, \tag{3.3.6a}$$

$$^<1^> = ^<1_0^> D^{-1} \,, \quad D = \text{diag.} \, (^<b^>_1{}^2, ^<b^>_2{}^2, ^<b^>_3{}^2) \,, ^<1_0^> \neq (^<1_0^>)^\dagger \tag{3.3.6b}$$

$$r^{<2>d} = (x \, ^<b^{>d}_1{}^2 \, x + y \, ^<b^{>d}_2{}^2 \, y + z \, ^<b^{>d}_3{}^2 \, z) D^{d-1} \,, \tag{3.3.6c}$$

$$^<1^{>d} = - \, ^<1_0^> \text{diag.} \, (^<b^>_1{}^{-2}, ^<b^>_2{}^{-2}, ^<b^>_3{}^{-2}) \,, \tag{3.3.6d}$$

Note the *reality* of the genospheres even though the basic unit is not real due to the cancellation of the nonhermitean scalar part.

The geometrically fundamental property is that *genospheres* (3.3.6a) *are perfect spheres when represented in their respective genospaces, while isodual genospheres* (3.3.6c) *are perfect isodual spheres also when represented in their respective isodual genospaces.*

In fact, jointly with the deformation of the semiaxes $1 \to \diamond_0 {}^{<}b^{>}{}_k{}^2$, where $\diamond_0 \neq \diamond_0{}^\dagger$ and the b's are real, the corresponding unit is deformed of the inverse amount $1 \to \diamond_0{}^{-1}{}^{<}b^{>}{}_k{}^{-2}$, thus preserving the original spherical structure. The same occurrence holds under isoduality although now referred to a novel geometrical notion, the isodual sphere with *negative–definite radius.*

In different terms, *the ellipsoidical deformations of the sphere solely occur when the genospheres is projected in our Euclidean space.*

The image under genotopies and their isoduals of a conventional Hilbert space $\mathcal{H}$ with inner product $< \psi \mid \phi >$ over $C(c,+,\times)$ follows similar lines. In fact, we have the *genohilbert* and *isodual genohilbert spaces*

$$^{<}\mathcal{H}^{>} : \quad <\hat\psi \,{}^{<}\hat{P}\, \hat\psi > \;=\; <\hat\psi \mid \diamond \mid \hat\phi > \,{}^{<}\hat{1}^{>} \in {}^{<}\hat{C}^{>}({}^{<}\hat{c}^{>},+,\diamond), \qquad (3.3.7a)$$

$$^{<}\mathcal{H}^{>d} : \quad <\hat\psi \,{}^{<}\hat{P}^d\, \hat\psi > \;=\; <\hat\psi \mid \diamond^d \mid \hat\phi > \,{}^{<}\hat{1}^{>d} \in {}^{<}\hat{C}^{>d}({}^{<}\hat{c}^{>d},+,\diamond^d), \qquad (3.3.7b)$$

note that, when the forward isotopic element is purely imaginary, say, $a>b = aib$, the corresponding genounit is $\hat{1}^{>} = -i$ and the forward genoinner product *coincides* with the conventional inner product, $< \hat\psi \mid\!\!> \mid \hat\phi >\hat{1}^{>} \equiv < \hat\psi \mid \hat\phi >$.

The lifting of all operations on an isohilbert space (Sect. 1.6.3) then admits simple yet significant genotopic extensions. Particularly important for hadronic mechanics is the property that *a quantum mechanical operator which is conventionally Hermitean (observable) remains so under forward or backward genotopies.*

The above important result can be proved as follows. Let R be the genotopic element for forward action > and S that for the backward action < under the conjugation $R = S^\dagger$. The condition of Hermiticity for an operator A on $\mathcal{H}^{>}$, called *genohermiticity*, is then given by

$$< \hat\psi \mid (> A > \mid \hat\psi >) = < \hat\psi \mid (R A R \mid \hat\psi >) = (< \hat\psi \mid < A <) \mid \hat\psi > = (\mid \hat\psi \mid S A S) \mid \hat\psi >,$$

$$R A R = (S A S)^\dagger, \qquad (3.3.8)$$

which is evidently verified for all conventionally Hermitean operators. under the assumed condition $R = S^\dagger$.

But genotopies have been built to represent the *nonconservation* of physical quantities such as the energy. We reach in this way, apparently for the

first time in a systematic way, *the Hermiticity-observability of nonconserved quantities such as the energy or angular momentum.*[32]

3.3.C: Lie-admissible theory and its isoduals.

A further essential structural element we should briefly recall from Ch. I.7 is the image of the Lie and of the Lie-isotopic theories under genotopies and their isoduals known under the name of *Lie-admissible theory.*

The image of enveloping operator algebras $\xi(L)$ of an n-dimensional Lie algebra L with basis X_k, k = 1, 2, ...,n, is given by two *universal enveloping operator genoassociative algebras* and related isoduals defined over their respective genofields and their genoduals, with infinite-dimensional basis

$$\hat{\xi}^>(L): \quad \hat{R}^>, \quad X_k, \quad X_i \diamond X_j, \ (i \le j, \quad X_i \diamond X_j \diamond X_k, (i \le j \le k), \dots (3.3.9a)$$

$$\hat{\xi}^{>d}(L): \quad \hat{R}^{>d}, \quad X^d_k, \quad X^d_i \diamond^d X^d_j, (i \le j, X^d_i \diamond^d X^d_j \diamond^d X^d_k, (i \le j \le k), \dots(3.3.9b)$$

The above bases then characterize, in a unique and unambiguous way, the *genoexponentiations* (3.1.16) and their isodual

$$\hat{e}^> \,^{i\,w\,X} = \{ e^{i\,X \diamond w} \} \,\hat{1}^> = \hat{1}^> \{ ^{i\,w \diamond X} \}, \tag{3.3.10a}$$

$$\hat{e}^{>d} \,^{i^d\,w^d\,X^d} := \{ e^{i^d\,X^d \diamond^d w^d} \} \,\hat{1}^{>d} = \{ e^{i\,X \diamond w} \} \,\hat{1}^{>d} = - \,\hat{e}^> \,^{i\,w\,X}$$
$$\tag{3.3.10b}$$

We then have the following four types of *genounitary operators* on the respective genoalgebras

$$\hat{U}^> \diamond \hat{U}^{>\dagger} = \hat{U}^{>\dagger} \diamond \hat{U}^> = \hat{1}^>, \tag{3.3.11a}$$

$$\hat{U}^{>d} \diamond^d \hat{U}^{>d\dagger} = \hat{U}^{>d\dagger} \diamond^d \hat{U}^{>d} = \hat{1}^{>d} = - \,\hat{1}^>, \tag{3.3.11b}$$

---

[32] Since the time evolution of quantum mechanics is characterized by only one free operator (the Hamiltonian), everything is represented with it, including nonconservations. As an example, dissipative processes in nuclear physics are represented by adding an *imaginary potential* iV to a Hermitean Hamiltonian H_0. This treatment of nonconservation has serious problematic aspects studied in Ch. I.7. in fact, all quantum mechanical Lie symmetries which are consistent for H_0 are lost under the nonhermitean extension $H_0 \to H$ because Heisenberg's product $AH_0 - H_0A$ becomes a triple product $AH^\dagger$ – HA (see Sect.s 1.7.1 and 1.7.2) which does not define any algebra as conventionally understood in mathematics, let alone a Lie algebra; and other problematic aspects. These occurrences imply that, starting with nucleons with well defined spin $\frac{1}{2}$ for H_0, the notion of "spin $\frac{1}{2}$" becomes meaningless for the extended Hamiltonian H, trivially, because the underlying SU(2) algebra cannot be any longer defined in a physically consistent way, that via the brackets of the time evolution.

$$<0> = <\hat{e}>^{i w X} , \qquad <0>^d = <\hat{e}>^d \ ^{i w X} = - <0> . \qquad (3.3.11c)$$

The action of genoalgebras on genohilbert spaces must be of the same type for consistency, resulting in the genotrasnformations

$$|\hat{\psi}> = \ <\hat{e}>^{i w X} \diamond |\hat{\psi}> = e^{i X \diamond w}|\hat{\psi}> , \qquad (3.3.12a)$$

$$<\hat{\psi}| = <\hat{\psi}| \diamond^d <\hat{e}>^d \ ^{i w X} = <\hat{\psi}|e^{i w \diamond X} . \qquad (3.3.12b)$$

Note the nontriviality of the theory. In fact, after the elimination of the genoproduct for computational simplicity, there is the appearance in the exponent of four, different, integrodifferential operators R, S, R^d and S^d.

The time evolution of an operator A $\in$ $<\hat{\xi}>$ is characterized by the fundamental equations of hadronic mechanics [1], the one-dimensional *Lie-admissible group* (Sect. 1.7.6)

$$A(<\hat{t}>) = \hat{e}>^{i H t} > A(0) < <\hat{e}^{-i t H} = 1> e^{i t > H} > A(0) < e^{i H < t} <1 =$$

$$= e^{i H > t} A(0) e^{-i t < H} = e^{i H R t} A(0) e^{-i t S H} \qquad (3.3.13)$$

with infinitesimal form characterizing *Lie-admissible algebras*

$$i \frac{<\hat{\partial}>A}{<\hat{\partial}>t} = i<\hat{1}> \frac{dA}{dt} = (A,H) := A < H - H > A = A R H - H S A , \qquad (3.3.14)$$

and nonconserved but Hermitean, thus observable Hamiltonians, as desired,

$$i \frac{<\hat{\partial}>H}{<\hat{\partial}>t} = (H,H) = H R H - H S H \neq 0 . \qquad (3.3.15)$$

A property important for the understanding of the axiomatic structure of hadronic mechanics is that *even though the algebras in the neighborhood of the identity are not isomorphic, the above Lie-admissible theory coincides at the abstract level with the conventional Lie theory.* The abstract identity of conventional enveloping operator algebras ξ and their genotopes $<\hat{\xi}>$ and $<\hat{\xi}>^d$ is evident and requires no further comment.

The above results can be understood by noting that the conventional Lie theory possesses a natural differentiation of its modular action to the right and to the left. In fact, the Lie group of unitary transformations

$$A(t) = e^{i H t} A(0) e^{-i t H} , \qquad (3.3.16)$$

is structurally dependent on *two different modular actions,* the modular associative action to the right {exp (iHt)}→A(0) and that to the left A(0)←{exp(−itH)}.

In the transition to our covering Lie–admissible group, these actions are generalized in two different ways $(\exp^>(iHt)) > A(0)$ and $A(0) < (\exp^<(-itH))$. However, the corresponding units are jointly lifted in an amount which is the inverse of the deformations $\times \to >$ and $\times \to <$. As a result, Lie–admissible groups coincide with conventional Lie groups at the abstract level when represented in their respective spaces. Visible differences only emerge when the former are projected in the space of the latter, as done in the last identity of Eq.s (3.3.13).

The case of Lie vz Lie–admissible algebras follows the same lines. First, one must understand that the natural ordering of Lie group (3.3.16) carries over to the familiar infinitesimal case

$$i\dot{A} = AH - HA. \qquad (3.3.17)$$

In fact, the product $AH = A \leftarrow H$ originates from the *left* modular action $A(0) \leftarrow (\exp(-itH))$, while the product $HA = H \to A$ originates from the *right* action $(\exp(iHt)) \to A(0)$.

In the transition to our covering Lie–admissible theory, the product $A \leftarrow H$ and $H \to A$ are differentiated into the inequivalent forms $A < H = ARH$ and $H > A = HSA$. However, they are individually referred to generalized units which are the inverse of the deformation of the product, $^<\!1 = R^{-1}$ and $1^> = S^{-1}$. As a result, Lie algebras and their Lie–admissible generalizations also coincide at the abstract level when represented in their respective spaces, and we shall write

$$(A < H)_{^<\!1} - (H > A)_{1^>} \approx (A \leftarrow H)_1 - (H \to A)_1, \qquad (3.3.18)$$

where each product is referred to the corresponding unit.

Again, visible differences emerge only when the Lie–admissible product is projected in the original Lie space, as done in the last of the identities (3.3.14), and therefore treated with the same, trivial, original unit I.

The above occurrences clarify the difference between *isotopies* and *genotopy*. In the former case the theory verifies the Lie algebra algebra axioms in both the conventional and generalized spaces. In the latter case, the Lie axioms are violated in conventional spaces to "induce" generalized axioms.

Despite the above similarities, the lifting of the Lie into the Lie–admissible theory is nontrivial. As an example, the genohilbert spaces are *isobimodules* (Sect. I.7.8) because of their modular isoassociative structure with the differentiation of the actions to the right and to the left. The representation theory of Lie–admissible algebras over isobimodular Hilbert spaces is one of the most complex and unexplored fields of contemporary pure and applied mathematics.

For additional properties of Lie–admissible formulations we refer the reader to Ch. I.7. Specific realizations will be introduced later on in this volume when studying the Lie–admissible generalizations of the Galilean, Lorentz and

Poincaré symmetries.

3.3.D: Genoprobabilities and their isoduals. The *genorobability density* is given by

$$<\hat{\mathscr{p}}> = |\hat{\psi}(t, r)|^{<\hat{2}>} = |\hat{\psi}\dagger(t, r) <> \psi(t, r)|^{<\hat{1}>} , \qquad (3.3.19)$$

with genonormalization

$$\int dV \,\hat{\psi}\dagger <> \hat{\psi} = 1 , \qquad (3.3.20)$$

and *genoprobabilities for obtaining the genoeigenvalues* $\hat{c}_k{}^d$

$$<\hat{w}>_k = {}^{<}w>_k {}^{<\hat{1}>} \in {}^{<}\hat{R}>({}^{<}\hat{n}>,+,<>), \quad {}^{<}w>_k = |< \hat{\psi}_k | <> | \hat{\psi} >|^2 . \qquad (3.3.21)$$

The sum of all such possible probabilities then yields again the underlying unit, the genounit,

$$\Sigma_k {}^{<}\hat{w}>_k = {}^{<\hat{1}>} . \qquad (3.3.22)$$

The corresponding properties under isodualities are left for study to the interested reader.

3.3.E: Basic axioms of the genotopic branch of hadronic mechanics. The *genoaxioms* of hadronic mechanics were indicated by Santilli [1] in 1978, they were first formulated on a conventional Hilbert space[33] by Myung and Santilli [10] in 1982 and then finalized by Santilli [2] in 1989 on genohilbert spaces over genofields. For the case of genounits whose Hermitean components are of Class I, they can be presented as follows.

AXIOM $^{<\hat{1}>}$: The states of the genotopic branch of hadronic mechanics, called <genostates>, are elements $| \hat{\psi} >$ of a right isobimodular Hilbert space $^{<}\hat{\mathcal{H}}>$ with genoschroedinger equation

$$- i \frac{^{<}\partial>}{^{<}\partial>_t} | \hat{\psi} > = H <> | \hat{\psi} > = {}^{<}\hat{E}> <> | \hat{\psi} >, \qquad (3.3.23)$$

where $^{<}\partial>/^{<}\partial>_t = {}^{<\hat{1}>}_t \partial/\partial t$, the quantity $^{<}\hat{E}> = {}^{<}E>^{<\hat{1}>}$ is an element of the genofield $^{<}\hat{C}>({}^{<}\hat{c}>,+,<>)$ and the eigenvalues $^{<}\hat{E}>$ are elements of the

---

[33] As recalled in the introductory comments of Ch. I.6, the formulation of hadronic mechanics on a *conventional* Hilbert space is fully consistent on *mathematical* grounds. However, it implies problematic aspects of *physical* character, such as the loss of Hermiticity–observability of the energy operator and other physical quantities whose preservation required the genotopies of Hilbert spaces and fields.

genofield $\langle \hat{C} \rangle (\langle \hat{c} \rangle, +, \langle \rangle)$ and the eigenvalues $\langle E \rangle$ are elements of the conventional field $C(c, +, \times)$.

AXIOM $\langle II \rangle$: Measurable physical quantities mechanics, called <genoobservables>, are characterized by Hermitean operators on $\langle \mathcal{H} \rangle$ which, as such, admit genoreal genoeigenvalues (Sect. I.6.3). Measured quantities are then characterized by ordinary real numbers

$$H \equiv H^{\hat{\dagger}} \equiv H^{\dagger}, \quad H \diamondsuit | \hat{\psi} \rangle = \langle \hat{E} \rangle \diamondsuit | \hat{\psi} \rangle = \langle E \rangle | \hat{\psi} \rangle \qquad (3.3.24)$$

AXIOMS $\langle III \rangle$: The operators of the genotopic branch of hadronic mechanics are constructed from time t and the genoconjugated coordinates r_i and momenta p_j, with <fundamental Lie–admissible rules> in unified notation $a = (r, p)$

$$(a^\mu, a^\nu) = a^\mu \langle a^\nu - a^\nu \rangle a^\mu = i \, \S^{\mu \nu} \quad \mu, \nu = 1, 2, \dots, 6n \, . \qquad (3.3.25)$$

where $\S^{\mu \nu}$ is a properly symmetrized, operator image of the classical Lie–admissible tensor of Eq.s (II.1.5.4).

AXIOM $\langle IV \rangle$: The time evolution of the genostates is characterized by genounitary transformations with the genohermitean Hamiltonian as the generator; the time evolution of genoobservables is characterized by a genoequivalent, one-dimensional, Lie-admissible group also with the Hamiltonian as the generator.

AXIOM $\langle V \rangle$: The values expected in the measurements of genoobservables are the <genoexpectation values>

$$\hat{\langle} \langle A \rangle \hat{\rangle} = \frac{\langle \hat{\psi} | \diamondsuit A \diamondsuit | \hat{\psi} \rangle}{\langle \hat{\psi} | \diamondsuit | \hat{\psi} \rangle} = \langle a \rangle \in C(c, +, \times), \qquad (3.3.26)$$

A primary difference between Axioms V and $\langle V \rangle$ is that isoexpectation values are conserved, while in genoexpectation values *are not* conserved in time, as the reader can verify. This is evidently *necessary* to represent open systems.

The *isodual genoaxioms* can be constructed accordingly. Specific examples, illustrations, applications and experimental verifications will be given throughout the rest of our analysis.

In summary, in this chapter we have assumed the axiom of conventional quantum mechanics as being *exactly* valid for the exterior problem of particles

in vacuum at large mutual distances, and introduced two sequential *realizations* of the same abstract axioms, one of isotopic and the other of genotopic type for mutual distances of the range of the strong interactions, according to the following chain:

1) Quantum mechanics with basic unit $\hbar = \hbar^\dagger = 1 > 0$ and enveloping operator algebra ξ on the field $F(a,+,\times)$ of real or complex numbers over a Hilbert space $\mathcal{H}$ for the representation of systems of particles in exterior conditions at large mutual distances in vacuum with positive–definite, conserved, total energy $H = K + V$ and only potential internal forces represented by V;

2) Isotopic branch of hadronic mechanics (Class I), with basic isounit $\hat{\hbar}(t, p, \dot{p}, ...) = \hat{\hbar}^\dagger > 0$ and enveloping operator isoalgebra $\hat{\xi}$ on the isofield $\hat{F}(\hat{a},+,*)$ of real or complex numbers over an isohilbert space $\hat{\mathcal{H}}$ for the representation of systems of particles in interior conditions at mutual distances of the order of the size of their wavepackets or charge–distributions with positive–definite, conserved, total energy $H = K + V$ under potential internal forces represented by V and contact–nonpotential interactions represented by $\hat{\hbar}$;

3) Genotopic branch of hadronic mechanics (Class I), with basic units $^>\hat{\hbar}^>(t, p, \dot{p}, ...) \neq (^>\hat{\hbar}^>)^\dagger$, $\hat{\hbar}^> = (^>\hat{\hbar})^\dagger$ for forward motion to future time > or in past time < and genoassociative algebras $^>\hat{\xi}^>$ on corresponding genofields $^>\hat{F}^>(^>\hat{a}^>,+,<>)$ over genohilbert spaces $^>\hat{\mathcal{H}}^>$ for the representation of one particle or of a system of particles with positive–definite but nonconserved total energy $H = K + V$ due to external interactions of nonpotential type represented by $^>\hat{\hbar}^>$.

Moreover, in this chapter we have identified the isodual image of the basic axioms of quantum mechanics for the representation of antiparticles in vacuum and then presented the corresponding chain of isodual isotopic and genotopic coverings according to:

4) Isodual quantum mechanics, with basic unit $\hbar^d = -\hbar = -1 < 0$ and isodual enveloping algebra ξ^d on the isodual field $F^d(a^d,+,\times^d)$ of real or complex numbers over an isodual Hilbert space $\mathcal{H}^d$ for the representation of systems of antiparticles in exterior conditions in vacuum at large mutual distances with negative–definite, conserved, total energy $H^d = -K-V$ under only isodual potential internal forces represented by $-V$;

5) Isodual isotopic branch of hadronic mechanics (Class II), with basic unit $\hat{\hbar}^d = -\hat{\hbar}(t, p, \dot{p}, ...) < 0$ and isodual enveloping isoalgebra $\hat{\xi}^d$ on the isodual isofield $\hat{F}^d(\hat{a}^d,+,*^d)$ of real or complex numbers over an isodual isohilbert space $\hat{\mathcal{H}}^d$ for the representation of systems of antiparticles in interior conditions at mutual distances of the order of their wavepackets or charge distributions, with negative–definite, conserved, total energy $H^d = -K-V$, internal isodual potential interactions represented by $-V$ and additional contact–nonpotential interactions represented by $\hat{\hbar}^d$;

6) Isodual genotopic branch of hadronic mechanics (Class II), with basic units $^>\hat{\hbar}^{>d}(t, p, \dot{p}, ...) = -{}^>\hat{\hbar}^>$ for backward motion in future time > or in past time < and isodual genoassociative algebras $^>\hat{\xi}^{>d}$ on corresponding isodual

genofields $\langle \hat{F} \rangle^d (\langle \hat{a} \rangle^d, +, <>^d)$ over isodual genohilbert spaces $\langle \hat{\mathcal{H}}_a \rangle^d$ for the representation of one antiparticle or of a system of antiparticles with negative–definite, nonconserved, total energy $H^d = -K-V$ due to external interactions of nonpotential type represented by $\langle \hat{\hbar} \rangle^d$.

We also have the following additional branches which are vastly unexplored at this writing:

7) Hadronic mechanics of Class III, which can be considered as the union of Classes I and II;

8) Hadronic Mechanics of Class IV, which is characterized by *singular* isounits or genounits; and

9) Hadronic Mechanics of Class V, characterized by generalized units of arbitrary structure, including discontinuous functions, distributions, lattices, etc., and which evidently includes all preceding branches as particular cases.

APPENDIX 3.A: ISOEQUIVALENCE OF ISOSCHROEDINGER AND ISOHEISENBERG REPRESENTATIONS

We now indicate the isoequivalence of the isotopies of Schrödinger and Heisenberg representations first studied by Myung and Santilli [3] in 1982 for the case of conventional derivatives and then studied by Santilli [2] in 1989 for the case with isoderivatives.

For clarity, let us indicate quantities belonging to the isoschrödinger representation with the subscript (1) and those belonging to the isoheisenberg representation with the subscript (2). The Hamiltonians in the two representations are interconnected by a isounitary transformation

$$H_{(1)} = \hat{U} * H_{(2)} * \hat{U}^{\dagger}. \tag{3.A.1}$$

States in the two representations are interconnected by the corresponding isounitary transformation

$$|\hat{t}>_{(1)} = \hat{U}*|\hat{0}>_{(2)}. \tag{3.A.2}$$

where $\hat{t} = tT_t$ is the isotime (Sect. II.3.2).

A step–by–step isotopy of the proof of equivalence of the two representation in conventional quantum mechanics (see, e.g., [9] and quoted references) then establishes the following

Theorem 3.A.1 [2,3]: *Under sufficient topological conditions, the isoschrödinger's equations*

$$i \, \frac{\partial}{\partial t} | \hat{t} >_{(1)} = H_{(1)} * | \hat{t} >_{(1)} , \qquad \frac{\partial}{\partial t} | 0 >_{(1)} = 0 , \qquad (3.A.3)$$

are *isoequivalent* to the *isoheisenberg equations*

$$i \, \frac{\partial}{\partial t} (A_{(2)} * | 0 >_{(2)}) = [A_{(2)} , \hat{H}_{(2)}] * | 0 >_{(2)} , \qquad \frac{\partial}{\partial t} | 0 >_{(2)} = 0 . \qquad (3.A.4)$$

under the isounitary transform

$$| \hat{t} >_{(1)} \; = = \hat{U} * | \hat{r} >_{(2)} , \qquad (3.A.5a)$$

$$\hat{U} = \hat{e}^{\, i \, H_{(1)} \, \hat{t}} . \qquad (3.A.5b)$$

The study of the generalization of the above theorem for the genotopies of Schrödinger and Heisenberg representations is recommended to interested readers.

APPENDIX 3.B: OKUBO'S NO–QUANTIZATION THEOREM AND THE PROBLEMATIC ASPECTS OF WEINBERG'S NONLINEAR MECHANICS

The single, most dominant, algebraic property of hadronic mechanics is the preservation of the *associative* character of the enveloping operator algebra, as suggested since the original proposal [1]. This is due to the fact that generalizations of quantum mechanics based on *nonassociative* enveloping operator algebras, such as *Weinberg's nonlinear mechanics* [11], are faced with rather serious problematic aspects studied in detail by Mignani, Jannussis and Santilli [12].

In particular, the equivalence of the isoheisenberg and isoschrödinger representations of the preceding appendix is crucially dependence on such an associative character. On the contrary, the corresponding representations of Weinberg's nonlinear mechanics are inequivalent because the underlying envelope is nonassociative [12].

As shown by Ktorides, Myung and Santilli [13], the generalization of the Poincaré–Birkhoff–Witt theorem for an enveloping algebra ξ with generic associative product $A \times B$, $(A \times B) \times C = A \times (B \times C)$, is not known for the general case of nonassociative envelope $\xi^{\odot}$ with generic product $A \odot B$, $(A \odot B) \odot C \neq A \odot (B \odot C)$, with the sole exception for the quasi–associative algebra called *flexible Lie–admissible algebras* (App. 1.4.A) with product $A \odot B = \lambda A \times B - \mu B \times A$, where λ, μ are scalars.

The first reason for the problematic aspects is that nonassociative algebras

do not generally have a left and right unit which is at the foundation of the Poincarè–Birkhoff–Witt theorem.

The second reason is that nonassociative envelopes generally imply the loss of ordering in the polynomials

$$X_i \odot X_j, \ i \leqq j, \ X_i \odot X_j \odot X_k, \ i \leqq j \leqq k, \ldots\ldots \tag{3.B.1}$$

The lack of a consistent infinite–dimensional basis then implies the impossibility of defining the generalized exponentiation in a unique and unambiguous way. In turn, this render impossible to generalize in a unique and consistent way the various physical laws dependent on the exponentiation, such as the Gaussian, Heisenberg's uncertainties, etc.

The third reason is the following property first studied by Okubo [14]:

Theorem 3.B.1 (Okubo's no–quantization theorem): *A quantization of classical systems based on nonassociative enveloping operator algebras implies inequivalent generalizations of Heisenberg and Schrödinger's representations.*

Recall that the central algebraic structure of Heisenberg's representation is the associativity of its envelope ξ with product A×B. Heisenberg's equation is then constructed as the attached antisymmetric algebra

$$i \dot{A} = A \times H - H \times A, \quad (A \times B) \times C = A \times (B \times C). \tag{3.B.2}$$

In the transition to the Schrödinger's representation, associativity also remains dominant for the right (or left) modular action of operators H, A, B on the states,

$$H |\psi> = E |\psi>, \quad A \times B \times |\psi> = A \times (B \times |\psi>) = (A \times B) \times |\psi>. \tag{3.B.3}$$

Now, the isoheisenberg equations can be identically reformulated via a nonassociative envelope ξ⊙ with realization

$$i \dot{A} = A * H - H * A = ATH - HTA \equiv A \odot H - H \odot A,$$

$$A \odot H = A \times R \times H - H \times S \times A = \text{nonass.}, \quad T = R + S \neq 0. \tag{3.B.4}$$

The important point is that hadronic mechanics would be inconsistent under such reformulation.

In fact, the reformulation would demand the construction of a nonassociative generalization of Schrödinger's equations of the type

$$H \odot | \hat{\psi} > \, = \, E \, | \psi > , \quad A \odot (B \odot | \hat{\psi} >) \, \neq \, (A \odot B) \odot | \hat{\psi} > , \qquad (3.B.5)$$

which does not exist at this writing and this illustrates Theorem 3.B.1.

It should be noted that Okubo [loc. cit.] proved the inequivalence of Heisenberg–type and Schrödinger–type representations even for the *weaker* case of *flexible Lie–admissible (or quasiassociative) algebras.*

The inequivalence of the generalized Heisenberg and Schrödinger representations in Weinberg's nonlinear theory [11] is then evident. In fact, the former is based on a *nonassociative* envelope with product

$$A \odot B \, = \, \frac{\partial A}{\partial \psi_k} \, \frac{\partial B}{\partial \bar{\psi}_k} \qquad (3.B.6)$$

while the latter is based on the *conventional right modular associative action*

$$H \, | \psi > \, = \, [\, - \Delta \, / \, 2m \, + \, V(r) \, + \, W(\psi \bar{\psi}) \,] \, | \psi > . \qquad (3.B.7)$$

In the hope of avoiding the above problematic aspects, Jordan [15] reformuated Weinberg's theory in a fully isotopic way (without any quotation of isotopic studies), only diguised in the form (see Eq.s (2.8), p.87, ref. [15])

$$A_{i \, k} \, T_{kn} \, B_{nj} \, - \, B_{ik} \, T_{kn} \, A_{nj} \, . \qquad (3.B.8).$$

Despite that, Jordan's reformulation itself is afflicted by other problematic aspects, such as the general loss of Hermiticity–observability (Lemma 11.C.1 below) because it is formulated on conventional Hilbert spaces and fields.

It is unfortunate that, following ref. [14] of 1982, Okubo was forced to terminate *physical* research in the field and restrict his attention to *mathematical* profiles.

APPENDIX 3.C: AXIOMATIC REFORMULATION OF q–DEFORMATIONS

As pointed out in App. I.7.A.1, during his Ph. D. studies back in 1967, this author [16] was the originator of the q–deformations (see, e.g., ref.s [17]) via the joint *Lie–admissible and Jordan–admissible mutations* of an associative algebra[34]

---

[34] It should be noted that virtually none of ref.s [17] quotes the origination of q–deformations in ref. [16] despite a number of independent communications of the latter paper to the authors of the former. In particular, virtually none of them identifies the joint Lie–admissible and Jordan admissible character of the deformations. Above all, virtually no paper in the rather vast literature of q–deformations quotes the origination of the abstract notion of Lie–admissibility by A. A. Albert in 1948 (see App. I.7.9.A.1).

$$(a , b) = p a b - q b a , \qquad p, q \in R(n,+,\times), \qquad a b = assoc. \qquad (3.C.1)$$

Subsequently, this author was forced to abandon the above formalism in favor of hadronic mechanics because of a number of virtually unsurmontable problematic aspects for genuine physical applications, such as: lack of form-invariance under their own transformation theory; loss of Hermiticity-observability under their own time evolution (see the Preface and the end of this appendix); lack of measurement theory due to the lack of a left and right unit; lack of uniqueness of generalized mathematical structure (such as exponentiation, Gaussian distribution, etc.), with consequential lack of uniqueness of related physical laws (such as generalized Heisenberg's principle); general loss of the special functions under time evolution; general loss of the Galilean and Einsteinian axioms, thus raising the problems of identifying plausible new axioms and establishing their physical validity; and others (see App. I.7.9.A).

In this appendix we outlie the reformulation of q-deformations in terms of the basic axioms of hadronic mechanics studied by Lopez [8] which is quite simple and resolves all problematic aspects known to this author.

Case I: q-deformations of associative envelopes, e.g.,

$$AB \Rightarrow q AB , \qquad a\,a^\dagger \Rightarrow q\,a\,a^\dagger , \qquad (3.C.2)$$

The axiomatic reformulation is then achieved by simply lifting the unit I into the isounit $\hat{1} = q^{-1}$, and constructing the chain of isotopics of fields, spaces, Lie algebras, etc. The isotopic character then ensures the form-invariance of the theory under arbitrary transformations (Sect. II.3.2).

Case II: q-deformations of eigenvalues of commutators, e.g.,

$$r p - p r = i f(q) \neq i , \qquad (3.C.3)$$

As proved by Jannussis [18], these formulations are noncanonical, thus lacking an axiomatic character when treated with *conventional* methods. The above q-deformations can be easily reformulated in the axiomatic isotopic form.

Assume $f(q)$ as the new isounit, $\hat{1} = f(q)$. The isotopic element is then given by $T = [f(q)]^{-1}$ with consequential isoeigenvalue equation

$$p * |\hat{\psi}> := p T |\hat{\psi}> = - i\hat{1} \nabla |\hat{\psi}> , \qquad (3.C.4)$$

under which commutator (3.C.3) is turned into the equivalent isoform

$$r * p - p * r = r [f(q)]^{-1} p - p [f(q)]^{-1} r = i\hat{1} = i f(q) . \qquad (3.C.5)$$

which is now form-invariant under time evolution.

A similar axiomatic reformulation occurs for creation and annihilation operators with noncanonical eigenvalues. Note that Lie–admissible formulations are *inapplicable* in this case because the energy is conserved or, equivalently, the brackets are antisymmetric.

Along similar lines, the axiomatic reformulation of q–deformations of Lie's second theorem with structure constants C_{ij}^k

$$X_i\,X_j - X_j\,X_i \;=\; C_{ij}^{\;k}\,X_k \quad \rightarrow \quad X_i\,X_j - X_j\,X_i \;=\; F_{ij}^{\;k}(t,\,q,\,...)\,X_k \qquad (3.C.6)$$

is obtained by assuming the functions $F_{ij}^{\;k}(q,\,...)$ as the isofunctions of the theory (Sect. 1.4.4), and then searching for a compatible isotopic element T and reformulation of the basis such that

$$\hat{X}_i\,T\,\hat{X}_j - \hat{X}_j\,T\,\hat{X}_i \;=\; \hat{F}_{ij}^{\;k}(t,\,q,\,...)\,T\,\hat{X}_k. \qquad (3.C.7)$$

The form–invariance of the theory under the time evolution is then ensured by its isotopic structure.

Case III: q-deformations of Lie-products, such as

$$r\,p - q\,p\,r = i\,f(q). \qquad (3.C.8)$$

The latter case requires the full use of the Lie–admissible formulations because the brackets are no longer totally antisymmetric. In fact, the multiplication to the right is isotopic, $p>r = pSr$, $S = q$, and that to the left is conventional, $r<p = rRp$, $R = 1$, resulting in the product $(p,\,r) = r<p - p>r$ which is flexible, Lie–admissible and Jordan–admissible , as first identified by this author back in 1967 (Ch. 1.7).

A necessary condition of consistency of the theory is that the fundamental rules [14]

$$(\,a^\mu,\,a^\nu\,) \;=\; a^\mu < a^\nu - a^\nu > a^\mu \;=\; i\,\omega^{\mu\alpha}\,\triangleleft_{\alpha}^{\;\nu\triangleright}, \quad a = (r,\,p) \qquad (3.C.9)$$

characterize a Lie–admissible tensor $S^{\mu\nu} = \omega^{\mu\alpha}\,\triangleleft_{\alpha}^{\;\nu\triangleright}$ in each selected direction of time > or <.

The above reformulation was first studied by Jannussis and his collaborators [19] on conventional fields. That on genofields requires the selection of one "time arrow" and then the interpretation of the function $f(q,...)$ in rules (3.C.9) as the genounits for that direction.

Note that the q–deformation of the second term in the l.h.s. is not axiomatic and must be lifted into the inverse of the selected genounit, resulting in the reformulations

$$r\,p\,-\,q\,p\,r\,=\,i\,f(q,...) \quad \Rightarrow \quad \begin{cases} r<p\,-\,p>q\,=\,r\,R\,p\,-\,p\,S\,r\,=\,i1^>, \\[4pt] 1^>\,=\,f(q,...)\,/\,q, \quad S=q\,/\,f(q,...), \quad R=f(q,...) \\[2pt] \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{or} \\[2pt] r<p\,-\,p>q\,=\,r\,R\,p\,-\,p\,S\,r\,=\,i{<}1, \\[4pt] {<}1\,=\,f(q,...)\,/\,q, \quad S=f(q,...), \quad R=q\,/\,f(q,...) \end{cases}$$
$$(3.C.10)$$

The entire theory must then be reformulated on genofields, genospaces, genotrasnformations, etc., for the selected direction of time.

However, as studied in the main text (Sect. II.3.3.B), *the Hermiticity-observability is lost for reformulation (3.C.10) because of the violation of the basic rule* $R = S^\dagger$. This implies, again, that the only axiomatically consistent q-deformations of conventional commutators for *q real* are those with the *Lie-isotopic* structure

$$q\,r\,p\,-\,q\,p\,r\,=\,i\,q^{-1}, \qquad\qquad (3.C.11)$$

while the only axiomatically consistent q-deformations for q-complex are those with the *Lie-admissible* structure

$$\bar{q}\,r\,p\,-\,q\,p\,r\,=\,i\,q^{-1} \quad \text{or} \quad i\,\bar{q}^{-1}. \qquad\qquad (3.C.12)$$

The axiomatic reformulation of other deformations can be done with one or the other of the above methods.

The best way to see the inevitability of the above axiomatic reformulation *even when not desired,* is by subjecting q-deformations to their own *nonunitary time evolutions*

$$U\,U^\dagger\,=\,f(q)\,=\,1\,\neq\,I, \qquad\qquad T\,=\,(U\,U^\dagger)^{-1}. \qquad (3.C.13)$$

In this case we have

$$U\,(r\,p\,-\,q\,p\,r)\,U^\dagger\,=\,r'\,R\,p'\,-\,p'\,S\,r\,=\,i\,R^{-1},\ S=qR. \qquad (3.5.14)$$

Starting from a conventional formulation of a q-deformation one ends up in the axiomatic form studied in this chapter.

The reader should be aware that, while reformulation (3.C.12a) is mathematically and physically consistent, reformulation (3.5.12b) is mathematically consistent, but physically problematic because it does not verify the conjugation condition $R = S^\dagger$ some other conjugation (Sect. 3.3).

An examples is now in order. The damped particle represented by

$$H(t)\,=\,e^{-\gamma\,t}H_0, \qquad H_0\,=\,\tfrac{1}{2}\,p_0{}^2, \qquad m=1, \qquad \dot{H}=-\gamma\,H. \qquad (3.C.15)$$

is *axiomatically* represented by the genotopic branch of hadronic mechanics

(because the system is nonconservative) via Eq. (3.C.12) with

$$R = -\tfrac{1}{2} i \gamma H_0^{-1}, \qquad S = R^\dagger. \qquad (3.C.16)$$

Eq. (II.3.3.15) then correctly reads $i\dot{H} = (R - S)H_0^2 = -i\gamma H_0$. The direct universality of the theory ensures the existence of axiomatic representations of other systems. Note that the Hamiltonian remains Hermitean thus observable, yet it is not conserved in time. Note that Lie–admissible formulation (3.C.14) does verify the conjugation $R = S^\dagger$ as necessary for an axiomatic characterization of irreversibility and that the genoeigenvalues are *real* for *imaginary* genotopic elements $R = iT$, $T = T^\dagger$. Other examples will be studied later on.

We close this appendix with the following important

Lemma 3.C.1 (Lopez's Lemma [8]): *Physical quantities of q– or quantum–deformations which are Hermitean–observable at the initial time, are generally nonhermitean–nonobervable at subsequent times.*

Proof. Under a nonunitary time evolution $UU^\dagger = \hat{1} \neq 1$, the (q–) quantum–envelope ξ with product (qAB) AB assumes the *necessary* isotopic form $A'*B' = A'TB'$, $A' = UAU^\dagger$, $B' = UBU^\dagger$, $(T = q(UU^\dagger)^{-1})$ $T = (UU^\dagger)^{-1}$. The condition of Hermiticity $H = H^\dagger$ at time t_0 in a conventional Hilbert space $\mathcal{H}$ at times $t > t_0$ becomes $H^\dagger = TH^\dagger T^{-1}$ (Proposition 1.6.3.2) which, as such, is generally violated. The same result holds for *all* theories with nonunitary time evolution. **q.e.d.**

APPENDIX 3.D: ISOTOPIES AND GENOTOPIES OF DISCRETE TIME THEORIES

Another application of hadronic mechanics is to the so–called *discrete–time theories* which was initiated by Caldirola [19] in 1941 with the equations

$$i\hbar[\psi(t) - \psi(t-\tau)] / \tau = H\psi, \qquad (3.D.1a)$$

$$i\hbar[\psi(t+\tau) - \psi(t-\tau)] / 2\tau = H\psi, \qquad (3.D.1b)$$

$$i\hbar[\tau(t+\tau) - \psi(t)] / \tau = H\psi. \qquad (3.D.1c)$$

where τ is a discrete time, called *chronon*, representing the duration of the interaction considered. The equations are advanced, oscillating and retarded and evidently recover the conventional Schrödinger equation at the limit $\tau \to 0$.

Discrete–time equations of Caldirola's type were studied in great details by Jannussis and his collaborators [20], by Wolf [21] who has initiated the study of the experimental verification of the possible discreteness of time, and others.

The field has nowadays expanded considerably because of its evident connection to *lattice theories* on the structure of space–time (see, ref.s [22] and

literature quoted therein).

Jannussis and his collaborators [20] have established that all discrete-time theories are a particular case of hadronic mechanics. In fact, they have shown that the oscillating equation (3.D.1b) admits the isoscrödinger/Lie-isotopic structure,

$$ i\hbar \frac{\partial}{\partial t} \hat{\psi}(t) = H \, T(\tau) \, \hat{\psi}(t), \qquad T(\tau) = \frac{\hbar}{\tau H} \arcsin \frac{\tau H}{\hbar}, \qquad (3.D.2) $$

while the advances and retarded equations (3.D.1a) and (3.D.1c) possess the genoschrödinger/ Lie-admissible structures

$$ i\hbar \frac{\partial \psi(t)}{\partial t} = H \, R(\tau) \, \psi(t), \qquad (3.D.3a) $$

$$ i\hbar \frac{\partial \psi(t)}{\partial t} = H \, S(\tau) \, \psi(t), \quad S(\tau) = R(-\tau), \qquad (3.D.3a) $$

$$ R(\tau) = \frac{\hbar}{\tau H} \arctan \frac{\tau H}{\hbar} + i \frac{\hbar}{2\tau H} \mathrm{Ln} \left(1 + \frac{\tau^2 H^2}{\hbar^2} \right). \qquad (3.D.3b) $$

The above reformulations have important implications. To begin, they imply the complete embedding of the discrete time structure in the space isotopic or genotopic elements. As an example, the isoexpectation value of the isounit $\hat{1}(\tau) = [T(\tau)]^{-1}$ of Eq. (3.C.2) is the number one

$$ \hat{<} \hat{1}(\tau) \hat{>} = <\hat{\psi} \,|\, T(\tau) \,\hat{1}(\tau)\, T(\tau) \,|\, \hat{\psi}> \,/\, <\hat{\psi} \,|\, T(\tau) \,|\, \hat{\psi}> = 1. \qquad (3.D.4) $$

In turn, this implies the remarkable consequence that discrete time theories are indeed admitted by the abstract axioms of quantum mechanics, although realized in their most general possible form. Similar, although more general results occur for the genoreformulations (3.C.3).

A complementary reformulation of open systems (3.D.1a) and (3.D.1c) (only) has been done by Jannussis, Skaltsas and Brodimas [23] by embedding the discrete structure of Caldirola's chronon in the time component of Lie-admissible equations. These latter studies (not reported here for brevity) are also significant inasmuch as they show the equivalence of the discrete chronon with a continuous complex "unit" of time. As such, the latter treatment can be fully reformulated via the time genofields at the foundation of the Lie-admissible branch of hadronic mechanics.

In summary, the studies by Jannussis and his associates [loc. cit.] have identified a dual reformulation of discrete time theories with hadronic mechanics via the complete embedding of the discrete time contribution either in the space isotopic or genotopic element or in their time counterpart. Either

reformulation shows the compatibility of discrete time theories with the abstract axioms of quantum mechanics, *provided* that they are expressed via generalized units incorporating the totality of the discrete time contributions.

The above results are not mere mathematical curiosities because they imply that *discrete time theories admit different numerical results when treated via quantum or hadronic methods*, as one can see from the fact that renormalizations of the wavefunctions, expectation values, etc., are numerically different in the quantum and hadronic treatments.

But discrete–time theories deal with a *structural generalization* of quantum mechanics. Such a generalization then requires a corresponding generalization of the basic methods used in the elaboration of the theory. At any rate, the use of quantum mechanical methods for the elaboration of discrete time theory is afflicted by a number of problematic aspects also common to other generalized theories, such as the general lack of preservation of the Hermiticity–observability in time due to the generally nonunitary character of the underlying time evolution, the lack of invariance of the basic unit with consequential inapplicability to actual measurements, etc.

APPENDIX 3.E: ISOTOPIES OF BERRY'S PHASE

Consider a quantum mechanical system interacting with its environment, which is represented by the Hamiltonian $H(R)(t)$ on a conventional Hilbert space $\mathfrak{IC}$ over $C(c,+,\times)$ with eigenvalues $E(R)$ and basis $\mid b(R) >$ where R is a slowly changing parameter. Under the validity of the adiabatic theorem, when the system is returned to the original eigenstate of H, the wave function can only change by a phase factor which has been explicitly computed by Berry [24] and can be written

$$\exp\{-i\hbar^{-1}\int E_n\,dt + i\gamma_n(C)\}, \quad \gamma_n(C) = i\int_C < b(R)\mid\nabla_R\mid b(R) > dR, \quad (3.E.1)$$

where C is the closed path in R–space, the first term in the exponential is the conventional one and the last term is the so–called *Berry's phase*.

Simon [25] has shown that the Berry phase represents an holonomy in the Hilbert space bundle over the parameter space, and introduced the *Berry connection*

$$A_n(R) = i < b(R)\mid\nabla_R\,b(R) >, \quad \gamma_b(C) = \int_C A_n(R)\,dR. \quad (3.E.2)$$

For additional theoretical studies one may consult ref.s [26].

The *isotopic lifting of Berry's phase*, or *isoberry's phase* for short, has been studied by Mignani [27]. It essentially implies the step–by–step reformulation of the conventional treatment via the isotopic branch of hadronic mechanics, i.e.,

the use of the isoscrödinger's equations on an isohilbert space $\mathcal{H}$ over isofields $\hat{C}(\hat{c},+,*)$, etc.

The main result is that the dynamical phase essentially remains unchanged, but Berry's connection is lifted into the form

$$\hat{\gamma}_n(C) = i\hat{1} \int_C T < b(R) \,|\, \nabla_R\, b(R) > dR \,, \qquad (3.E.3)$$

which evidently coincides with the conventional connection for T = constant, but shows a significant deviation for T = T(R, ...).

As we shall see in Vol. III, the conventional Berry phase is experimentally measurable via neutron interferometry and other techniques and essentially represents a *point–like* structure in adiabatic evolution. The isotopy of Berry's phase has then important experimental implications because it permits measures of the expected *extended* character of the systems and the presence of internal nonlocal–nonhamiltonian forces. In the final analysis, as noted in ref. [27], only hadronic mechanics can account in a consistent and axiomatic way for a deviation from Berry's phase.

In closing we point out that ref.s [27] use the old version of the isoscrödinger's equation, that with *conventional* time derivative. The results of ref.s [27] must be therefore re-elaborated with the isotime derivative to achieve a full axiomatic form.

REFERENCES

1. R. M. SANTILLl, Hadronic J. $\underline{1}$, 574 (1978)
2. R. M. SANTILLl, Hadronic J. Suppl. $\underline{4B}$, Issues no.s 1 and 2 (1989)
3. H. C. MYUNG and R. M. SANTILLl, Hadronic J. $\underline{5}$, 1277 (1982)
4. R. MIGNANl, Hadronic J. $\underline{4}$, 608 (1981); $\underline{5}$, 1120 (1982)
5. A. J. KALNAY, Hadronic J. $\underline{6}$, 1 (1983)
6. A. J. KALNAY and R. M. SANTILLl, Hadronic J. $\underline{6}$, 1798 (1983)
7. R. MIGNANI, H. C. MYUNG and R. M. SANTILLl, Hadronic J. $\underline{6}$, 1873 (1983)
8. D. F. LOPEZ, Hadronic J. $\underline{16}$, 429 (1993)
9. D. BOHM, *Quantum Theory*, Dover Publications, New York (1979 reprint);
   E. MERZBACHER, *Quantum Mechanics*, Wiley and Sons, New York (1961);
   P. ROMAN, *Advanced Quantum Theory*, Addison–Wessley, Reading MA (1965)
10. H. C. MYUNG and R. M. SANTILLl, Hadronic J. $\underline{5}$, 1367 (1982)
11. S. WEINBERG, Ann. Phys. $\underline{194}$, 336 (1989)
12. A. JANNUSSIS, R. MIGNANI and R. M. SANTILLl, Ann. Fond. de Broglie $\underline{18}$, 371 (1993)
13. C. N. KTORIDES, H. C. MYUNG and R. M. SANTILLl, Phys. Rev. $\underline{D22}$, 892 (1980)
14. S. OKUBO, Hadronic J. $\underline{5}$, 1667 (1982)
15. T. F. JORDAN, Ann. Phys. $\underline{225}$, 83 (1993)

16. R. M. SANTILLI, Lettere Nuovo Cimento **51**, 570 (1967)

17. L. C. BIEDENHARN J. Phys. A **22**, L873 (1989); A. J. MACFARLANE, J. Phys. A **22**, L4581 (1989); T. L. CURTIS, B. FAIRLIE and Z.K. ZACHOS, Editors, *uantum Groups*, World, Scientific, Singapore (1991); MO–LIN GE and BAO HENG ZHAO, Editors, *Introdu–ction to Quantum Groups and Integrable Massive Models of Quantum Field Theory*, World Scientific (1991); Yu. F. SMIRNOV and R. M. ASHEROVA, Editors, *Proceedings of the Fifth Workshop Symmetry Methods in Physics*, JINR, Dubna, Russia (1992)

18. A. D. JANNUSSIS, G. N. BRODIMAS, V. PAPATHEOU and A. LEODARIS, Lett. Nuovo Cimento **36**, 545 (1983)

19. P. CALDIROLA, Nuovo Cimento **18**, 393 (1941); P. CALDIROLA, Suppl. Nuovo Cimento **3**, 297 (1956); P. CALDIROLA, Lett. Nuovo Cimento **16**, 151 (1976); P. CALDIROLA and E. MONTALDI, Nuovo Cimento B **53** (291 (1979);

20. A. JANNUSSIS, A. LEODARIS, P. FILIPPAKIS AND T. FILIPPAKIS, Lett. Nuovo Cimento **29**, 437 (1980); A JANNUSSIS, G. N. BRODIMAS, V. PAPATHEOU and A. LEODARIS, Lett. Nuovo Cimento **31**, 533 (1981); A. JANNUSSIS, A. STRECLAS, A. LEODARIS, N. PATARGIAS, V. PAPATHEOU, P. FILIPPAKIS, V. ZISIS AND N. TSAGAS, Lett. Nuovo Cimento **34**, 571 (1982); A. JANNUSSIS, A. LEODARIS, P. FILIPPAKIS, and TH. FILIPPAKIS, Nuovo Cimento B **67**, 161 (1982); A. JANNUSSIS, G. N. BRODIMAS, V. PAPATHEOU A and A. LEODARIS, Lett. Nuovo Cimento **17**, 545 (1983); A. JANNUSSIS, G. N. BRODIMAS, V. PAPATRHEOU, G. KARAYANNIS, P. PANAGOPOULOS and H. IOANNIDOU, Lett.Nuovo Cimento **38**, 181 (1983); A. JANNUSSIS, Lett. Nuovo Cimento **42**, 129 (1985); A. JANNUSSIS, D. SKALTSAS and G. BRODIMAS, Hadronic J. **13**, 415(1990)

21. C. WOLF, Phys. Lett. A **123**, 208 (1987); Nuovo Cimento **100B**, 141 (1987), **103B**, 649 (1989) and **105B**, 805 (1990); Ann. Fond. de Broglie **15**, 189 (1990); Eur. J. Phys. **10**, 197 (1987); Hadronic J. **11**, 227 (1988), **13**, 22 (1990), **15**, 149 (1992) and in press

22. T. D. LEE, Phys. Lett. **122B**, 217 (1983); K. G. WILSON, Phys. Rev. **D10**, 2445 (1974); M. CREUTZ, Phys. Rev. D. **15**, 1128, D. FINKELSTEIN, Intern. J. Theor. Phys. **27**, 473 (1985)

23. A. JANNUSSIS, D. SKALTSAS and G. BRODIMAS, Hadronic J. **13**, 415 (1990)

24. M. V. BERRY, proc. Roy. Soc. , London, A 392, 45 (1984)

25. B. SIMON, Phys. rev. Lett. 51, 2167 (1983)

26. F. WILCZEK and A. ZEE, Phys. Rev. Lett. **52**, 2111 (1984); Z. BIALYNICKA–BIRULA , Phys. Rev. D **35**, 2383 (1987); M. S. SIRAM and V. SRINIVASAN, J. Phys. A **20**, L1071 (1987)

27. R. MIGNANI, in *Proceedings of the Fourth Workshop on Hadronic Mechanics and Nonpotential Interactions,* M. Mijatovic, Editor, Nova Science, Commack, N.Y. (1988); and in *Hadronic Mechanics and Nonpotential Interactions,* H. C. Myung, Editor, Part II, Nova Science, Commack, N.Y. (1993)

4: BASIC PHYSICAL LAWS

4.1: STATEMENT OF THE PROBLEM

In this chapter we shall identify the most salient physical laws of hadronic mechanics which are implied by the basic axioms of the preceding chapter.

In appraising the content of this chapter, the first point the reader should keep in mind is the difference in physical systems studied by quantum and hadronic mechanics. The corresponding disparity in physical laws is then intended to represent precisely those physical differences.

Quantum mechanics represents a system of particles of the *exterior dynamical problem*, that is, while moving in the homogeneous and isotropic vacuum at large mutual distances under only local-potential interactions. Hadronic mechanics represents instead particles of the *interior dynamical problem*, that is, while moving within the hyperdense inhomogeneous and anisotropic medium composed of the wavepackets of other particles (called *hadronic medium* [1]), thus experiencing local-potential as well as nonlocal-nonpotential interactions.

In the consideration of physical laws such as Heisenberg's uncertainties, Pauli's exclusion principle, etc., one should therefore keep in mind that, in the former case, particles are *unconstrained* in vacuum while, in the latter case, they are *constrained* by the surrounding wavepackets. The assumption for the latter conditions of exactly the same physical laws of the former conditions would imply the evident suppression of the physical differences to be described.

The second point the reader should keep in mind is that quantum mechanics is generally perceived as representing only *one* class of systems, the closed-isolated ones, thus resulting in only *one* set of laws.

By comparison, hadronic mechanics has been constructed with *two* branches to represent *two* classes of systems, the isotopic branch for the representation of systems with nonlocal-nonpotential internal forces as isolated from the rest of the Universe and time evolution

$$i\dot{A} = [A\,\hat{,}\,H] = A*H - H*A = A T H - H T A. \qquad (4.1.1)$$

and the genotopic branch for the representation of open–nonconservative–
irreversible systems of particles under external interactions with time evolution

$$i \dot{A} = (A, H) = A < H - H > A = A R H - H S A. \qquad (4.1.2)$$

It is then evident that *hadronic mechanics possesses two sets of physical
laws*. In fact, laws valid for the antisymmetric brackets [A,^H] cannot possibly
hold for the brackets (A, H) which are neither antisymmetric nor symmetric. It is
also evident that the laws of the latter admit the laws of the former as a
particular case.[35]

However, on formal grounds, *hadronic mechanics possesses only one set of
laws for particles in genospaces*, because all isotopic and quantum laws are
particular cases.

The third point the reader should keep in mind is that *quantum mechanics
possesses one set of laws for both particles and antiparticles, while hadronic
mechanics implies different, antiautomorphic laws for particles and
antiparticles*. This is due to the structural uniqueness of quantum mechanics
which therefore forces the adaptation of the polyhedric physical reality to the
theory at hand. On the contrary, hadronic mechanics has been built with a
multiple structure so as to adapt the theory to the physical reality considered.

A first, preliminary study of the basic laws of hadronic mechanics was
done in the original proposal of 1978 by Santilli [1] to build hadronic mechanics,
with particular reference to the inapplicability of Pauli's exclusion principle for
hadrons under *external* strong interactions. The generalized Heisenberg's
uncertainties were studied by Santilli in ref. [2] of 1981 and finalized in the recent
paper [3]. The hadronic formulation of the time–reversal invariance was studied
by the author in paper [4] of 1983 and that of parity in paper [5] of 1984.
Comprehensive studies on the isotopies and genotopies of Bosonic creation and
annihilation operators were conducted by Jannussis and his associates [6] as
reported in Appendix II.4.A. The isotopies of Fermionic creation and annihilation
operators and related Pauli's principle were studied by Aringazin in ref. [7] of
1990. The extension of these studies to the supersymmetric formulation of
hadronic mechanics was studied by Ntibashirakandi and Callebaut in the recent
paper [8] reviewed in App. II.4.B. No additional contribution has appeared in print
at this writing (early 1994) on the generalized physical laws, specifically, under
isofields and genofields, isospaces and genospaces, etc.

By contrast the *conventional* literature on possible generalizations of

---

[35] At a deeper study, the perception that quantum mechanics describes only closed–
isolated systems is erroneous because quantum mechanics also represents open
nonconservative systems, e.g., in nuclear physics, via a nonhermitean Hamiltonians and
related time evolution $i \dot{A} = (A, H, H^\dagger) = AH^\dagger - HA$. However, as mentioned earlier, the
latter formulation are afflicted by a host of problems of consistency.

quantum mechanics, those based on conventional fields, conventional Lie theory, conventional functional analysis, etc., is so vast to discourage even a partial outline. We here limit ourself to the sole quotation of Selleri's studies [9] and vast references listed therein, with particular reference to the celebrated *Einstein–Podolsky–Rosen argument* [10].

Intriguingly, as we shall see in this chapter, *hadronic mechanics can be interpreted as* a *"completion" of quantum mechanics essentially along the vision of Einstein, Podolsky, Rosen and others.* More particularly, the lifting from conventional eigenvalue equations to their covering isotopic (or genotopic) form with an *infinite set of different eigenvalues for the same operator* H (Sect. I.6.2)

$$ H \,|\, \psi > \; = \; E_0 \,|\, \hat{\psi} > \; \rightarrow \; H*|\, \hat{\psi} > \; = \; H \, T \,|\, \hat{\psi} > \; = \; E_T \,|\, \hat{\psi}>, \quad E_T \neq E_0, \quad (4.1.3) $$

constitutes the first explicit and concrete *realization of "hidden variables"*. In actuality, lifting (4.1.3) implies the realization of "hidden variables" in terms of *"hidden operators"* (see App.. II.4.C).

We see in this way that the isotopic operator T acquires the new meaning of expressing "hidden degrees of freedom" of quantum mechanics. In fact, as now familiar, the conventionally modular action $H|\, \psi >$ and its isotopic covering $H*|\, \hat{\psi} >$ coincide at the abstract level. In this sense, the latter is "hidden" in the abstract treatment of the former.

The isotopic methods permit further generalizations beyond the transition from "hidden variables" to "hidden operators". In fact, the topological structure of the "hidden operator" T is unrestricted by the theory, provided that it is positive-definite. In particular, its restriction to local–differential descriptions is grossly unwarranted, mathematically (because contrary to the natural admission of integral realizations) and physically (because contrary to the nonlocal conditions occurring in the physical reality).

Hadronic mechanics therefore permits the natural generalization of the current theory of *local realism* [9] into a covering theory submitted by the author [11] under the name of *isolinear–isolocal–isocanonical realism* also reviewed in App. II.4.C.

By no means these epistemological aspects are mere curiosity. On the contrary, they have fundamental physical implications. The best illustration is given by conventional studies on "parity violation in weak interactions". To be mathematically correct, such a statement must be completed by adding "under the simplest possible realization of fields, Hilbert spaces and eigenvalues equations", those of the form $H|\, \psi > \; = \; E|\, \psi >$, $E \in R(n,+,\times)$, $|\, \psi > \in \mathcal{H}$.

As we shall see, the assumption of more general *realizations* of the *same* axioms $H*|\, \hat{\psi} > \; = \; \hat{E}*|\, \hat{\psi} >$, $\hat{E} \in \hat{R}(\hat{n}<+*)$, $|\, \hat{\psi} > \in \mathcal{H}$, implies a realistic possibility of achieving an *exact parity symmetry under weak interactions*, evidently for closed–isolated conditions.

4.2: BASIC PHYSICAL LAWS OF THE LIE-ISOTOPIC BRANCH OF HADRONIC MECHANICS

4.2.A: Isotopies of Heisenberg's uncertainties and their isodualities. As well known, the conventional Heisenberg's uncertainties for coordinates/momenta, angular momentum components and energy/time (see, e.g., ref.s [12])

$$\Delta r^i \Delta p_j \gtrless \tfrac{1}{2} \hbar \delta_{ij}, \tag{4.2.1a}$$

$$\Delta L_i \Delta L_j \gtrless \tfrac{1}{2} \hbar \Delta L_k, \quad i < j < k, \qquad i, j, k = 1, 2, 3, \tag{4.2.1b}$$

$$\Delta E \Delta T \gtrless \tfrac{1}{2} \hbar, \tag{4.2.1c}$$

are true physical pillars of quantum mechanics. However, they were conceived and have been experimentally confirmed only for the physical conditions of the original conception, those of the *exterior dynamical problem* of particles in vacuum.

Doubts on the final character of the conventional formulation of these uncertainties date back to their very conception, the most notable being those by Einstein–Podolsky–Rosen [10] and other founders of quantum mechanics (see review [9] and quoted literature), although these doubts are all still referred to the validity of quantum uncertainties for the exterior dynamical problem in vacuum.

In these volumes we accept the *exact* validity of quantum mechanics and, therefore, of the quantum uncertainties in the exterior problem in vacuum, and study instead the broader physical conditions of the *interior dynamical problem* of particles within hadronic media.

The basic issue addressed in this section is therefore whether an electron moving in vacuum when a member of an atomic cloud has exactly the same uncertainties as when in the core of a collapsing star.

The belief that quantum mechanics holds everywhere for whatever physical conditions beyond those of its original conception and verification leads to the following contradiction first identified in ref. [3].

Consider an astrophysical body undergoing gravitational collapse. The center–of–mass of such object evidently verifies classical mechanics and, therefore, is fully determined.

Consider now an elementary particle in its interior and suppose that it keeps verifying the same uncertainties as in vacuum. At the limit of gravitational collapse of the body into a singularity, the following contradiction clearly emerges:

Paradox of Heisenberg's uncertainties for gravitational singularities [3]: *Particles in the interior of a star collapsed all the way into a geometric singularity, if assumed to obey quantum mechanics, possess uncertainties which are in contradiction with the deterministic character of that singularity.*

In fact, the astrophysical body is classical and its center-of-mass can therefore be determined exactly. But then particles in its interior cannot obey Heisenberg's certainties.

As we shall see in this section, the basic axioms of hadronic mechanics resolve the above paradox. The main idea is elementary and consists in the prediction that structurally different physical conditions reflect in predictably different uncertainties such to recover classical determinism at the limit of gravitational singularities.

In Vol. III we shall then point out that the current experimental evidence for interior conditions, even though preliminary, supports rather clearly the hadronic generalization of quantum uncertainties, but again, only in the *interior* conditions of their conception and physical applicability.

To begin, let us define the *isouncertainty* ΔA of a hadronic observable A via the expression

$$(\Delta \hat{A})^2 = (\Delta A \hat{1}) T (\Delta A \hat{1}) = (\Delta A)(\Delta A)\hat{1} = (\Delta A)^2 \hat{1} =$$

$$= \hat{1}\int \hat{\psi}^\dagger T (A - <A>)^2 T \hat{\psi} = \int \hat{\psi}^\dagger T (A - <A>) T (A - <A>) T \hat{\psi}, \qquad (4.2.2)$$

where: $\Delta \hat{A} = (\Delta A) \hat{1} \in \hat{R}(\hat{n},+,\times)$ is an isonumber while $\Delta A \in R(n,+,\times)$ is an ordinary number; we assume the isonormalization

$$< \hat{\psi} \hat{|} \hat{\psi} > = < \hat{\psi} | T | \hat{\psi} > \hat{1} = \hat{1}, \quad \text{or} \quad \int dv\, \hat{\psi}^\dagger T \hat{\psi} = 1; \qquad (4.2.3)$$

and set $\hbar = 1$ hereon.

The isotopic generalization of the familiar quantum mechanical procedure to derive Heisenberg's uncertainties [12], including the use of the *isoschwartz inequality for two (well behaved) functions f and g* (Sect. I.6.2)

$$(\int dv\, | f |^2) * (\int dv\, | g |^2) \geq |\int dv\, \bar{f} g |^2, \qquad (4.2.4)$$

yields the following result for two isoobservables A and B [3]

$$[(\Delta \hat{A}) * (\Delta \hat{B})]^2 = (\Delta A \hat{1}) T (\Delta A \hat{1}) T (\Delta B \hat{1}) T (\Delta B \hat{1}) = [(\Delta A)(\Delta B)]^2 \hat{1} \geq$$

$$\geq |\hat{1}\int dv\, \hat{\psi}^\dagger T (A - \hat{<} A \hat{>}) T (B - \hat{<} B \hat{>}) T \hat{\psi} |^2 \geq$$

$$\geq | \tfrac{1}{2} \hat{1} \int dv\, \hat{\psi}^\dagger T ((A - \hat{<}A\hat{>})\hat{,} (B - \hat{<} B \hat{>})) \hat{\psi} |^2 +$$

$$+ \, | \, \tfrac{1}{2} \widehat{1} \int dv \, \hat{\psi}^\dagger \, T [\, (A - \hat{<} A \hat{>}) \hat{,} \, (B - \hat{<} B \hat{>}) \,] \, T \hat{\psi} \, |^2 \, \geqq$$

$$\geqq \tfrac{1}{2} \widehat{1} \, | \int dv \, \hat{\psi}^\dagger \, T [\, A \hat{,} B] \, T \psi \, |^2 \, = \, \tfrac{1}{2} \widehat{1} \, | \int dv \, T [A \hat{,} B] \, T \hat{\psi} \, | \; ^2 \, , \qquad (4.2.5)$$

that is,

$$\Delta A \;\; \Delta B \; \geqq \; \tfrac{1}{2} \, | \int dv \, \hat{\psi}^\dagger \, T [A \hat{,} B] \, T \hat{\psi} \, | \; = \; \tfrac{1}{2} \, | \, \hat{<} [A \hat{,} B] \hat{>} \, | \, , \qquad (4.2.6)$$

where we have used the isoexpectation values of Axiom V.

Consider first the case of the position and *momentum* operators r, p in one dimension. Then, their isouncertainties are given by

$$\Delta r \;\; \Delta p \; \geqq \; \tfrac{1}{2} \, | \, < [r \hat{,} p] > \, | \, . \qquad (4.2.7)$$

From Eq. (II.2.3.9), the momentum operator has the realization (for isounits independent of r)

$$p * | \psi > \; = \quad p \, T \, | \psi > = \, - i \, \hbar \, \frac{\partial}{\partial r} | \psi >, \quad \hat{\hbar} = \hbar \widehat{1} \, , \; \hbar = 1, \qquad (4.2.8)$$

thus yielding the isocommutation rules for r–contravariant, r $= x^1$ (Axiom III)

$$[r \hat{,} p] \; = \; r \, T \, p \, - \, p \, T \, r \; = \; i \, \hat{\hbar} \; = \; i \, T^{-1} . \qquad (4.2.9)$$

The *isouncertainties for coordinates and momenta in one space dimension* are then given by

$$\Delta r \;\; \Delta p \; \geqq \; \tfrac{1}{2} \, | \, \hat{<} [r \hat{,} p] \hat{>} \, | \; = \; \tfrac{1}{2} \, | \, \hat{<} \hat{\hbar} \hat{>} \, | \; = \; \tfrac{1}{2} \, \hbar \, | \int dr \, \hat{\psi}^\dagger \, T \widehat{1} T \hat{\psi} \, | \; = \tfrac{1}{2} \, \hbar \, , \qquad (4.2.10)$$

and we have proved the following

Lemma 4.2.1 [3]: *The numerical value $\tfrac{1}{2}\hbar$ of Heisenberg's uncertainties in one space dimension for a contravariant coordinate and a covariant linear momentum remains invariant under isotopies of Class I, i.e., the following physical law*

$$\Delta r \;\; \Delta p \; \geqq \; \tfrac{1}{2} \, \hat{<} \hat{\hbar} \hat{>} \; = \; \tfrac{1}{2} \, \hbar \qquad (4.2.11)$$

is a true axiom of quantum mechanics because it persists identically for the covering hadronic mechanics.

The interpretation of isolaw (4.2.11) is the following. Locally, the

isouncertainties are given by

$$\Delta r \, \Delta p \Big|_{\text{locally}} \geq \tfrac{1}{2} \, \hat{\hbar}(t, \, p, \, \dot{p}, \, ...) \, , \tag{4.2.12}$$

and therefore exhibit an explicit dependence on local quantities. However, globally the isouncertainties are given by form (4.2.11), thus recovering conventional values. The lack of difference with quantum mechanics emerges from the fact that *the measured quantities are the isoexpectation values*, thus yielding $\tfrac{1}{2} \langle \hat{\hbar} \rangle \equiv \tfrac{1}{2} \hbar$.

The reader familiar with the *functional isoanalysis* of hadronic mechanics (Ch. 1.6) known that the local behaviour (4.2.12) is *necessary* from the isofourier transforms of a wavefunction $\psi(r)$ into its conjugate $\hat{\psi}(p)$, see Sect. 1.6.6, Eq.s (1.6.6.19).

The case in more than one dimension is different because it implies the appearance of "hidden" degrees of isotopic freedom evidently absent in quantum mechanics. Consider in this respect one particle in three–dimensional isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{\Re})$ with co*variant* momentum components p_k and co*ntravariant* coordinates r^k or co*variant* coordinates $r_k = \hat{\delta}_{ki} r^i$, k, i = 1, 2, 3. Assume further that the isounit of Class 1 is diagonalized

$$\hat{1} = \text{diag.} (\hat{1}_{11}, \hat{1}_{22}, \hat{1}_{33}) = \text{diag.} (T_{11}^{-1}, T_{22}^{-1}, T_{33}^{-1}) \, , \tag{4.2.13}$$

which is always possible owing to its positive–definiteness (see Vol. III for numerous explicit examples of the diagonal elements).

Then the non–null fundamental isocommutation rules are

$$[r^{i} \hat{,} p_j] = i \, \hbar \hat{1}^{i}_{j} \, , \tag{4.2.14a}$$

$$[r_i \hat{,} p_j] = i \, \hbar \, \delta_{ij}, \quad ri = \delta_{ij} \, r^j = T_{ii} \, r^i \; (\text{no sum}) \, . \tag{4.2.14b}$$

We reach in this way the following

Lemma 4.2.2 [3]: *The isouncertainties in three–dimensions are given by*

$$\Delta r^k \, \Delta p_k \geq \tfrac{1}{2} | \int dv \, \hat{\psi}^{\dagger} \, T \, [r^{k} \hat{,} p_k] \, T \, \hat{\psi} = \tfrac{1}{2} \hbar | \int dv \, \hat{\psi}^{\dagger} \, T \hat{1}^{k}_{k} \, T \, \hat{\psi} \neq \tfrac{1}{2} \hbar \, , \tag{4.2.15a}$$

$$\Delta r_k \, \Delta p_k \geq \tfrac{1}{2} | \int dv \, \hat{\psi}^{\dagger} \, T \, [r_k \hat{,} p_k] \, T \, \hat{\psi} = \tfrac{1}{2} \hbar | \int dv \, \hat{\psi}^{\dagger} \, T \, T \, \hat{\psi} \neq \tfrac{1}{2} \hbar \, , \tag{4.2.15b}$$

$$\int dv \, \hat{\psi}^{\dagger} \, T \, \hat{\psi} = 1. \tag{4.2.15c}$$

and they do not generally coincide with the conventional uncertainties of quantum mechanics.

The analysis of this section therefore confirms a fundamental implication of hadronic mechanics, that *the isotopies of Planck's constant* $h \rightarrow \hbar$ *imply a corresponding, necessary isotopy of Heisenberg's uncertainty principle.*

Note that such generalization also exist in one dimension, provided that the coordinate r is assumed to be covariant.

We now study the isouncertainties at the limit of gravitational collapse all the way into a singularity. Consider in this respect the space-component (only) of the interior problem of any gravitational theory on a conventional Riemannian space $\Re(r,g,R)$ reformulated into the equivalent isoeuclidean form on $\hat{E}(x,\hat{\delta},\hat{\Re})$ of Sect. 1.3.7,

$$R(r,g,R) \approx \hat{E}(r,\hat{\delta},\hat{R}), \quad g(r) = T(r)\,\delta \equiv \hat{\delta}, \quad \hat{1} = [\,T(r)\,]^{-1}. \tag{4.2.16}$$

As a specific example, one can assume the space component of the familiar *Schwartzchild's line element*

$$ds^2 = (1 - 2M/r)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\;d\phi^2), \tag{4.2.17}$$

for which

$$T = \text{diag.}\,\{\,(1 - 2M/r)^{-1}\,,\,r^2\,,\,r^2\sin^2\theta\,\}. \tag{4.2.18}$$

It is evident that, at the limit of gravitational collapse all the way to a singularity at $r = 0$, the above isotopic element becomes null and the isounit becomes singular

$$\text{Lim}_{\,r\to 0}\,T \;\equiv\; 0\,, \quad \text{Lim}_{\,r\to 0}\,\hat{1} \;\equiv\; \infty\,, \tag{4.2.19}$$

namely, hadronic mechanics assumes the singular character of Kadeisvili Class IV (see Ch. 11.9 for details).

But then at that limit (and at that limit only), all isouncertainties are identically null,

$$\text{Lim}_{T\to 0}\;\Delta A\;\Delta B \geq \text{Lim}_{T\to 0}\;\tfrac{1}{2}\,|\textstyle\int dv\;\hat{\psi}^\dagger\,T\,[A \,\hat{,}\,B]\,T\,\hat{\psi}\,|\;=0\,, \tag{4.2.20}$$

thus resolving the paradox of Heisenberg's uncertainties presented earlier in this section. At a deeper inspection, the above results are due to the collapse of the basic isoeigenvalues into the null value,

$$\text{Lim}_{T\to 0}\,H*|\hat{\psi}> \;=\; \text{Lim}_{T\to 0}\,H\,T\,|\hat{\psi}> \;\equiv 0, \tag{4.2.21}$$

with consequential collapse of all isoexpectation values

$$\text{Lim}_{T\to 0} \hat{<} A \hat{>} = \text{Lim}_{T\to 0} <\hat{\psi} | T A T | \hat{\psi}> \equiv 0 . \qquad (4.2.22)$$

The recovering of classical determinism via the isoexpectation value of a singular isounit $\hat{h} \to \infty$ is then due to the mechanism

$$\text{Lim}_{T\to 0, \hat{1}\to\infty} \hat{<} \hat{1} \hat{>} = \text{Lim}_{T\to 0, \hat{1}\to\infty} <\hat{\psi} | T \hat{1} T | \hat{\psi}> \equiv 0 , \qquad (4.2.23)$$

and we have the following nonrelativistic property

Lemma 4.2.3 [3]: *The hadronic isouncertainties recover classical determinism at the limit of gravitational collapse into a singularity.*

The interested reader may study the full space–time extension of the above results following the isorelativistic theories studied later on inCh.s II.8 and II.9.

Evidently, the difference between the isotopic and the conventional uncertainties must be verified via experiments. We assume the reader is familiar with the fact that none of the currently available experimental verifications of Heisenberg's uncertainties is applicable to isouncertainties (4.2.15), trivially, because all the former apply in vacuum, while the latter require the immersion of the particle within a hadronic medium. As we shall see in Vol. III, new physical conditions must be identified for genuine tests of the isouncertainties.

Deviations from quantum mechanical uncertainties are more transparent in the case of angular momentum components

$$L_k = \epsilon_{kij} r^i p_j , \qquad \hat{L}_k = \epsilon_{kij} r^i p_j , \qquad (4.2.24)$$

whose definition remains unchanged in hadronic mechanics.[36]

As we shall study in more details in Ch. II.6, isoeigenvalue (4.2.8) and isocommutators (4.2.9) for diagonal isounits (4.2.13) imply the following rules

$$[L_i \hat{,} L_j] = L_i T L_j - L_j T L_i = i \epsilon_{ijk} \hat{1}_{kk} L_k \text{ (no sum)} , \qquad (4.2.25a)$$

$$[\hat{L}_i \hat{,} \hat{L}_j] = \hat{L}_i T \hat{L}_j - \hat{L}_j T \hat{L}_i = i \epsilon_{ijk} \hat{L}_k, \qquad (4.2.25b)$$

and we have the following

---

[36] Recall from Ch. I.3 that the basis of a vector space remains unchanged under isotopies. Therefore, the basis of the rotational algebra, the angular momentum components, remain unchanged under isotopies as it must be because it represent physical quantities. Note also that the reinterpretation of the r-components as isonumbers does not alter the definition because $L_k {}^*| \hat{\psi}> = \epsilon_{kij} \hat{r}_i {}^*p_j {}^*| \hat{\psi}> \equiv (\epsilon_{kij} r_i p_j) {}^* || \hat{\psi}>.$

Lemma 4.2.4: *The isouncertainties for the hadronic angular momentum components are given by*

$$\Delta L_i\, \Delta L_j \;\geq\; \tfrac{1}{2}\,|\,\hat{<}\,[\,L_i\,\hat{,}\,L_j\,]\,\hat{>}\,| \;=\; \tfrac{1}{2}\,\hbar\,|\,\hat{<}\hat{1}_{kk}\,L_k\,\hat{>}\,| \;\neq\; \tfrac{1}{2}\,|<L_k>|\,. \quad (4.2.26a)$$

$$\Delta\hat{L}_i\, \Delta\hat{L}_j \;\geq\; \tfrac{1}{2}\,|\,\hat{<}\,[\,\hat{L}_i\,\hat{,}\,\hat{L}_j\,]\,\hat{>}\,| \;=\; \tfrac{1}{2}\,\hbar\,|\,\hat{<}\,\hat{L}_k\,\hat{>}\,|\,. \quad (4.2.26b)$$

The interpretation of the above result is elementary. In the conventional quantum case, particles are free to orbit because they move in vacuum. In the covering hadronic case particles cannot freely orbit because they move within a physical medium. The deviations from the conventional uncertainties in Eq.s (4.2.26a) therefore represent the deviations from free motion (see Ch. 11.6 for more details). The preservation of the conventional uncertainties for Eq.s (4.2.16b) is only apparent because the quantities representing the non conservation of the orbit are contained in the covariant expressions $r_k = T_{kk}r^k$ (no sum). In different terms, the *physical* linear momentum is the quantity L, while $\hat{L}$ has a mere *mathematical* character.

To put it explicitly, deviations from conventional uncertainties in the angular momentum are necessary to prevent approximations of the type of the perpetual motion within a physical environment, e.g., that a proton orbits in the core of a star with conserved angular momentum. In fact, as we shall see, the factor $\hat{1}_{kk}$ represents precisely the nonconservation of the k–component of the angular momentum, e.g., for $\hat{1}_{kk} = \exp(-\gamma t)$.

The derivation of the uncertainties of the energy is also simple. Recall from Axiom I that the energy isooperator and related isocommutation rules are given by

$$E = i\hbar\hat{1}_t\, \partial\,/\,\partial t\,, \quad [\,E\,\hat{,}\,T\,] = E\,T\,t - t\,T\,E = i\hbar\hat{1}_t\,, \quad (4.2.27)$$

where $\hat{1}_t \neq 1$ is the isounit of time. We therefore have the following

Lemma 4.2.5: *The isouncertainties for the energy and time are given by*

$$\Delta E\ \Delta T \;\geq\; \tfrac{1}{2}\,\hbar\,|\,\hat{<}\hat{T}_t\hat{>}\,| \;\neq\; \tfrac{1}{2}\,\hbar\,. \quad (4.2.28)$$

We therefore confirm the expected property that *the change in the unit of time implies a corresponding change in the uncertainties* energy/time. This latter aspect can be more effectively studied within relativistic settings where we have the four vectors $X = \{x, c_0 t\}$ and $P = \{p, E/c_0\}$, c_0 being the speed of light in vacuum, with related four–dimensional isounit $\hat{1} = \mathrm{diag.}\ (\hat{1}_{11}, \hat{1}_{22}, \hat{1}_{33}, \hat{1}_{44})$. The necessary alteration of the uncertainties in energy/time can then be derived from space–time symmetrization of the space components even *in the absence of a redefinition of the unit of time.*

The *isodual isouncertainties* ΔA^d are defined on an isodual field of real numbers $R^d(n^d+,\times d)$ and, as such, they are *negative-definite*. It is then easy to prove the following

Lemma 4.2.6: *The isodual isouncertainties of nonrelativistic hadronic mechanics with diagonal isounit are given by*

$$\Delta r^{id} \times^d \Delta p_j{}^d \;\leq\; -\tfrac{1}{2}\,\hbar\,|\;\hat{\langle} 1^j{}_j \hat{\rangle}\,|\,, \tag{4.2.29a}$$

$$\Delta L_i{}^d \times^d \Delta L_j \;\leq\; -\tfrac{1}{2}\,|\;\hat{\langle} 1_{kk}\,L_k\,\hat{\rangle}\,|\,,\qquad i<j<k\,, \tag{4.2.29b}$$

$$\Delta E^d \times^d \Delta T^d \;\leq\; -\tfrac{1}{2}\,\hbar\,|\;\hat{\langle} 1_t{}^d \hat{\rangle}\,|\;\neq\; -\tfrac{1}{2}\hbar\,. \tag{4.2.29c}$$

The interested reader can work out additional aspects on uncertainties.

As a closing comment we would like to indicate that, following ref. [2], numerous generalizations of quantum uncertainties have been introduced in the literature, such as the so-called *squeezed states* and others (see Appendix 4.E). However the latter generalizations, on one side, are based on a structural generalization of quantum mechanics while, on the other side, preserve the quantum mechanical formalism unchanged, thus resulting in a host of problematic aspects in regards to form-invariance under the time evolution, lack of preservation of Hermiticity-observability at all times, loss of causality, etc.

The generalized uncertainties studied in this chapter avoid these problems precisely because based on a structural generalization of the *formalism* of quantum mechanics which is compatible, by construction, with the generalized uncertainties. These results are evident from the role of the isounit for the isouncertainty, its isohermiticity and its invariance at all times.

4.2.B: Isotopies of the superposition principle and their isodualities. Another important principle of quantum mechanics, which originates from the linearity of the theory of operator on Hilbert spaces, is the *linear superposition principle* [12]. It essentially states that, if a system can assume any of the states ψ_k, $k = 1, 2, ..., n$, then the following linear superposition of said states is also an admissible state

$$\Psi = \sum_k c_k\,\psi_k\,,\qquad c \in C\,, \tag{4.2.30}$$

under normalization and statistical weighs

$$\int \Psi^\dagger \Psi\,dv = 1\,,\qquad \sum_k |c_k|^2 = 1\,. \tag{4.2.31}$$

Now, it is well known that theories which are nonlinear in the wavefunctions and Hamiltonian of the type $H(t, r, p, \psi, \psi^\dagger)$ are afflicted by a chain

of problematic aspects, such as violation of the above superposition principle, lack of preservation of the Hermiticity of the Hamiltonian at all times, loss of causality, etc.[37]

We have stressed throughout our presentation that one of the objectives of hadronic mechanics (in its general form with isounits depending on local coordinates of Ch. II.8) is to represent the most general known nonlinear dependence in the wavefunctions $\hat\psi$ and $\hat\psi^\dagger$ and their derivatives of arbitrary order $\partial\hat\psi$, $\partial\hat\psi^\dagger$, $\partial\partial\hat\psi$, $\partial\partial\hat\psi^\dagger$, etc.

This is inherent in the theory of isolinear operators on isohilbert spaces (Sect,. I.6.3) and simply permitted by the factorization in the Hamiltonian of all nonlinear terms in the wavefunctions and their representation into the isotopic element

$$H(t, r, p, \hat\psi, \hat\psi^\dagger, \partial\hat\psi, \partial\hat\psi^\dagger, ...) = H_0(t, r, p)\, T(\hat\psi, \hat\psi^\dagger, \partial\hat\psi, \partial\hat\psi^\dagger, ...), \quad (4.2.32)$$

with equation

$$i\, 1_t \frac{\partial}{\partial t}\,\hat\psi(t, r) = H * \hat\psi(t, r) \equiv H_0\, T(\hat\psi, \hat\psi^\dagger, \partial\hat\psi, \partial\hat\psi^\dagger, ...)\,\hat\psi(t, r). \quad (4.2.33)$$

It is remarkable that, despite such a general nonlinearity, hadronic mechanics does admit the following generalized principle of easy verification:

Lemma 4.2.7 (Isosuperposition principle): *If $\hat\psi_k$, k = 1, 2, ..., n, is a collection of possible isostates as solution of the isoscrödinger's equation, then the isotopic sum*

$$\hat\Psi = \sum_k \hat c_k * \hat\psi_k \equiv \sum c_k \hat\psi_k, \quad \hat c_k \in \hat C(\hat c, +, *), \quad c_k \in C(c, +, \times), \quad (4.2.34a)$$

$$\int \hat\Psi^\dagger * \hat\Psi\, dv = I, \qquad \sum_k |c_k|^2 = I, \quad (4.2.34b)$$

is also a possible isostate.

This important result is a consequence of the notion of *isolinearity* of Sect. I.4.2, namely, the capability of turning all possible nonlinear systems into an *identical* form verifying the abstract axiom of linearity in isospace. This also illustrates the "direct universality" of hadronic mechanics of Sect. I.7.9.

Note that factorization (4.2.32) must also be used for the effective Hamiltonian with a logarithmic term appearing after the naive isoquantization, Eq. (II.2.3.4b). In fact, the preservation of such a nonlinear terms in the

---

[37] The systems considered are generally open-nonconservative and their time evolution is therefore generally nonunitary. The problematic aspects here referred to then follows from a simple application of non unitary transformations.

Hamiltonian would violate the isosuperposition principle. This essentially implies the use of the conventional Hamiltonian and a redefinition of the isotopic element to incorporate said nonlinear terms,

$$H^{eff} (t, r, p, \hat{\psi}, \hat{\psi}^\dagger, ...) \, \hat{T}(t, r, p, \hat{\psi}, \hat{\psi}^\dagger, ... = H(t, r, p) \, \hat{T}(t, r, p, \hat{\psi}, \hat{\psi}^\dagger, ...) . \quad (4.2.35)$$

Note finally that, in most practical applications, the isotopic elements can be averaged into constants, by providing an approximate yet significant and experimentally verifiable representation of the nonlinearity in the wavefunctions and in their derivatives. Even in this simplest possible treatment, nonlinearities in the wavefunctions have non trivial implications, such as a shift in the eigenvalues, departures from conventional laws, etc.

An example is here in order. Consider the *isoplanewaves* (II.3.2.37)

$$\hat{\psi}_p(t, r) = \hat{N} * \hat{e}^{i(p\,r} * \hat{e}_t^{-\, t\,E)} = N \, e^{i(p\,T\,r - t\,T_t\,E)} . \quad (4.2.36)$$

To reach an expression of the normalization factor N, we must use the *isodelta functions* of Sect. I.6.4, the case considered being continuous. We therefore have

$$\int \hat{\psi}_p^\dagger * \hat{\psi}_{p'} \, dr = N^2 \int \{\hat{e}^{-i(p\,r - \, t\,E)}\} * \{\hat{e}^{\, i(p'\,r - t\,E)}\} \, d\,r =$$

$$= N^2 \int \hat{e}^{-i\,(p - p')\,r} \, d\,r = N^2 \, \hat{\delta}_1(p - p'), \quad (4.2.37)$$

where the usual integration range has been ignored for simplicity, $\hat{e}$ is the isoexponentiation, and $\hat{\delta}$ is the isodirac delta of the first kind. The isonormalization constant N therefore preserves the conventional value

$$N = (2\,\pi)^{-3/2} . \quad (4.2.38)$$

The isosuperposition principle can then be illustrated as follows. Consider the *isofourier transform of the first type* of Sect. I.6.6. Then, we can write

$$\hat{\psi}(t, r) = (2\pi)^{-3} \, T \int_{-\infty}^{+\infty} d^3p \, \hat{\phi}(t, p) \, \hat{e}^{\, i\,p\,r} , \quad (4.2.39)$$

with inverse

$$\hat{\phi}(t, p) = (2\pi)^{-3} \, T \int_{-\infty}^{+\infty} d^3r \, \hat{\psi}(t, r) \, \hat{e}^{-\, i\,p\,r}. \quad (4.2.40)$$

It is then possible to prove the identities

$$\int_{-\infty}^{+\infty} |\hat{\psi}(t, r)|^2 \, d^3r = \int_{-\infty}^{+\infty} |\hat{\phi}(t, p)|^2 \, d^3p = 1 . \quad (4.2.41)$$

The probability densities for coordinates and momenta are therefore given respectively by

$$\hat{w}(t, r) = |\hat{\psi}_2(t, r)|^2, \qquad \hat{w}(t, p) = |\hat{\phi}_2(t, p)|^2, \qquad (4.2.42)$$

in full agreement with isoprinciple (4.2.34).

In conclusion, we have learned in this section that theories of the contemporary literature which are nonlinear in the wavefunctions and which were consequently believed to violate the superposition principle with a number of consequential problematic aspects, can be turned into an *identical* isotopic form which verifies the principle in an axiomatic way.

The isodual formulation of the isosuperposition principle is suggested for study by the interested reader.

4.2.C: Aringazin's isotopies of Pauli's exclusion principle and their isodualities. We now study the isotopies of another fundamental law of quantum mechanics, *Pauli's exclusion principle* [12] according to which two or more electrons (or, more generally, Fermions) in the same state cannot have the same quantum numbers.

The emerging covering principle, here called *isoexclusion principle*, was first investigated by Aringazin [7] via the isotopies of the Fermionic creation and annihilation operators. The isoprinciple is important because it shows that the conventional exclusion principle persists for a closed–isolated hadronic state of *two* particles and for the hadronic restricted case of *three* particles and provides, apparently for the first time, an explanation of its physical origin precisely in the nonlocal–nonhamiltonian interactions.

However, as we shall see, Pauli's principle no longer holds in general for a system of *four* or more particles in hadronic conditions.

Recall that quantum mechanics cannot treat (in first quantization) the interaction among the electrons of a given atomic orbit because this would imply adding a potential to the Hamiltonian with consequential gamma–exchanges that would de–stabilize the state. The consequence is essentially that electrons can <exclude> the properties of each other without interactions of any type. This consequence implies an evident uneasiness which has persisted for generations (see Selleri's studies in this respect [9]).

A possible solution of the issue rests precisely in the alternatives local–Hamiltonian vs nonlocal–nonhamiltonian interactions. Hadronic mechanics "completes" quantum mechanics via contact interactions due to mutual wave–overlapping which admit no meaningful representation with the Hamiltonian and are represented instead by the isotopic element T or isounit $\hat{1} = T^{-1}$. In particular, the interactions due to mutual wave–overlappings are precisely the interactions experienced by the electrons in the same orbit of Pauli's principle. Hadronic mechanics therefore offers a possibility for a quantitative understanding of the origin of Pauli's principle.

Consider the case of two electrons in the same state. In its simplest possible

formulation, the conventional Pauli principle can be expressed via the Fermionic creation operator a and annihilation operator $a^\dagger$ characterized by the following anticommutation rules on a conventionally associative envelope ξ over a conventionally modular Hilbert space $\mathcal{H}$ with normalized basis $|b>$

$$\{a, a^\dagger\} = a\,a^\dagger + a^\dagger a = I, \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0. \tag{4.2.43}$$

Then, the occupation number operator $N = aa^\dagger$ has the eigenvalue $n \in R(n,+,\times)$

$$N|b> = a\,a^\dagger|b> = n|b>, \tag{4.2.44}$$

which, from properties (4.2.43), implies

$$[N, a] = Na - aN = -a, \quad [N, a^\dagger] = a^\dagger. \tag{4.2.45}$$

Then it can be easily proved that the only possible eigenvalue are $n = \pm 1$,

$$N^2|b> = NN|b> = n|b>, \quad n = \pm 1, \quad <b|b> = 1. \tag{4.2.46}$$

By following Aringazin [7], the isotopies of the preceding formulation can be done as follows. Consider the Lie–isotopic branch of hadronic mechanics of Class 1. Its *isofermionic creation operator* $\hat{a}$ and the *annihilation operator* $\hat{a}^\dagger$ on an isoassociative envelope $\hat{\xi}$ acting on an isomodular Hilbert space $\hat{\mathcal{H}}$ with isonormalized isobasis $|\hat{b}>$ and common isotopic element T, can be defined by the following *isoanticommutation rules*

$$\{\hat{a}\,\hat{,}\,\hat{a}^\dagger\} = \hat{a} * \hat{a}^\dagger + \hat{a}^\dagger * \hat{a} = \hat{a}\,T\,\hat{a}^\dagger + \hat{a}^\dagger\,T\,\hat{a} = \hat{1} = T^{-1}, \tag{4.2.47a}$$

$$\{\hat{a}\,\hat{,}\,\hat{a}\} = \{\hat{a}^\dagger\hat{,}\hat{a}^\dagger\} = 0. \tag{4.2.47b}$$

Note the isotopy of Eq.s (4.2.43) into (4.2.47a) which ensures their local isomorphism by construction because of the lifting $1 > 0 \rightarrow \hat{1} > 0$.

Next, Aringazin [loc. cit.] introduces the *isofermionic isooccupation number* $\hat{N} = \hat{a} * \hat{a}^\dagger$ with isoeigenvalue $\hat{n}_T \in \hat{R}(\hat{n},+,*)$ depending on the "hidden", operator degree of freedom T,

$$\hat{N} * |\hat{b}> = \hat{a} * \hat{a}^\dagger * |\hat{b}> = \hat{n}_T * |\hat{b}> \equiv n_T|\hat{b}>, \quad ,\hat{b}|T\hat{b}> = 1. \tag{4.2.48}$$

By using Eq.s (4.2.47), it is easy to prove the following isotopy of Eq.s (4.2.45)

$$[\hat{N}\hat{,}\hat{a}] = \hat{N} * \hat{a} - \hat{a} * \hat{N} = -a, \quad [\hat{N}\hat{,}\hat{a}^\dagger] = \hat{a}^\dagger. \tag{4.2.49}$$

The isotopy of the final equations (4.2.46) then follows,

$$\hat{N}^2 * | \hat{b} > \; = \; \hat{N} * \hat{N} * | \hat{b} > \; = \; \hat{n}_T * | \hat{b} > \; = \; n_T | \hat{b} > \,, \qquad (4.2.50)$$

i.e., $n_T = \pm 1$ for all infinitely possible isotopic element T, yielding the result:

Lemma 4.2.8 (Isoexclusion principle for two particles [7]): *Two spinorial particles in the same stable orbit of a closed-isolated isostate cannot have the same characteristic numbers exactly as in conventional quantum mechanics*

It should be indicated for completeness that, as proved by Aringazin [loc. cit.], the above result holds also when the isotopic element T of the isofield and of isoenvelope is different than the isotopic element G of the isohilbert space.

The extension to more than two spinors in the same hadronic state is mathematically elementary, but physically nontrivial. Mathematically, the isostate of three particles with individual states $|\hat{N}_k >$, with quantum isonumbers $\hat{N}_k$, $k = 1, 2, 3$, is given by the isotensorial products of the type $|\hat{N}_i >* | \hat{N}_j >* | \hat{N}_k >$, $i \neq j \neq k$. The totally antisymmetric isostate $|\hat{N}_i , \hat{N}_j , \hat{N}_k >_{(-)}$ is then constructed exactly as in the conventional case [12]. It is then easy to see that

$$| \hat{N}_i , \hat{N}_j , \hat{N}_k >_{(-)} \; = \; 0 \quad \text{for any two equal isoeigenvalues}. \qquad (4.2.52)$$

However, this property *does not* necessarily imply that Pauli's principle holds also for hadronic states with three or more particles. The reasons are of physical nature and concern the very notion of Fermion in hadronic mechanics. As we shall see in Ch.s II.8 and II.10, the conventional spin $\frac{1}{2}$ is indeed preserved for a bound state of two particles under nonlinear–nonlocal–nonhamiltonian. However, for more general particles the interactions considered imply a deformation, called *isorenormalization*, of all the intrinsic characteristic of particles, including spin. This alters the conventional physical interpretation of rules (4.2.52).

In other words, the technical issues which must be addressed as a necessary pre–requisite for a quantitative study of the isotopies of Pauli's principle for an arbitrary number of particles is the isotopy of the notion of spin or, more generally, of spinor. Evidently, such an issue requires the study of the isorepresentation of $S\hat{U}(2)$ and of the isotopies of Dirac equation studied later on in this volume.

As well known, a fundamental property of the conventional Pauli's principle is that *two electrons can only be in the same orbit in a singlet state*, evidently because the triplet state is forbidden by the principle.

As we shall see, this property becomes fundamental for all two–body hadronic states and applicable not only to Fermions but also to Bosons. In fact, mutual penetration and wave–overlapping at short distances render triplet

couplings highly unstable irrespective of whether the particles are Fermions or Bosons. In turn, this property is important for the novel predictions of hadronic mechanics studied in Vol. III.

The isodualities of Lemma 4.2.8 are straightforward and left to the interested reader.

4.2.D: Isotopies of causality and their isodualities. One of the most serious criticisms moved against nonlocal interactions is that they violate the fundamental principle of quantum causality, i.e., that the effect must follow the cause in particle events [12].

One of the most important physical applications of hadronic mechanics is that of permitting a fully causal treatment of nonlocal interactions. This is due to the notion of *isolocality* of Sect. 1.4.2, which permits the *transformation of all possible nonlocal interactions into an "identical" form verifying the axioms of locality in isospace*. As now familiar, this is achieved by embedding all nonlocal-integral terms in the isotopic element T.

Technically, the preservation of causality under nonlocal interactions is permitted by the *realization of (connected) Lie–isotopic groups via isounitary operators*. In fact, the conventional causality of quantum mechanics is a manifestation of the unitary time evolution of a quantum state [12]

$$| t > = U(t, t_p) | t_o >,$$ (4.2.53)

with familiar Lie group properties

$$U(t, t) = I, \qquad U(t', t) \, U(t, t_o) = U(t, t_o) \, U(t, t_o) = U(t', t_o),$$ (4.2.54a)

$$U(t, t_o) \, U(t_o, t) = I.$$ (4.2.54b)

However, the above structure is local–differential, as well known. *The very essence of the Lie–isotopic theory (Vol. 1) is that of generalizing the conventional Lie formulation into the most general possible, nonlinear and nonlocal form which, being axiom–preserving, also preserved the principle of causality.*

In fact, the isotopic lifting of time evolution (4.2.53) is given by

$$| t > = \hat{U}(t, t_o) * | t_o > = \hat{U}(t, t_o) \, \hat{T}(t, p, \dot{p}, \hat{\psi}, \partial\hat{\psi}, ...) | t_o >,$$ (4.2.55)

and now ensures the direct representation of nonlocal interactions embedded in the isotopic element, while its causality is ensured by the rules for a Lie–isotopic group in their isounitary realization

$$\hat{U}(t, t) = \hat{1}, \qquad \hat{U}(t', t) * \hat{U}(t, t_o) = \hat{U}(t, t_o) * \hat{U}(t', t) = \hat{U}t', t_o),$$ (4.2.56a)

$$\hat{U}(t, t_o) * \hat{U}(t_o, t) = \hat{1}.$$

(4.2.56b)

The study of further causality aspects of hadronic mechanics (e.g., the intriguing connection between analyticity of the hadronic S–matrix) is left to the interested reader.

The study of causality under isoduality is also left to the interested reader, with the understanding that a theory is causal in the latter case when the effect *precedes* the causes in our space time.

4.2.E: Isotopies of the measurement theory and their isodualities. As well known, *quantum measurements require the necessary existence and prior identification of the basic unit.* The basic unit of quantum mechanics is the trivial value 1. It is important to understand that *the basic unit of the measurement theory of hadronic mechanics, called "isomeasurement theory" remains the trivial number 1.*

This is due to the fact that, following Axioms V, the isoexpectation value of the generalized unit $\hat{1}$ is 1,

$$\hat{\langle} \hat{1} \hat{\rangle} = \frac{\langle \hat{\psi} | * \hat{1} * | \hat{\psi} \rangle}{\langle \hat{\psi} | * | \hat{\psi} \rangle} = 1.$$

(4.2.57)

The reader is then encouraged to verify that all various aspects of the quantum mechanical theory of measurements admit consistent isotopies as well as isodualities.

A necessary pre–requisite for the above consistency is that the theory admits *an associative enveloping algebra with a correct left and right unit.* In fact, such a property is at the foundation of the uniqueness of the exponentiation and consequential invariance of the unit under the time evolution.

As now familiar, the above properties are fully verified by hadronic mechanics. However, all generalizations of quantum mechanics which do not admit a unique and consistent left and right unit in the enveloping operator algebra cannot possibly admit a consistent measurement theory, that is, they cannot be applied to real physical events. This is the case of nonlinear theories of the type H(t, r, p, ψ, $\psi^\dagger$, ...),[38] nonlinear theories of Weinberg's type,[39] q–deformations[40], and others.

---

[38] Because their time evolution is nonunitary, and, as such, do not preserve the basic unit of the theory, $U \, 1 \, U^\dagger \neq 1$.

[39] Because their envelopes are *neral Lie–admissible nonassociative algebras* (App. 1.7.9B) which, as such, do not admit any unit at all, whether left or right.

[40] Because they deform the associative product AB → A*B = qAB but preserve the old unit 1 of AB which, as such, is no longer the correct left and right unit of the theory and

4.2.F: Isotopies of time-reversal and their isodualities. A visual observation of Jupiter reveals a majestic realization of the following dichotomy:

> Exact validity of the time–reversal invariance for the center–of–mass trajectory of the planet; vs

> The manifest irreversibility of the interior dynamics, such as in vortices with nonconserved angular momenta.

The corresponding occurrence in particle physics can be seen in the neutron stars for which the time–reversal invariance of their center of mass is evident. Equally evident is their irreversible interior dynamics.

ORIGIN OF IRREVERSIBILITY

Jupiter or a neutron star
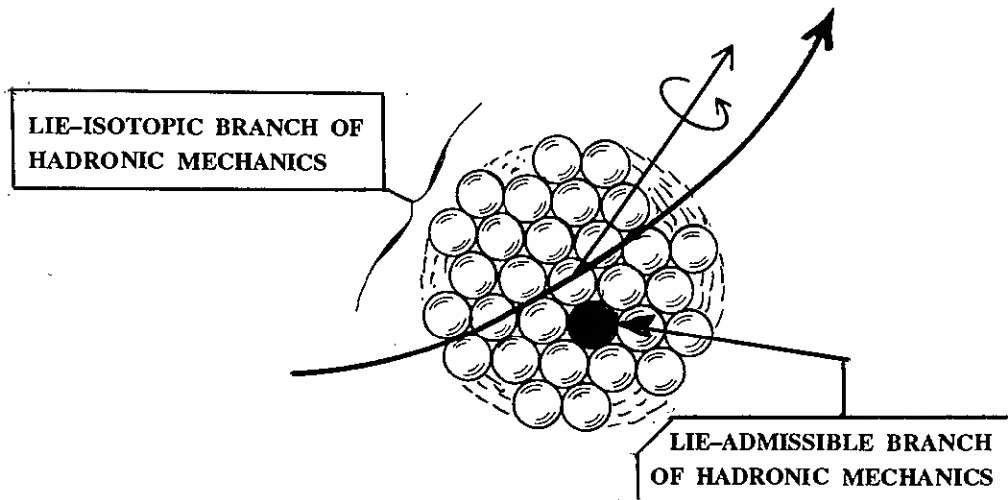
FIGURE 4.2.F.1: Quantum mechanics is structurally reversible, i.e., reversible for reversible Hamiltonians. This occurrence has left the problem of the origin of macroscopic irreversibility essentially unresolved. Hadronic mechanics alters this historical problem by identifying the origin of irreversibility in the ultimate structure of matter, and then deriving macroscopic irreversibility as a mere consequence. This novel approach submitted by Santilli in [4] is permitted by the Lie–admissible branch of hadronic mechanics which represents open–nonconservative interior conditions in a structurally irreversible way, that is, in a way which is irreversible irrespective of whether the Hamiltonian is reversible or not. This origin of irreversibility is also mandated by clear insufficiencies of quantum mechanics identified by the *No–Reduction Theorems* of Ch. 1.1 according to which a macroscopic object in interior conditions (such as a space–ship during re–entry) *with monotonically decaying irreversible orbit,* simply cannot be reduced to a finite collection of quantum mechanical particles *each*

cannot possibly be preserved in time.

in stable-reversible conditions. The physical origin of this inconsistency is the absence in quantum mechanics of the interactions experienced by the space-ship, the contact, zero-range, nonlocal-integral forces due to motion of an extended object within a physical medium. Hadronic mechanics restores precisely the same forces at the particle level to such an extend that the forces acting on a space-ship during re-entry and those acting on a proton moving in the core of a star are analytically equivalent. These forces are then implemented in a dual way, first via the Lie-isotopic branch to recover the time-reversal invariance of the center-of-mass trajectories of isolated systems and, second, for open-nonconservative interior conditions to represent the irreversibility of physical reality.

A quantitative representation of this dichotomy is one of the reasons which stimulated the construction of hadronic mechanics [I] with the dual branches:

> The Lie-isotopic branch with its structural reversibility and

> The Lie-admissible branch, with its structural irreversibility.

It is a rather general *theoretical* and *experimental belief* at this time that "strong interactions are time-reversal invariant". This statement is correct if completed with the sentence *"in their center-of-mass treatment"*, in which case the property needs no experimental verification because established already at the classical level for all closed-isolated systems (Figure 4.2.F. I).

The point is that all orthodox experiments on time-reversal invariance available until now (see Volume III) are conducted in the center-of-mass of the system. In the transition to the *open-nonconservative* conditions suggested in ref. [I], we have such a structurally different setting to render inapplicable the experimental results for the center-of-mass [4].

The validity of the time-reversal invariance is known to depend on the validity of the *theorem of detailed balancing* [12]. In turn, such a theorem centrally depends on the *unitary* character of the time evolution. Now in the center-of-mass, such time evolution is indeed unitary because of the conservation of the energy, thus implying the validity of the theorem of detailed balancing and a consequential exact time-reflection symmetry.

The isotopic extension of this setting implies no new occurrence. In fact, the isounitary time evolution law implies the validity of an isotopic form of the theorem of detailed balancing, yet time-reversal invariance persists in its entirety as it is the case for all other space-time and internal symmetries (see next subsection for parity, Ch.s II.6, II.7 and II.8 for space-time symmetries and Vol. III for internal symmetries).

The *time-reversal isooperator* $\hat{\tau}$ can be defined by

$$\hat{\tau}^{\dagger} * \hat{\psi}(t, r) * \hat{\tau} = \hat{\psi}(-t, r) . \tag{4.2.58}$$

Consider the *closed* reaction $a + A \rightarrow b + B$ in its center-of-mass consisting of a polarized beam of nucleons a with spin s in interaction with

nuclei A, say, of a target, which are unpolarized and of spin s_A. Let $b + B \rightarrow a + A$ be the time reversal reaction. Let A_j be the analyzing power of the forward reaction and P_j the polarization of the backward reaction. It is then easy to see the existence of the following

Lemma 4.2.9: *The hadronic principle of isodetailed balance is given by*

$$(1 + \sum_j P_j * A_j)^{-1} (k_i/k_j) 2 (2s_A + 1) X_{if} =$$

$$= (1 + \sum_j P_j * A_j)^{-1} (k_i/k_j) 2 (2s_B + 1) X_{if} , \qquad (4.2.59a)$$

$$\frac{\text{Analyzing power of forward reaction}}{\text{Polarization backward reaction}} = \frac{1}{1} = 1, \qquad (4.2.59b)$$

where X_{ij} are the elements of the transition matrix and the k's are wave numbers.

We reach in this way the following important

Theorem 4.2.1 [5]: *The center-of-mass trajectory of a closed-isolated system of particles with nonlinear-nonlocal-nonhamiltonian internal interactions verifies the time-reversal symmetry when represented with the isotopic branch of the hadronic mechanics because the analyzing power of the forward reaction equals the polarization of the backward reaction.*

However, when we consider only one (or more) strongly interacting particle and assume all others as *external*, then the time evolution of that particle is generally *nonunitary*, because of the lack of conservation of its energy and the general instability of its orbit, thus implying the predictable time-asymmetry treated via the Lie-admissible approach as the origin of irreversibility (see Sect. II.4.3.F).

The isodualities of the time-reversal invariance imply no structural novelty, with the understanding that, in this latter case, the map is from our past into our future time.

4.2.G: Isotopies of space-reversal and their isodualities. An important property studied in these volumes is that *the Lie-isotopic theory permits the reconstruction of exact space-time and internal symmetries when believed to be broken.*

In Vol. 1 we have shown that the isotopies restore the exact rotational symmetry when believed to be broken by ellipsoidical deformations of the sphere. In this volume we shall show that the isotopies restore the exact Lorentz

symmetry when believed to be broken by deformations of the Minkowski metric expected in the interior of hadrons (see Ch. II.8). In Vol. III we shall show that the isotopies restore the exact isospin symmetry in nuclear physics under weak and electromagnetic interactions.

In a scenario of this type it then becomes inevitable to ask the question *whether hadronic mechanics can reconstruct the exact space-reflection symmetry for <weak> interactions.*

The issue has fundamental character because it is directly related to the basic assumption of hadronic mechanics, the generalization of Planck's unit $\hbar \rightarrow \hat{\hbar}$. Currently available experimental data in weak interactions, such as

$$\frac{\text{Rate}\,(\,K^{\circ}_{L} \rightarrow e^{+} + \pi^{-} + \nu\,)}{\text{Rate}\,(\,K^{\circ}_{L} \rightarrow e^{-} + \pi^{+} + \nu\,)} = 1.00648 \pm 0.00035 \qquad (4.2.60)$$

are certainly interpreted in a correct way when assumed to characterize parity violation within the context of conventional quantum theories.

The issue addressed in this subsection is whether measures (4.2.60) can also be interpreted as indicating *the possible need for a different unit for the weak interactions, that is, a generalized Planck's constant* $\hat{\hbar}$. In fact, the notion of parity violation for measures (4.2.60) is centrally dependent on the (generally tacit) assumption of the trivial unit $\hbar = 1$. However, if a different unit is consequently assumed, then parity could be restored as an exact symmetry.

This problem was studied by Santilli in paper [5], but no additional studies appeared thereafter. With the understanding that the issue is essentially open at this writing, the "young mind of all ages" indicated in the Preface may be intrigued by a brief presentation of the problem. It is understood that the final resolution of the issue can only be searched within the context of the isotopies of quantum field theories which are beyond the scope of these volumes.

The mechanism of *isotopic reconstruction of exact symmetries* is rather simple. Suppose that a given Hamiltonian $H_0 = K + V$ has a quantum mechanical Lie symmetry G,

$$G\,H_0\,G^{\dagger} \equiv H_0\,. \qquad (4.2.61)$$

Suppose now that, because of certain physical reasons, the Hamiltonian H_0 must be implemented into the form $H = H_0 + V' = K + V + V'$. If the new potential V' breaks the symmetry G

$$G\,H\,G^{\dagger} \neq H\,, \qquad (4.2.62)$$

it is rather universally believed nowadays that "the system violates the symmetry G". This is precisely the case of the current view that "weak interactions violate parity".

Hadronic mechanics establishes that these views are not generally correct. The tacit assumption of this belief is that all possible interactions are of Hamiltonian type, which is evidently not true at both the classical and operator levels.

In fact, given the original system H_0 in the Euclidean space $E(r,\delta,R)$, one can select the isotopic element T in such a way that the equations of motion of the system represented by $H = H_0 + V'$ in $E(r,\delta,R)$ are *identical* (and not equivalent) to the equations of motion of H_0 in the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, $\hat{R} \approx R\,\hat{1}$, $\hat{1} = T^{-1}$.

It is therefore generally possible to embed the totality of the symmetry breaking term V' in the isotopic element T of the theory.

If the above conditions are met and T is of Class 1, then the original symmetry is exact because it is now computed at the isotopic level $\hat{G}$ and results to be locally isomorphic to the original symmetry G. As we shall see in details in the subsequent chapters, the symmetry transformations (rotation, Lorentz, etc.) are irreconcilably lost, but the symmetry remains exact in its axioms, and merely realized in a nonlinear–nonlocal–noncanonical way.

Theorem 4.2.2 (Reconstruction of exact Lie symmetries [5]): *Consider a quantum mechanical system represented by the Hamiltonian H_0 in the Euclidean space $E(r,\delta,R)$, and suppose that it possesses an exact Lie symmetry G. Suppose that the original G–invariant system H_0 must be implemented with a potential V' which violates the original symmetry in $E(r,\delta,R)$*

$$G H_0 G^\dagger \equiv H_0 \rightarrow G H G^\dagger = G(H_0 + V')G^\dagger \neq H.$$ (4.2.63)

Then, under sufficient continuity, boundedness and regularity conditions, there always exists an isotopy of Kadeisvili Class I of the original field, Euclidean space, Hilbert space and symmetry G with the same positive–definite isotopic element T

$$F(n,+,\times) \rightarrow \hat{F}(\hat{n},+,*); \quad E(r,\delta,R) \rightarrow \hat{E}(r,\delta,\hat{R}), \; \mathcal{H} \rightarrow \hat{\mathcal{H}}, \; G \rightarrow \hat{G}, \quad (4.2.64)$$

such that the old Hamiltonian H_0 written on $\hat{\mathcal{H}}$ over $\hat{E}(r,\delta,\hat{R})$ and denoted $\hat{H}_0$ characterizes exactly the same system as that described by H on $\mathcal{H}$ over $E(r,\delta,R)$ under which the original symmetry is restored as being exact because $\hat{G}$ leaves invariant $\hat{H}_0$ by construction and $\hat{G} \approx G$.

Proof. The original systems $H| >$ can always be identically written in the isotopic form $\hat{H}_0*| > = \hat{H}_0 T| >$ for the isotopic element

$$T = H_0^{-1} H,$$ (4.2.65)

Then, the original symmetry G always admits the isotope $\hat{G}$ constructed with respect to to the new unit $\hat{1} = T^{-1}$ which leaves invariant the original Hamiltonian by construction,

$$\hat{G} * \hat{H}_0 * \hat{G}^\dagger \equiv \hat{H}_0 . \tag{4.2.66}$$

The local isomorphism $G \approx \hat{G}$ follows from the fact that both Hamiltonians H_0 and H are positive–definite and, therefore, the isotopic element $T = H_0^{-1} H$ is positive–definite. **q.e.d.**

As indicated earlier, we shall have ample opportunities to illustrate this theorem with continuous space–time and internal symmetries. In this section we would like to present a tentative study of the application of Theorem 4.2.2 to parity.

The *isotopic space–reversal operator*, or *isoparity operator*, denoted $\hat{\pi}$, in the Lie–isotopic branch of hadronic mechanics of Class 1 is defined by

$$\hat{\pi} * r * \hat{\pi}^\dagger = - r, \tag{4.2. 67a}$$

$$\hat{\pi} * p * \hat{\pi}^\dagger = - p , \tag{4.2.67b}$$

$$\hat{\pi} * J * \hat{\pi}^\dagger = J , \tag{4.2.67c}$$

where r, p and J are the position, momentum and angular momentum operators, respectively, under the isounitarity condition

$$\hat{\pi} * \hat{\pi}^\dagger = \hat{\pi}^\dagger * \hat{\pi} = \hat{h} = \hat{1} , \quad \hat{h} = 1 , \tag{4.2.68}$$

and the additional condition

$$\hat{\pi}^2 = \hat{\pi} * \hat{\pi} = \hat{1} . \tag{4.2.69}$$

In most practical cases (although not necessarily all), the isoparity operator can be expressed via the simple factorization

$$\hat{\pi} = \pi \hat{1} , \tag{4.2.70}$$

where π is the conventional parity operator, in which case the above properties are trivial because reduced to the conventional ones, e.g.,

$$\hat{\pi} * r * \hat{\pi}^\dagger \equiv \pi r \pi^\dagger = - r , \quad \text{etc.} \tag{4.2.71}$$

Consider now a π-invariant Hamiltonian H_O,

$$\pi\, H_O\, \pi^\dagger \;\equiv\; H\,, \qquad H_O \;=\; K(p) + V(r) \;=\; p^2/2m \;+\; V(r)\,, \qquad (4.2.72)$$

on a conventional Hilbert space $\mathcal{K}$ with eigenvalues E_O,

$$H_O\,|\psi> \;=\; E_O\,|\psi>\,. \qquad\qquad (4.2.73)$$

Suppose that a parity violating (PV) potential V_{PV} is added to H_O resulting in the new system

$$\pi\, H\, \pi^\dagger \;=\; \pi\,(\,H_O + w\,V_{pv}\,)\,\pi^\dagger \;\neq\; H\,, \qquad H\,|\psi> \;=\; E\,|\psi>\,, \qquad (4.2.74)$$

where w is a coupling constant.

It is then evident that, at least on formal grounds under sufficient topological conditions (boundedness, continuity, regularity, etc.), an isotopy *always* exist for which

$$H\,|\psi'> \;\equiv\; \hat{H}_O\, T\,|\psi'> \;=\; \hat{H}_O *|\psi'> \;=\; E\,|\psi'>\,. \qquad (4.2.75a)$$

$$\hat{H}_O \;=\; K_{\hat{E}} + V_{\hat{E}} \;=\; p * p\,/\,2m \;+\; \hat{V}(r), \qquad\qquad (4.2.75b)$$

and $\hat{V}$ means that the potential is now computed in the isospace, with expression such as $r^2 = r\,T\,r$, etc. The formal solution[41]

$$T \;=\; \hat{H}_O^{-1}\,H\,, \qquad\qquad (4.2.76)$$

then implies the reconstruction of the exact parity invariance, because the parity violating law (4.2.74) is turned into the parity exact form[42]

---

[41] It should be recalled from Vol. I that, while the Hamiltonian must necessarily be written in the correct form in isospace, this is not the case for the isotopic element T and isounit $\hat{1}$ because they are ou*tside* the original fields, spaces, etc.

[42] We assume the reader is now familiar with the fact that isooperator on an isospace do not act on the isotopic element which, as such, is left unrestricted. Even though the isotopic element $T = H^{-1}\,H_O$ is manifestly *noninvariant* under parity, the isokinetic energy is indeed invariant because

$$\hat{\pi} * p^2 * \hat{\pi}^\dagger \;=\; \hat{\pi} * p * p * \hat{\pi}^\dagger \;=\; (-p)*(-p) \;\equiv\; p * p \;=\; p^2\,.$$

As a matter of fact, this point illustrates the mechanism of isotopic reconstruction of exact symmetries when conventionally broken. Note also that, while p*p is invariant under isoparity, *it is not invariant under the conventional parity,*

$$\hat{\pi} * \hat{H}_0 * \hat{\pi}^\dagger \equiv \hat{H}_0 . \tag{4.2.77}$$

Ref. [5] provided the following approximate expression for the isotopic element

$$T = [I + \tfrac{1}{2} w \{ H_0^{-1}, V^{PV} \}]^{\tfrac{1}{2}} \approx I + (w/4) \{ H_0^{-1}, V^{PV} \}, \tag{4.2.78}$$

where the curl brackets represent conventional anticommutators.

The reconstruction of parity can then be completed via the factorization of the parity–violating term Q_{pv} in the states according to the rule

$$|\psi'> = Q_{pv} |\psi_0 >, \tag{4.2.79}$$

and the assumption of the isohilbert space

$$\mathcal{H}_G : \; <\psi_0 \hat{T} \Phi_0 > \; = \; <\psi_{p0} | G | \phi_0 > \; \hat{T} \in \hat{C}(\hat{c},+,*), \tag{4.2.80a}$$

$$G = T Q_{pv}, \quad \hat{T} = T^{-1} . \tag{4.2.80b}$$

This completes the review of paper [5]. As indicated earlier, the final resolution of the issue requires the construction of the *hadronic field theory* which is unavailable at this writing.

The study of the exact *isodual isoparity symmetry* for antiparticles is an instructive exercise for the interested reader.

The interested reader can also study a number of additional isotopies of basic quantum mechanics laws, principles and insights along the lines of the preceding sections.

4.3: BASIC PHYSICAL LAWS OF THE LIE-ADMISSIBLE BRANCH OF HADRONIC MECHANICS

Quantum mechanics was theoretically and experimentally established via the study of a particle under *external electromagnetic interactions*. In fact, the celebrated Dirac equation for the hydrogen atom *does not* represent the two-body system electron–proton, but represents instead the electron under the *external* field of the proton.

A main viewpoint attempted to convey with the original proposal [1] to

$$\pi p^2 \pi^\dagger = \pi p * p \pi^\dagger = (-p) \pi T \pi\dagger (-p) \neq p^2$$

evidently because T is not invariant under parity by assumption.

build hadronic mechanics is that structural advances in strong interactions require a similar approach, that is, the study of one particle under *external strong interactions*, such as a proton in the core of a star considered as external. In fact, the isotopies of the Dirac equation studied later on in Ch. 11.10 are aimed precisely at the representation of one particle under external electromagnetic and strong interactions.

A fundamental physical difference occurs in the transition from electromagnetic to strong interactions when considered as external. The orbits of individual particles under external electromagnetic interactions are *stable*, thus preserving the symmetries of the system as a whole. For instance, the Lorentz symmetry applies not only to the hydrogen atom as a close-isolated system, but also to each of its constituents.

On the contrary, the orbits of particles under external strong interactions are *unstable* in the general case, as it is evidently the case for the proton in the core of a star, thus implying the general loss of the symmetry of the system as a whole when considered as closed-isolated. In particular, this loss is interpreted in these volumes as the origin of irreversibility in nature.

The only exceptions to the above general rules known to this author are the two-body and the restricted three-body with nonlocal-nonhamiltonian internal forces in which each orbit must be individually stable because of certain dynamical conditions from the global stability. This evidently includes quark orbits for *individual* mesons and baryons, but *not* the orbits of a generic quark (or those in a meson) when in the core of a star.

In this section we shall study *the general physical laws* of a *particle under external strong interactions.* Their specialization to the two-body and three-body case will be studied in Vol. III. The main result is that

Symmetries of external strong interactions. *With the exception of the two-body and restricted three-body cases, when strong interactions are realized with a nonlinear-nonlocal-nonhamiltonian component due to mutual wave-overlapping and considered as external, they generally violate <all> Lie and Lie-isotopic symmetries and conservation laws of the same system completed into a closed-isolated form, and verify instead covering Lie-admissible symmetries for the characterization of time-rate-of-variations of physical quantities.*

It is at this point that the physical implications of the nonpotential-nonhamiltonian character of the strong interactions begin to emerge. The contemporary theory of strong interactions is based on the conjecture that they are entirely derivable from a potential. Under this assumption the orbits of the individual particles remain stable under external interactions, irrespective of whether electromagnetic or strong.

The loss of conventional space-time symmetries and physical laws under

external strong interactions occurs only when they possess a *nonhamiltonian* or *nonlagrangian component.*

It is at this point that the complementarity between the Lie–isotopic and the Lie–admissible theories emerge in their full light. In fact, such a complementarity ensures that a closed–system can verify the *Lorentz–isotopic symmetry* as a whole in a way fully compatible with the verification of the *Lorentz–admissible symmetry* by the individual constituents (see Ch. II.8).

4.3.A: Genotopies of Heisenberg's uncertainties and their isoduals. A particle under external nonhamiltonian strong interactions is in generally open, nonconservative and irreversible conditions. As such, its study requires *four different uncertainties,* two for motion forward in past and future times <> and two for motion backward in past and future times $<>^d$ (Sect. II.3.3).

It is easy to see that both the conventional and isotopic uncertainties are inapplicable to all of them. Consider, for instance, the forward direction to future times >. The *forward genouncertainties* $\Delta A^>$ of a genooperator A are then defined as in Eq. (4.2.2) with the replacement of the isotopic element T with the forward genotopic element $S^>$. Properties (4.2.5) then formally hold, by reaching the general result

Lemma 4.3.1: *The forward genouncertainties* $\Delta A^>$ *and* $\Delta B^>$ *of two hadronic operators* A *and* B *with Lie–admissible brackets*

$$(A, B) = A < B - B > A = A R B - B S A, \tag{4.3.1}$$

are given by

$$\Delta A^> \Delta B^> \geq \tfrac{1}{2} \left| \int dv \, \hat{\psi}^\dagger S^> (A S^> B - B S^> A) T^> \hat{\psi} \right.$$

$$= \tfrac{1}{2} \left| \int dv \, \hat{\psi}^\dagger > [A \hat{,} B]^> > \psi \right| = \tfrac{1}{2} \left| \hat{<} [A\hat{,}B]^> \hat{>} \right|. \tag{4.3.2}$$

In different terms, the *Lie–isotopic* content ASB – BSA of the product ARB – BSA persists in the uncertainties. But the applicable algebra is now Lie–admissible. Therefore, none of the physical results of Sect. 4.2.A applies. As an example, consider the case of coordinate and momentum in one dimension verifying the forward Lie–admissible rule

$$r < p - p > r = i \hat{h}^> = i 1^>, \quad \hbar = 1. \tag{4.3.3}$$

To compute the genouncertainties, one must compute the forward Lie–isotopic rule attached to the above Lie–admissible form for each case at hand

$$r > p - p > r = i Q^> \neq i 1^>. \tag{4.3.4}$$

The forward genouncertainties are then given by

$$\Delta r^> \, \Delta p^> \; \geqq \; \tfrac{1}{2} \, | \, \hat{<} \, Q^> \, \hat{>} \, | \; \neq \; \tfrac{1}{2} \, . \tag{4.3.5}$$

By recall that the forward genoexpectation value of the forward genounit $1^>$ is I (Axiom $^<V^>$), the inapplicability of the result of Sect. 4.2.A for the one dimensional case then follows from the difference in the Lie–admissible rule (4.3.3) and Lie–isotopic ones, i.e., from $1^> \neq Q^>$ (see App. II.4.E for examples).

A similar situation occurs for all other uncertainties. Note that the above genouncertainties can be *greater* or *smaller* than the quantum mechanical ones, depending on the local physical conditions of the medium in which the particle is immersed.

The isodual genouncertainties are then uniquely determined by isoduality.

4.3.B: Genotopies of the superposition principle and their isodualities. It is easy to see that the isosuperposition principle of Sect. 4.2.B persists in its entirety under genotopies and their isodualities and we have the following

Lemma 4.3.2 (Genosuperposition principle): *If $\hat{\psi}_K^>$ is a set of forward genostates solution of the forward genoschrödinger equation, then the genotopic sum*

$$\hat{\Psi}^> = \textstyle\sum_k \hat{c}^> > \hat{\psi}_K^> \, , \qquad \hat{c}^> \in \hat{C}^>(\hat{c}^>,+,>) \, , \tag{4.3.6}$$

is also a solution of the same equation.

Similar results evidently occur in the remaining three cases.

4.3.C: Genotopies of Pauli's principle and their isodualities. It is easy to see that *two particles in the same state under external nonhamiltonian strong interactions do not obey Pauli's exclusion principle* as first studied in ref. [1], whether in its conventional or isotopic form of Sect. II.4.2.B.

This is evidently due to the fact that the "Fermionic" creation and annihilation operator now obey the Lie–admissible rules

$$a < a^\dagger + a^\dagger > a = i^{<}1^> . \tag{4.3.7}$$

The genoeigenvalues $n^>$ are then no longer restricted to the ± 1, as one can verify.

This is a crucial point of proposal [1] recommended for experimental verification, that is, the expected general inapplicability of Pauli's exclusion principle under *external* strong interactions which will be considered again in Vol. III.

4.3.D: Genotopies of causality and their isoduals. It is easy to see that genotopic formulations are causal in each of their four possible directions in time. This is a consequence of the fact that each of these directions forms a Lie-admissible group which, as one recalls from Ch. 1.7, is composed of two different *Lie-isotopic* groups for multiplication to the right and to the left interconnected by Hermiticity.

In fact, the time evolution of a forward genostate $|t_0^> >$ is given by

$$|t^> > = \hat{U}^>(t, t_0) > |t_0^> >,$$ (4.3.8)

where $\hat{U}^>(t, t_0)$ verifies rules (4.2.56) for a Lie-isotopic group in the forward direction toward future time. Causality then follows.

A similar situation occurs for the remaining directions in time.

4.3.E: Genotopies of the Measurement theory and their isodualities. It is also easy to see that the measurement theory admits a consistent genotopic formulation in which the basic unit remains the conventional one.

This is due to the property from Axiom $^<V^>$ that the genoexpectation value of the forward genounit is the conventional value 1,

$$<\hat{1}^> \hat{>} = \frac{<\hat{\psi}|>\hat{1}^> > |\hat{\psi}>}{<\hat{\psi}|>|\hat{\psi}>} = 1.$$ (4.3.9)

The invariance of the genounit $\hat{1}^>$ at all times can also be easily proved. The full applicability of a measurement theory then follows as in the conventional quantum case.

A similar situation then occurs for the remaining directions in time. The above property is important because it establishes that the *nonconservation* of genohermitean operators such as the energy can indeed be measured. In turn, the possibilities of measuring time-rate-of-variations of physical quantities have fundamental experimental implications for novel tests studied in Vol. III.

4.3.F: Genotopies of time-reversal and their isodualities. The reader familiar with the preceding studies can evidently expect that *hadrons under external strong interactions generally violate the time-reversal symmetry.*

The above property is inherent in the very structure of the Lie-admissible equations

$$i \dot{A} = A < H - H > A,$$ (4.3.10)

which are *time-reversal noninvariant irrespective of whether the Hamiltonian is time-reversal invariant or not* (Figure 4.2.F.1) This property indicates the need of scientific caution before stating that the symmetries of a Hamiltonian are the symmetries of the system.

The above irreversibility was established by Santilli in ref. [4] as follows.

Consider the *open* reaction a + A → b + B consisting of a polarized beam of nucleons a with spin s in interaction with *external* nuclei A of a fixed target, which are unpolarized and of spin s_A. Let b + B → a + A   be the time reversal *open* reaction.

Since the physical conditions are open–nonconservative by assumption, we have the applicability of the Lie-admissible formulations. We therefore have two *geno-time–reversal operators* $\hat{T}^>$, one for the reaction forward to future time and the other for in past time, and their isoduals $\hat{T}^{>d}$ interconnected by genohermiticity, with rules of the type

$$\hat{T}^> > \hat{\psi}(t, r) <^< \hat{T} = \psi(-t, r), \tag{4.3.11}$$

Similarly, we have four analyzing powers $^<A_j^>$ and $^<A_j^{>d}$, four polarizations $^<P_i^>$ and $^<P_i^{>d}$, etc.

Suppose now that the interactions considered is of sufficiently low energy to ensure the preservation of the conventional spin s, conventional wave numbers k, etc. (i.e., all intrinsic quantities are Lie and only the kinematical quantities are Lie–admissible).

Then we have the following

Lemma 4.3.3: *The genotopies of the principle of detailed balancing are given by*

$$(1^> + \textstyle\sum_j P^>_j > A^>_j)^{-1} (k_i/k_j) 2 (2s_A + 1) X^>_{if} =$$

$$= (^<1 + \textstyle\sum_j {}^<P_j < {}^<A_j)^{-1} (k_i/k_j) 2 (2s_B + 1) {}^<X_{if} . \tag{4.3.12}$$

The difference

$$1^> \neq {}^<1 , \tag{4.3.13}$$

alone is then sufficient to establish the general irreversibility of open strong interactions.

Additional simple, but tedious calculations yield the following

Theorem 4.3.1 (Irreversibility of open strong interactions [3]): *The ratio between the analyzing power* $A^>_j$ *for the open forward reaction of nucleons* a *interacting with external nuclei* A *and the polarization* $^<P_j$ *of the open backward reaction of hadrons* b *interacting with external nuclei* B, *is equal to the ratio between the genounits of the corresponding forward and back reactions*

$$\frac{A^>}{^<P_j} = \frac{1^>}{^<1} , \tag{4.3.14}$$

|

The irreversibility of the system then follows from the lack of unit value of such ratio, as in Eq. (II.4.2.59b).

Note the full recovering of the exact time reversal invariant for the Lie–isotopic branch of hadronic mechanics (Sect. II.4.2.F) for which we have the particular case

$$ 1^> \equiv {}^< 1 \equiv \hat{1} \equiv \hat{1}^\dagger . \tag{4.3.15} $$

This is the reason why it is at times best to compute properties at the covering Lie–admissible level because it automatically yields as particular cases the simpler Lie–isotopic and Lie subcases.

Note that there is no need of experiments to verify irreversibility under the assumed conditions. In fact, the validity of the theorem of detailed balancing requires the *unitarity* of the time evolution. The same theorem is also known to be violated under *nonunitary* time evolutions which are precisely those under consideration here. Experiments are only needed to measure the *numerical value* of the ratio $A^> / {}^< P$ for given open physical conditions, that is, to measure the amount of irreversibility for the characteristics at hand.

Note also that such tests would provide a direct measure of the ratio of the genounits $1^> / {}^< 1$, which is evidently of fundamental character for hadronic mechanics because measuring specific deviation from quantum mechanics (see Vol. III).

The study of the isodual genotopy is left to the interested reader.

4.3.G: Genotopies of space-reversal and their isodualities. It is an instructive exercise for the interested reader to prove the following:

Theorem 4.3.2: *Given a closed-nonhamiltonian system of particles whose center-of-mass treatment verifies the isoparity symmetry, the individual particles generally violate the same symmetry .*

The above occurrence is evidently the space–counterpart of the time property of the preceding section.

The result also indicates the need of scientific caution in stating either the preservation or the violation of parity. Hadronic mechanics permits the reconstruction of the exact parity for weak interactions *under the clearly stated conditions that they are treated in their center-of-mass.* In fact, parity is violated for part of the same system when considering the rest as external.

Numerous additional physical laws and properties of quantum mechanics can be subjected to Lie–admissible generalization, but their study is left to the interested reader for brevity. We here merely quote the studies by: Nishioka [14] on a Lie–admissible treatment of Yukawa nonlocal theories; Jannussis, Brodimas and Mignani [15] on a Lie–admissible treatment of quantum groups; Veljanoski

and Jannussis [16] on a Lie–admissible; treatment of probabilities for open systems; Jannussis and Mignani [17] on a Lie–admissibie treatment of open systems; Jannussis et al. [18] on a Lie–admissibie formulation of the density matrix; and other quoted therein.[43]

APPENDIX 4.A: ISOTOPIES AND GENOTOPIES OF BOSONIC ALGEBRAS

As It is well known, the conventional, quantum mechanical *algebra of Bosonic creation and annihilation operators*, or *Bosonic algebra* for short (see, e.g., ref. [12])

$$a\,a^\dagger - a^\dagger a = 1 = \text{diag. } (1,\,1,\,1,\,\dots),\qquad (4.A.1)$$

possesses an *axiomatic structure*, i.e., a structure derivable from primitive quantum mechanical axioms which implies: the form–invariance under the time evolution of the theory, the unitary transforms

$$U\,U^\dagger = U^\dagger U = 1,\qquad a' = U\,a\,U^\dagger,\,a'^\dagger = U\,a^\dagger\,U^\dagger,\qquad (4.A.2a)$$

$$U\,(a\,a^\dagger - a^\dagger a)\,U^\dagger = a'\,a'^\dagger - a'^\dagger a' = U\,1\,U^\dagger = 1.\qquad (4.A.2b)$$

the Hermiticity–observability of the *occupation number operator* $N = a^\dagger a$; the preservation of such Hermiticity–observability at all times; the existence of a left and right unit 1 in the enveloping algebra with consequential applicability of the measurements theory; etc. In short, the axiomatic structure of the conventional Bosonic algebra ensures its consistent physical applications at all times.

Following the initiation back in 1967 by this author [19] of the jointly Lie–admissible and Jordan admissible, (p, q)–deformations of associative algebras back in 1967 (see also App. II.3.C)

$$(a,b) = p\,a\,a^\dagger - q\,a^\dagger a,\qquad (4.A.2)$$

a large number of (p = 1) q–deformations of the Bosonic algebra have appeared in the literature (see, e.g., ref.s [20] and papers quoted therein), such as:

$$a\,a^\dagger - q\,a^\dagger a = I,\qquad a\,a^\dagger - f(q)\,a^\dagger a = 1,\qquad (4.A.3a)$$

$$a\,a^\dagger - a^\dagger a = f(q)\,1,\qquad \lambda(q)\,a\,a^\dagger - \mu(q)\,a^\dagger a = 1,\qquad (4.A.3b)$$

---

[43] Note that these contributions are generally based on conventional fields, vector and Hilbert spaces thus requiring their reformulation on the more recent genofields, genovectors and genohilbert spaces to achieve an axiomatic structure. in accordance with hadronic mechanics.

q-deformations implying the Hopf algebra, combination of conventionally quantized coordinates and momenta and deformed creation-annihilation operators, and others.

It is well known (see, e.g., Lopez [21]) that deformations (4.A.3) are *noncanonical*, thus implying that their time evolution is *nonunitary*. In turn, the nonunitarity of the time evolution implies the following rather serious problematic aspects of the above deformations which prevent any realistic possibility of physical application:

1) *Lack of form-invariance under their own time evolution*, as one can easily verify (see also below);

2) *General loss of the Hermiticity-observability of the number operator* N = a†a *under the time evolution of the theory because of Lopez's Lemma II.3.C.1;*

3) *Loss of the measurement theory*, due to the preservation of the original unit I but its lack of invariance under the time evolution, $UIU^† = \hat{1} \neq I$;

4) *Lack of uniqueness of generalized physical laws*, due to the lack of uniqueness of exponentiation and other operations;

5) *Loss of the special functions and operations under time evolution;*
and others problematic aspects.

These are the reasons why this author abandoned his original approach (4.E.1) in favor of the more adequate formulations of hadronic mechanics. In this appendix we show that, *hadronic mechanics permits an axiomatic reformulation of operator-deformed Bosonic algebras*, that is, a reformulation which preserves all properties of the conventional quantum formulation.

The greatest number of contributions on Bosonic algebras of Lie-isotopic and Lie-admissible type has been done by Jannussis and his associates [22]. Their axiomatic formulation has been reached in the recent paper [23]. The emerging formulations are here called *Jannussis' isotopic and genotopic algebras.*

The *only* deformed Bosonic algebra known to this author at this time which possess the above axiomatic structure are given by the Lie-isotopic algebras (*Jannussis' isobosonic algebras*)

$$\hat{a}\, T(q, ...)\, \hat{a}^† - \hat{a}^†\, T(q, ...)\, \hat{a} = [\, T(q, ...)\,]^{-1} = \hat{1}, \quad T = T^†, \quad (4.A.4)$$

and by the more general Lie-admissible algebras (*Jannussis' genobosonic algebras*) for motion forward to future time

$$\hat{a}\, {}^<T(q, ...)\, \hat{a}^† - \hat{a}^†\, T^>(q, ...)\, \hat{a} = [\, T^>(q, ...)\,]^{-1} = \hat{1}^> , \qquad (4.A.\,5)$$

or motion forward from past time

$$\hat{a}\, {}^<T(q, ...)\, \hat{a}^† - \hat{a}^†\, T^>(q, ...)\, \hat{a} = [\, {}^<T(q, ...)\,]^{-1} = {}^<\hat{1} , \qquad (4.E.5)$$

under the condition

$$^<T(q, ...) = [\, T^>(q, ...)\,]^† , \qquad (4.A.6)$$

which is necessary for the preservation of Hermiticity–observability, of a Lie-admissible group structure with consequential preservation of causality, etc. (Sect. III.3.3), where T, $^<$T and T$^>$ are in general *operators* verifying the needed conditions of regularity, boundedness, etc., but otherwise possess an unrestricted functional dependence on the parameter q as well as any other needed quantity.

The most direct way of identifying the Lie–isotopic structure (4.A.4) is by submitting structure (4.A.I) to nonunitary transformations, according to the rule identified in the original proposal [I]

$$U U^\dagger = 1 = 1^\dagger \neq 1, \quad T = (U U^\dagger)^{-1} = T^\dagger, \quad a' = U a U^\dagger, \quad a'^\dagger = U a^\dagger U^\dagger, \quad (4.A.8a)$$

$$U (a a^\dagger - a^\dagger a) U^\dagger = U a U^\dagger T U a^\dagger U^\dagger - U a^\dagger U^\dagger T U a U^\dagger =$$

$$= a' T a'^\dagger - a'^\dagger T a' = U I U^\dagger = 1 = T^{-1}. \quad (4.A.8b)$$

As indicated earlier, the Lie–isotopic structure therefore emerges *even when not desired*.

Once the latter structure has been reached, it is *necessary* for the above axiomatic structure to treat it with the formalism of the Lie–isotopic branch of hadronic mechanics (Sect.s II.3.2 and II.4.2), i.e., on isofields $\hat{F}(\hat{n},+,*)$ (F = R, C), isoenveloping algebras $\hat{\xi}$ on $\hat{F}$, isohilbert spaces $\hat{\mathcal{H}}$ on $\hat{F}$, etc. In particular, this implies that *all* multiplications must be isotopic, e.g., the new number operator is $\hat{N} = a^\dagger T a$, the action of any operator A on the isostates $| >$ must be isomodular, $AT| >$, etc.

The necessity of the hadronic formalism can be easily proved. For instance, an additional *conventional* nonunitary transform of isostructure (4.A.4) would imply its lack of form–invariance because the isotopic operator T would not be preserved,

$$U (\hat{a} T \hat{a}^\dagger - \hat{a}^\dagger T \hat{a}) U^\dagger = \hat{a}' T T' T \hat{a}'^\dagger - \hat{a}'^\dagger T T' T \hat{a}'; \quad (4.A.9)$$

similarly, the *conventional* occupation number operator $N = a^\dagger a$ would be Hermitean in the original space $\mathcal{H}$ but such Hermiticity is not preserved in time; the same situation occurs for the isotopic operator $\hat{N} = a^\dagger T a$; etc.

The use of the formalism of hadronic mechanics resolves *all* the above problems. First, the formalism implies the identity of the conventional and isotopic Hermiticity (we shall therefore drop the superscript $^\wedge$ in the operation $\dagger$). An arbitrary *nonunitary* operator U, can always be written for isostructure (4.A.4) in the *identical* isotopic form

$$U U^\dagger = 1 \neq I, \quad U = \hat{U} T^{1/2}, \quad U U^\dagger = \hat{U} T \hat{U}^\dagger = U^\dagger U = 1, \quad (4.A.10)$$

under which isostructure (4.A.4) remains invariant,

$$\hat{U} T (\hat{a} T \hat{a}^\dagger - \hat{a}^\dagger T \hat{a}) T \hat{U}^\dagger = \hat{a}' T \hat{a}'^\dagger - \hat{a}'^\dagger T \hat{a}' = \hat{U} T 1 T \hat{U}^\dagger = 1. \quad (4.A. II)$$

The original operator $N = a^\dagger a$ is then lifted into the *isooccupation operator* $\hat{N} = \hat{a}^\dagger T \hat{a}$ which is indeed Hermitean–observable in the isohilbert space $\mathcal{H}$ with isotopic element T. Such Hermiticity–observability is also preserved at all times from its own invariance $\hat{U} T \hat{a}^\dagger T \hat{a} T \hat{U}^\dagger = a'^\dagger T a'$, as well as from the invariance of the underlying isoinner product $< | T | > \equiv < |T\hat{U}^\dagger T U T| >$. The isotopic theory possesses a generalized unit $\hat{1}$ which is invariant under the most general nonunitary–isounitary transforms, $\hat{1}' = \hat{U} T \hat{1} T \hat{U}^\dagger \equiv \hat{1}$, etc.

A *particular case* of isostructure (4.A.4) can be reached via the simple, but effective map of the *Klimyk rule* (Lemma 1.4.7.5)

$$a \to \hat{a} = a\,T^{-1}, \quad a^\dagger \to \hat{a}^\dagger = T^{-1} a^\dagger, \quad T = T^\dagger, \quad [T, a] = [T, a^\dagger] = 0, \quad (4.A.12)$$

under which the original algebra (4.A.1) is mapped precisely into a structure of type (4.A.4),

$$a\,a^\dagger - a^\dagger a = 1 \to \hat{a}\,T\,\hat{a}^\dagger\,T - \hat{a}^\dagger\,T\,\hat{a}\,T = 1 \to \hat{a}\,T\,\hat{a}^\dagger - \hat{a}^\dagger\,T\,\hat{a} = T^{-1}. \quad (4.A.13)$$

The condition $[T, a] = [T, a^\dagger] = 0$ implies that *the Klimyk rule applies for all realizations of the isotopic element T via ordinary functions.* In particular, the rule generally implies the preservation of the original eigenvalues under an internal degree of freedom represented by T. In fact, this is the general occurrence for a particular class of isorepresentations of Lie–isotopic algebras called *standard* (Sect. 1.4.5). Almost needless to say, the quantum mechanical eigenvalues are altered for the general case in which T is an operator such that $[T, a] \neq 0$, $[t, a^\dagger] \neq 0$.

The Lie–admissible structure (4.A.5) or (4.A.6) describes an *irreversible* generalization of the conventional Bosonic algebra. The theory therefore requires: a selection of the direction of time in which the theory is elaborated; the identification, e.g., of the product $\hat{a}^< T \hat{a}^\dagger = \hat{a} < \hat{a}^\dagger$ with motion forward from past time and the product $\hat{a} T^> \hat{a}^\dagger = \hat{a} > \hat{a}^\dagger$ with motion forward to future time; the identification of a generalized unit in each direction of time, $\hat{1}^> = (T^>)^{-1}$ and $^< \hat{1} = (^< T)^{-1}$ and all the various aspects of the Lie–admissible branch of hadronic mechanics (see Sect.s 11.3.3 and 11.4.3).

The important result is that the essential isotopic lines hold in their entirety in each direction of time, including form invariance, achievement of Hermiticity–observability under nonconservative conditions, its preservation at all times, invariance of the generalized unit, etc.

As a particular case we can introduce *two Klimyk rules,* one for motion forward to future times and one for motion forward from past time and write

$$\hat{a}^> = a\,(T^>)^{-1}, \qquad \hat{a}^{>\dagger} = a^\dagger\,[T^>]^{-1}, \quad [T^>, \hat{a}^>] = [T^>, \hat{a}^{>\dagger}] = 0, \quad (4.A.14a)$$

$$^<\hat{a} = a\,(^<T)^{-1}, \qquad {}^<\hat{a}^\dagger = a^\dagger\,[^<T]^{-1}, \quad [^<T, {}^<\hat{a}] = [^<T, {}^<\hat{a}^\dagger] = 0, \quad (4.A.14b)$$

$$T^> = (^< T)^c, \text{e.g.,} \quad T^> = (\,^<T\,)^\dagger, \qquad (4.A.14c)$$

Note that the lack of conjugation (4.A.14c) implies not only the loss of Hermiticity, but also that of a Lie–admissible group structure, with consequential chain of problematic aspects, such as loss of causality, lack of axiomatic representation of irreversibility and others.

For the general case we refer the interested reader to paper [23].

APPENDIX 4.B: ARINGAZIN–NTIBASHIRAKANDI–CALLEBAUT SUPERSYMMETRIC HADRONIC MECHANICS

As is well known, the *supersymmetric (susy) quantum mechanics* has been studied by various authors [24]. In this appendix we present the *supersymmetric hadronic mechanics* or *isosupersymmetric (iss) mechanics* introduced by Ntibashirakandi and Callebaut [8] following Aringazin's [7] isotopies of anticommuting creation and annihilation operators (see Aringazin's studies for the classical counterpart in Appendix 11.1.D).

Iss systems are described by N odd *isosupercharges* Q^i, $i = 1, 2, ..., N$, and an even *isosupersymmetric Hamiltonian* $\hat{H}_{iss}$ with isotopic anticommutation rules

$$\{Q^i, Q^j\} = Q^i * Q^j + Q^j * Q^i = Q^i T Q^j - Q^j T Q^i = 2\,\hat{\delta}^{ij} * \hat{H}_{ss}, \quad (4.B.1a)$$

$$[\hat{H}_{iss}, Q^i] = H_{iss} * Q^i - Q^i * \hat{H}_{iss} = 0, \quad \hat{H}_{iss} = Q^i * Q^i, \qquad (4.B.1b)$$

where $\hat{\delta}^{ij}$ is the isokronecker delta and the conservation of the isosupercharges is evident.

Let us consider first the two–dimensional case according to [8]. Consider the following realization of the isosupercharges

$$Q = (Q^1 + i Q^2)/\sqrt{2}, \quad Q^\dagger = (Q^1 - i Q^2)/\sqrt{2}. \qquad (4.B.2)$$

Then, the isosuperalgebra becomes

$$\{Q, Q^\dagger\} = 2\hat{H}_{iss}, \quad \{Q, Q^\dagger\} = \{Q^\dagger, Q^\dagger\} = 0, \qquad (4.B.3a)$$

$$[Q, \hat{H}_{iss}] \, [Q^\dagger, \hat{H}_{iss}] = 0. \qquad (4.b.3b)$$

Note that the above relations are not isoindependent because the conservation of the isosupercharge follows from the isonilpotency of Q and $Q^\dagger$ and the assumed form of $\hat{H}_{iss}$.

For the case of the *isosuperoscillator* in one dimension, the isosupercharges are realized by

$$\hat{Q} = (p + ir) * \hat{\sigma}^-, \quad \hat{Q}^\dagger = (p - ir) * \hat{\sigma}^+, \tag{4.B.4}$$

where the variables (r, p) are isobosonic and $(\hat{\sigma}^-, \hat{\sigma}^+)$ are isofermionic with properties

$$[r, \hat{} p] = i\hat{1}, \tag{4.B.5a}$$

$$\{\hat{\sigma}^+, \hat{} \hat{\sigma}^-\} = \hat{1}, \quad [\hat{\sigma}^+, \hat{} \hat{\sigma}^-] = \hat{\sigma}_3, \quad \hat{\sigma}^\pm * \hat{\sigma}^\pm = 0. \tag{4.B.5b}$$

The isosupersymmetric Hamiltonian is then given by

$$\hat{H}_{iss} = \tfrac{1}{2}(p * p + r * r) + \tfrac{1}{2}\hat{\sigma}_3 =$$

$$= \tfrac{1}{2}(p * p + W' * W') + i[p, \hat{} W'] * \hat{\sigma}_3 / 2, \tag{4.B.6a}$$

$$W = \tfrac{1}{2}r^2, \quad \Omega' = \partial_r * W, \tag{4.B.6b}$$

where the mass and frequency are assumed to be the unity.
The isosupercharges can then be written

$$\hat{Q} = (p + iW') * \hat{\sigma}^-, \quad \hat{Q}^\dagger = (p - iW') * \hat{\sigma}^+. \tag{4.B.7}$$

The latter reformulation is more general than the preceding one because it applies for an arbitrary potential W in one dimension.
The isosupercharges can also be realized in the form

$$\hat{Q} = \hat{B}^- * \hat{\xi}^+, \quad \hat{Q}^\dagger = \hat{B}^\dagger * \hat{\xi}^-, \tag{4.B.8a}$$

$$\hat{B}^\pm = \hat{B}^1 \pm i\hat{B}^2, \quad \{\hat{\xi}^+, \hat{} \hat{\xi}^-\} = \hat{1}, \quad \hat{\xi}^\pm * \hat{\xi}^\pm = 0. \tag{4.B.8b}$$

or the additional one

$$\hat{Q}^1 = (\hat{B}^1 * \hat{\phi}^1 - \hat{B}^2 * \hat{B}^2)/\sqrt{2}, \quad \hat{Q}^2 = (\hat{B}^1 * \hat{\phi}^1 + \hat{B}^2 * \hat{\phi}^2)/\sqrt{2}, \tag{4.B.9a}$$

$$\hat{\phi}^1 = \hat{\xi}^+ + \hat{\xi}^-, \quad \hat{\phi}^2 = i(\hat{\xi}^+ - \hat{\xi}^-), \tag{4.B.9b}$$

which define the *isoclifford algebra*

$$\{\hat{\phi}^a, \hat{} \hat{\phi}^b\} = 2\,\delta^{ab}, \quad a, b = 1, 2. \tag{.4.B.10}$$

The isosupersymmetric Hamiltonian can then be written

$$\hat{H}_{iss} = \tfrac{1}{2}(\hat{B}^1 * \hat{B}^1 + \hat{B}^2 * \hat{B}^2) + \tfrac{1}{2}[\hat{B}^1 \hat{,} \hat{B}^2] * \hat{\Phi}_1 * \hat{\Phi}^2 =$$

$$= \tfrac{1}{2}\{\hat{B}^+ \hat{,} \hat{B}^-\} - [\hat{B}^1 \hat{,} \hat{B}^2] * [\hat{C}^+ \hat{,} \hat{C}^-]/4 . \qquad (4.B.11)$$

The latter realization permits the formulation of the isosupersymmetric hadronic mechanics in three dimensions [8] which can be summarized as follows. Introduce the isosupercharges

$$\hat{Q} = \sum_k \hat{B}_k^+ * \hat{\xi}_k^-, \quad \hat{Q}^\dagger = \sum_k \hat{B}_k^- * \hat{\xi}_k^+, \qquad (4.B.12)$$

where

$$\{\hat{\xi}_i^+ \hat{,} \hat{\xi}_j^-\} = \hat{\delta}_{ij}, \qquad \{\hat{\xi}_i^\pm \hat{,} \hat{\xi}_j^\pm\} = 0, \qquad (4.B.13a)$$

$$\hat{B}_k^1 = (\hat{B}_k^+ + \hat{B}_k^-), \quad \hat{B}_k^2 = (\hat{B}_k^+ - \hat{B}_k^-)/2i . \qquad (4.B.13b)$$

It is then straightforward to obtain the *isoclifford algebra in three dimensions*

$$\{\hat{\Phi}_i^a \hat{,} \hat{\Phi}_j^b\} = 2\hat{\delta}_{ij} * \hat{\delta}^{ab}, \quad \hat{\Phi}_k^1 = \hat{\xi}_k^+ + \hat{\xi}_k^-, \quad \hat{\xi}_k^2 = i(\hat{\xi}_k^+ - \hat{\xi}_k^-). \qquad (4.B.14)$$

which permits the following *isosupersymmetric Hamiltonian* in three dimension

$$\hat{H}_{iss} = \tfrac{1}{2}\sum_k [p_k * p_k + (\partial_k * W) * (\partial_k * W)] + \tfrac{1}{2}\sum_{ij}(\partial_i * \partial_j * W) * [\hat{\xi}_i^+ \hat{,} \hat{\xi}_j^-]$$
$$(4.B.15)$$

The further assumptions

$$\{\hat{\Phi}_i^a \hat{,} \hat{\Phi}_j^a\} = 2\hat{\delta}_{ij}, \quad \{\hat{\Phi}_i^1 \hat{,} \hat{\Phi}_j^2\} = 2\Gamma_{ij} \qquad (4.B.16a)$$

$$\hat{\xi}_i^+ = \hat{\sigma}_i \otimes \hat{\sigma}^+, \quad \hat{\xi}_i^- = \hat{\sigma}_i \otimes \hat{\sigma}^+, \qquad (4.B.16b)$$

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = 2\epsilon_{ijk}\hat{\sigma}_k, \qquad (4.B.16c)$$

lead to the Hamiltonian

$$\hat{H}_{iss} = \tfrac{1}{2}\sum_k [p_k * p_k + (\partial_k * W) * (\partial_k * W)] + \tfrac{1}{2}\sum_{ij}(\partial_i * \partial_j * W) * [\hat{\xi}_i^+ \hat{,} \hat{\xi}_j^-] -$$
$$- \tfrac{1}{2}\sum_{ij}[(\partial_i * W) * p_j - (\partial_j W) * p_i]\Gamma_{ij} , \qquad (4.B.17)$$

which can be finally written in the form

$$\hat{H}_{iss} = \tfrac{1}{2}\sum_k (p_k * p_k + r_k * r_k) - \tfrac{1}{2}(3 + 4L * S) * \hat{\sigma}_3, \qquad (4.B.18)$$

where L and S are the hadronic angular momentum and spin (Ch. II.6) and the $\hat{\sigma}_k$ are the *standard isopauli matrices* (I.4.7.53) (see also Ch. II.6).

The physical implications of supersymmetric hadronic mechanics are not trivial. In fact, as we shall see in Vol. III, the formalism permits a consistent construction of unstable hadrons as the chemical synthesis of ligther hadrons generally with the lowest decay mode, which is permitted by the supersymmetric hadronic mechanics but prohibited by the conventional quantum mechanical form.

APPENDIX 4.C: ISOLOCAL REALISM

As is well known, the criticisms of quantum mechanics by Einstein–Podolsky–Rosen (E–P–R) [10] and by others led to a new branch of physics known under the name of *local realism* (see, e.g., the studies by Selleri [9] and vast literature quoted therein) which is formulated via conventional fields, conventional metric spaces, conventional functional analysis, etc.

In this appendix we outline the elements of a generalization of local realism submitted by Santilli [11] under the name of *isolocal realism,* which is based on the covering theory of isonumbers, isospaces, isofunctional isoanalysis, and the other isotopic methods.

The main result is that the conventional linear (in the wavefunction), local and canonical formulation of quantum mechanics is indeed an *incomplete theory* considerably, although not completely, along the E–P–R argument, in the sense that its abstract axioms can be "completed" into an isotopic form which is nonlinear (In the wavefunctions and their derivatives), nonlocal–integral and noncanonical, yet coincides with the conventional formulation at the abstract level.

4.C.1: Isolinearity, isolocality and isocanonicity. Let us recall the following fundamental notions introduced in Vol. 1:

Definition 4.C.1: *Let $\hat{\xi}$ be an isoassociative enveloping algebra of operators A, B, C, ...with isoproduct and isounits*

$$A*B := ATB, \quad \hat{1} = T^{-1} > 0, \quad \hat{1}*A = A*\hat{1} \equiv A, \forall A \in \hat{\xi}, \quad (4.C.1)$$

of Kadeisvili's Class I on an isohilbert space $\hat{\mathcal{H}}$ with isoinner product over the field of isocomplex numbers

$$<\hat{\psi}\hat{\dagger}\hat{\Phi}> := <\hat{\psi}|T|\hat{\Phi}>\hat{1} \in \hat{C}(\hat{c},+,*), \quad (4.C.2)$$

where $<\hat{\psi}| \hat{\Phi}>$ is the conventional inner product. Then, the theory is called <isolinear> because it verifies on isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{R})$ the same abstract axioms of a conventionally linear theory (see Sect.

I.6.3)

$$\hat{O} * (\hat{n} * r + \hat{n}' * r') = \hat{n} * (\hat{O} * r) + \hat{n}' * (\hat{O} * r'), \quad \hat{n}, \hat{n}' \in \hat{R}, \qquad (4.C.3a)$$

$$(\hat{n} * \hat{O} + \hat{n}' * \hat{O}') * r = \hat{n} * (\hat{O} * r) + \hat{n}' * (\hat{O}' * r), \quad \hat{O} * (\hat{O}' * r) = (\hat{O} * \hat{O}') * r.$$
$$(4.C.3b)$$

while the projections of the transformations in the original space
E(r,δ,R) are nonlinear in the wavefunctions and their derivatives

$$r' = \hat{O} * r = \hat{O} T r = \hat{O} T(t, r, p, \dot{p}, \psi, \psi^\dagger, \partial\psi, \partial\psi^\dagger, ...) r. \qquad (4.C.4)$$

The theory is also called <isolocal> on Ê(r,δ,R̂), because it is everywhere local–differential, except in the dependence of the unit (see Sect. I.1.1.4). When projected in the original space E(r,δ,R) , the theory is nonlocal–integral as in (4.C.4) Finally the theory is called <isocanonical> because it is derivable via conventional variational principles, although formulated in isospace Ê(r,δ,R̂) (see Sect. II.1.4). The same theory is however noncanonical when projected in the conventional Euclidean space, e.g., because nonlinear in the velocities and accelerations.

We should recall that the above isolinear, isolocal and isocanonical structure is primarily intended for the quantitative treatment of *interior dynamical problems* of particles, such as a proton in the core of a star (or, along similar lines, hadronic and nuclear structures) in which we expect a superposition of local–differential–potential and nonlocal–integral–nonpotential interactions, the latter ones due to the mutual penetration of the wavepackets of the constituents [1].

Nevertheless, applications also exist for the *exterior dynamical problem,* i.e., a particle moving in vacuum under only action–at–a–distance local–differential interactions. In this latter case the isolinearity and isocanonicity of the theory are evidently un–necessary, but its isolocal structure is still significant, e.g., for a possible deeper understanding of the origin of the exclusion principle (Sect. II.2.C) of for the direct representation of *nonspherical* charge distributions and all their infinitely possible *deformations* (see Vol. III).

4.C.2: Isotopic completion of quantum mechanics. Recall that the E–P–R argument essentially states that [9,10] *quantum mechanics is an incomplete theory because its description of physical reality does not include all elements of reality, while every element of physical reality should be precisely represented in a complete theory.*

To begin, we point–out the inapplicability under isotopies of the

conventional Dirac δ-function and the need for its replacement with the *isodelta function* (Sect. I.6.4)

$$\delta(r - q) = \frac{1}{2\pi} \int du \ \exp_{\hat\xi} [\ i\ u\ (r - q)\ ] = \frac{\hat{1}}{2\pi} \int du \ \exp[\ i\ u\ T\ (r - q)\ ], \quad (4.C.5a)$$

$$\delta(r - r') = \int dz \ \delta(r - z) * \delta(z - r'), \quad (4.C.5b)$$

where the isounit is assumed to be independent of the variable of integration for simplicity.

The isodelta function is the true foundation of hadronic mechanics inasmuch as it embodies its nonlocal structure. Recall that the best expression of the local structure of quantum mechanics is precisely given by the conventional δ-function which is everywhere null except at the point q where it is infinite. Such an infinity can be eliminates by the isodelta function and spread over the region of space occupied by the wavepacket of the particle via a suitable selection of the T operator.

Next, the conventional, right, modular, associative action of the operator Q on its eigenfunction u(q; r), Qu(q; r) in the notation of [8] is no longer valid in hadronic mechanics, and must be lifted into the right, modular, isoassociative action

$$Q * \delta(r - q) = \hat{q} * \delta(r - q) = q\ \delta(r - q), \qquad \hat{q} \in \hat{R}, \quad q \in R, \quad (4.C.6)$$

The above expression is another representation of the nonlocality of isotopic theories. In fact, *the quantity* q *now represents the center-of-mass of the particle, with integral corrections due to its extended structure represented by the isotopic operator* T (see Fig. 1.1.4.1).

Next, the conventional planewaves lose any mathematical consistency in hadronic mechanics and must be replaced with the (space component of the) *isoplane wave* (II.3.2.37),

$$\hat{v}(p; r) = \exp(\ i\ pTr), \quad (4,C.7)$$

(where the factors are ignored for simplicity), which represents, e.g., the deformation of a conventional plane wave caused by its immersion within an inhomogeneous and anisotropic physical medium.

Next, the conventional momentum operator is no longer applicable, in favor of the isotopic form

$$P * \hat{v}(p; x) = -i\hat{1}\ \nabla_r\ \hat{v}(p; x) = p\ \hat{v}(p; x), \quad (4,C.8)$$

By following [8], consider now an ensemble of two isoparticles 1 and 2, that is, particles characterized by two, generally different isounits $\hat{1}_\alpha$, $\alpha = 1, 2$. In this case, the ensemble is characterized by the ordinary tensorial product of the

various quantities

$$T_{tot} = T_1 \times T_2, \qquad \hat{1}_{tot} = \hat{1}_1 \times \hat{1}_2 = (T_1 \times T_2)^{-1}, \qquad (4.C.9a)$$

$$\hat{\xi}_{tot} = \hat{\xi}_1 \times \hat{\xi}_2, \qquad \hat{C}_{tot} = \hat{C}_1 \times \hat{C}_2, \qquad \mathcal{K}_{tot} = \mathcal{K}_1 \times \mathcal{K}_2, \quad etc. \qquad (4.C.9b)$$

The system 1+2 can then be described by the structure

$$\Phi(q_0; r_1, r_2) = \int dq' \, \hat{c}(q') * u_1(q'; r_1) * u_2(q_0 + q'; r_2), \qquad (4.C.10a)$$

$$\Phi(p_0; r_1, r_2) = \int dp' \, \hat{c}(p') * u_1(p'; r_1) * u_2(p_0 - p'; r_2), \qquad (4.C.10b)$$

where the $\hat{c}$'s are statistical *isoweights*, that is, weights referred to $\hat{1}$ as the isounit (since they are isonumbers, they can be simply written as $\hat{c} = c\,\hat{1}_{tot}$), while the isotopic product is now characterized by the element T_{tot}.

As in quantum mechanics, a measurement of the position q on 1 yields the *isoprobability*

$$|\hat{c}(q')|^2 = |\hat{c}(q') * \hat{c}(q')| = |c(q')|^2 \hat{1}_{tot}, \qquad \cdot \qquad (4.C.10b)$$

where the property $\hat{1}_{tot} > 0$ has been used. Once q' is known, the corresponding position on 2 is given by $q_0 + q'$ as in the quantum mechanical case, with similar conclusions for p. By again following the conventional treatment [8], the system 1+2 can be represented with the isowavefunction

$$\hat{\Psi}(q_0, p_0; r_1, r_2) = k \int dp' \, e^{i(r_1 - r_2 + q_0) T_{tot} p'} = 2\pi k \, \hat{\delta}(r_1 - r_2 + q_0) =$$

$$= 2\pi k \int dq' \, \delta_{\hat{A}_{tot}}(r_1 - q') \, T_{tot} \, \delta_{\hat{A}_{tot}}(q' - r_2 + q_0) \qquad (4.C.11)$$

where the composition rules for the isodirac function have been used.

The implications can now be identified. In fact, *the isotopic completion of quantum mechanics implies the admission of nonlinear, nonlocal and nonhamiltonian interactions represented by the isotopic element T, even when all conventional quantum mechanical potentials are identically null, H = K, V = 0.*

This novel concept of interactions was first shown to originate at the *classical* level (see the representation of conventional electromagnetic interactions with a *null* electromagnetic potential of App. I.1.A and then proved to persist under isotopic quantization in Ch. I.2).

The implications for the E-P-R argument are intriguing. In fact, commuting quantities are traditionally believed to be independent. On the contrary, *in the isotopic completion of quantum mechanics isocommuting quantities can be mutually interacting.* The understanding is that such interactions are structurally different than those of action-at-a-

distance/potential type. The best illustration is precisely Pauli's exclusion principle indicated earlier.

In summary, quantum mechanics can be considered as an incomplete theory in the sense that (as merely referring to one among several aspects) it does not contain the element of reality given by the nonlocal structure of interactions expected form the mutual wave overlapping. Hadronic mechanics does contain that element of reality via its direct representation in terms of the isotopic element.

Additional elements of reality are indeed permitted by hadronic mechanics, such as the representation of the nonspherical shape of a charge distribution, all its infinitely possible deformations, and other representations outside the technical; capabilities of quantum mechanics.

It should be stressed, that hadronic mechanics *is not* intended to represent *all* elements of reality (which is a subjective issue, to begin with). After all, physics is a discipline that will never admit final theories. Hadronic mechanics merely provides *one* particular type of completion of quantum mechanics for the representation of the nonlocal element of reality, that of axiom-preserving type.

The behaviour of the uncertainties under isotopies has been studied in Subsection II.4.2.A which is an integral part of the isolocal realism. As one recalls, hadronic mechanics "completes" quantum mechanics into a full determinism, but only at the limit of gravitational singularities.

We can therefore say that the isotopic completion of quantum mechanics does indeed allow the recovering of the deterministic element of reality under certain limiting conditions. The understanding is that such determinism *is not* recovered by hadronic mechanics for all physical conditions of particles.

4.C.3: Isotopic formulation of "hidden variables". It is easy to see that von Neumann's theorem on hidden variables [25] is *inapplicable* (and not "violated") under isotopies. In fact, one of the assumptions of the theorem is the *uniqueness* of the spectrum of eigenvalues of a Hermitean operator. This assumption is correct for quantum mechanics, but inapplicable under isotopies because the *same* Hermitean operator admits infinitely many different spectra of eigenvalues in hadronic mechanics, evidently depending on the assumed isounit.

Consider, for instance, a quantum Hamiltonian $H = H^\dagger$ and its quantum eigenvalue E_0, $H| \psi > = E_0| \psi >$. Under isotopies, the same Hamiltonian H (which remains Hermitean as now familiar and with real eigenvalues) admits instead the infinitely different spectra of eigenvalues E_T depending on the isotopic element T (or isounit $\hat{1}$)

$$ H * |\hat{\psi}> = H T |\hat{\psi}> = \hat{E}_T* |\hat{\psi}> = E_T |\hat{\psi}>, \quad E_T \in R, E_T \neq E_0 . \quad (4.C.12) $$

As a matter of fact, one of the primary applications of hadronic mechanics

is the identification of the *alteration* of a quantum mechanical spectrum of eigenvalues E_0 of a given operator H which is expected in a system of interacting particles at distances smaller than their wavepacket/wavelength/charge-distribution (see Vol. III for various cases)

Another application is the reconstruction of the *exact* su(2)–isospin symmetry under *electromagnetic and weak interactions* into equal proton and neutron masses in isospace and others (see also Vol. III).

Moreover, isoeigenvalue equations provide an explicit realization of the "hidden variables", e.g., with the simplest possible identification of the isotopic element as the nowhere null function $T(\lambda)$, in which case

$$H * |\hat{\psi}> = E_\lambda |\psi>, \quad E_\lambda = E_0 [T(\lambda)] \Gamma^1 . \qquad (4.C.13)$$

In actuality, Eq.s (4.C.12) provide an *operator* realization of "hidden variables". We can therefore say that the isotopic completion of quantum mechanics is based on "hidden operators" T whose diagonal elements are "hidden functions".

In fact, most of the novel applications of quantum mechanics can be interpreted as a realization of the theory of "hidden variables" of the above type. An explicit realization is provided in the next section.

4.C.D: Isotopic generalization of Bell's inequality. We now pass to the behavior under isotopies of the celebrated Bell's inequality [26]

$$D_{Max\ QM} = | P(a, b) - P(a, b') | + | P(a', b) + P(a', b') |_{Max\ QM} \leqq 2, \qquad (4.C.14a)$$

$$P(a, b) = <S_{1-2} | (\sigma_1 \cdot a) \times (\sigma_2 \cdot b) | S_{1-2} > = - a \cdot b. \qquad (4.C.14b)$$

where we have used for clarity the same notation of [8].

As well known, the above inequality is crucially dependent on the conventional, Pauli's realization of the adjoint (two–dimensional) representation of the spin algebra su(2). The inapplicability of inequality (4.C.14) under isotopies is then consequential, owing to the novel structure of the adjoint isorepresentations of the Lie–isotopic algebra sû(2) which have been classified into the following *regular, irregular and standard isopauli matrices* (see Ch. II.6).

For our needs at this point it is sufficient to consider the *standard isopauli matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix} \qquad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix} \qquad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} , \qquad (4.C.15a)$$

$$T = diag. (\lambda, \lambda^{-1}), \quad \Delta = det\ T = I, \qquad (4.C.15b)$$

$$[\hat{\sigma}''_i \hat{,} \hat{\sigma}''_i] = i \epsilon_{iik} \hat{\sigma}''_k, \quad \hat{\sigma}''_3 * | b >= \pm | b > , \qquad \hat{\sigma}''^2_* | b > = 3 | b > , \qquad (4.C.15c)$$

where one should keep in mind the Lie-isotopic product.

As we shall see in Ch. 11.6, the inapplicability of Bell's inequality under the regular and irregular isopauli matrices is evident from the alteration of the eigenvalues and needs no treatment. In this subsection we show that the inequality does not hold even for the case of the special isopauli matrices in which, as one can see, we have *conventional structure constants and eigenvalues*, yet the concrete appearance of the "hidden variable" λ in the very structure of spin $\frac{1}{2}$. Note that λ can be an arbitrary, nonlinear and integral, nowhere null function of the local variables, wavefunctions and their derivatives, because it is completely unrestricted by the isotopic theory once positive-definite.

Consider two *isoparticles*, i.e., particles obeying hadronic mechanics, yet having conventional spin $\frac{1}{2}$ characterized by Eq.s (4.C.15). Even though their spin is the same, there is no necessary reason to restrict their isotopic degrees of freedom λ to be the same (e.g., because their density may be different) and we can assume

$$\text{Particle 1:} \quad T = \text{diag.}(\lambda, \lambda^{-1}), \quad \Delta = \det T = 1, \quad \text{spin } \tfrac{1}{2}, \quad (4.C.16a)$$

$$\text{Particle 2:} \quad T' = \text{diag.}(\lambda', \lambda'^{-1}), \quad \Delta' = \det T' = 1, \quad \text{spin } \tfrac{1}{2}. \quad (4.C.16b)$$

Next, consider the composite system of two isoparticles 1 and 2 characterized by the isounit

$$\hat{1}_{tot} = \hat{1}_1 \times \hat{1}_2 = T_{tot}^{-1} = (T \times T')^{-1}. \quad (4.C.17)$$

To properly recompute the isotopies of Bell's inequality it is necessary to identify the isonormalized basis $|\hat{S}_{1-2}>$, that is, the basis of the total spin of the particles 1 and 2 normalized to $\hat{1}_{tot}$,

$$<\hat{S}_{1-2}|\hat{S}_{1-2}> = <\hat{S}_{1-2}|Q_{tot}|\hat{S}_{1-2}>\hat{1}_{tot} = \hat{1}_{tot}, \quad (4.C.18)$$

A simple isotopy of the conventional case (see, e.g., ref. [27], Sec. 17.9) leads to the *isobasis for the singlet state*

$$|\hat{S}_{1-2}> = \frac{1}{\sqrt{2}}\left\{ \begin{pmatrix} \lambda^{-\frac{1}{2}} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ \lambda'^{\frac{1}{2}} \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda'^{-\frac{1}{2}} \\ 0 \end{pmatrix} \right\}. \quad (4.C.19)$$

It is a tedious but instructive exercise for the interested reader to verify isonormalization condition (4.C.118) by constructing the adjoint of basis (4.C.19), by sandwiching the quantity $T_{tot} = T \times T'$, by contracting only quantities of the same particle, and then multiplying the scalar results of the two different

particles, much along the conventional case [18].

Next, recall that the conventional scalar product $\sigma \cdot a$, where a is a three-vector, has no mathematical or physical meaning in the two-dimensional complex isoeuclidean isospace $\hat{E}(z,\bar{z},\hat{\delta},\Re)$ (Ch. I.3) underlying $s\hat{u}(2)$ and must be replaced by the isoscalar product

$$\hat{\sigma}T\,a = \begin{pmatrix} a_z & (a_x - i\,a_y) \\ (a_x + i\,a_y) & -a_z \end{pmatrix}. \qquad (4.C.20)$$

The tedious but straightforward repetition of the conventional procedure [27] under standard isotopies then leads to

$$<\hat{S}_{1-2}\,|\,(T \times T')\,\{\,(\hat{\sigma}*a)\times(\hat{\sigma'}*b)\,\}\,(T \times T)\,|\,\hat{S}_{1-2}> =$$

$$= -\,a_x b_x - a_y\,b_y - \tfrac{1}{2}(\lambda\,\lambda'^{-1} + \lambda^{-1}\lambda')\,a_z\,b_z. \qquad (4.C.21)$$

Consider now unit vectors a, b, a', b' along the z-axis. Then Bell's inequality [8,17]

$$D_{Bell} = Max\,|\,P(a,b) - P(a,b')\,| + |\,P(a',b) + P(a',b')\,| \le 2, \qquad (4.C.22a)$$

$$P(a,b) = <S_{1-2}\,|\,(\sigma_1 \cdot a)\times(\sigma_2 \cdot b)\,|\,S_{1-2}> = -\,a\cdot b, \qquad (4.C.22b)$$

admits the following isotopic image under the covering $s\hat{u}(2)$ symmetry

$$D_{Bell} \le D_{Max}{}^{HM} = \tfrac{1}{2}(\lambda\,\lambda'^{-1} + \lambda^{-1}\lambda')\,D_{Bell}. \qquad (4.C.23)$$

Now, the factor $\tfrac{1}{2}(\lambda\,\lambda'^{-1} + \lambda^{-1}\lambda')$ can be easily proved to admit values bigger than one. This establishes the statement made earlier in this subsection, to the effect that Bell's inequality is not universally valid, but holds, specifically, for the conventional, linear, local and canonical realization of quantum mechanics and of the related su(2) algebra. The proof for arbitrary orientations of the unit vectors follows the conventional one [8] and it is here omitted for brevity.

4.C.5: Completion of hadronic into Newtonian mechanics. Intriguingly, the isotopies permit a completion of quantum mechanics, which is again significantly along the E-P-R argument [11]. Recall that [8]

$$D_{Max}{}^{Class.} = Max\,|\,a\cdot b - a\cdot b'\,| + |\,a'\cdot b + a'\cdot b'\,| = 2\sqrt{2} > 2, \qquad (4.C.24)$$

and that $D_{Bell} < D_{Max}{}^{Class.}$, thus preventing the completion of quantum mechanics into a deterministic theory.

However, under isotopic liftings, one can assume a classical isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ (representing, say, motion of extended objects within physical media) with isotopic scalar product

$$a * b = a^t T b = a_x g_{11} b_x + a_y g_{22} b_y + a_z g_{33} b_z. \qquad (4.C.25)$$

Then, *there always exists a realization of* $\hat{E}(r,\delta,\Re)$ *under which we have the identity of the maximal operator and classical values,*

$$\hat{D}_{Max}^{HM} \equiv \hat{D}_{Max}^{Classical}, \qquad (4.C.26)$$

as it is the case for the above orientation of the unit vectors, and values

$$g_{11} = g_{22} = 1, \quad g_{33} = \tfrac{1}{2}(\lambda\lambda'^{-1} + \lambda^{-1}\lambda') = \sqrt{2}. \qquad (4.C.27)$$

In conclusion, the traditional obstacles against the completion of quantum mechanics (von Neumann theorem, Bell's inequality and all that) are inapplicable under isotopies, thus opening the door for its isotopic completion. The latter results to be considerably, although not exactly along the E–P–R argument.

Note that the studies of this subsection refer to closed–isolated systems with conventional total conservation laws as a condition to apply isotopic techniques. The study of the further generalization of the isorealism into the *genorealism* possessing the broader Lie–admissible structure for open–nonconservative systems is left to the interested reader.

APPENDIX 4.D: ISOTOPIES AND ISODUALITIES OF BOHM'S EXTERIOR MECHANICS

In the main text of these volumes we have assumed quantum mechanics as being exactly valid for the exterior problem in vacuum and constructed a covering mechanics for interior conditions via isotopic and genotopic methods.

In this appendix we would like to point out that the isotopic and genotopic techniques also apply to any theory other than quantum mechanics. An illustrative example is given by *Bohm's exterior mechanics* [28] (see also the review in ref. [20]) which is a reformulation of quantum mechanics, thus being an "exterior theory" in vacuum according to our terminology, aiming at full determinism via the following three basic assumptions:

I) The conventional 3N–dimensional Schrödinger equation on time t, the the space coordinates r and momenta p

$$i \hbar \, \partial_t \, \psi(t, r) = H(t, r, p) \, \psi(t, r), \qquad (4.D.1)$$

II) A deterministic law on the actual coordinates R(t) of the particles as a function of the probability current J computed at R,

$$\frac{dR_k(t)}{dt} = \frac{J_k(t, R)}{|\psi(t, R)|^2}, \qquad (4.D.2a)$$

$$J = i\hbar(\psi \nabla\psi^\dagger - \psi^\dagger \nabla\psi) / 2m, \qquad (4.D.2b)$$

III) The assumption that the probability for the particles to be located at R is equal to

$$\mathcal{P}(R) = |\psi(t, R)|^2 = |\hat{\psi}^\dagger(t, R) \psi(t, R)|, \qquad (4.D.3)$$

(see ref.s [29,30] for details).

The isotopies of Bohm's mechanics are straightforward and given by

I') The 3N-dimensional isochrödinger equation on time t, the the space coordinates r and momenta p

$$i\hbar \partial_t \hat{\psi}(t, r) = i\hbar \partial_t \hat{\psi}(t, r) = H * \hat{\psi}(t, r) = H(t, r, p) T(t, p, \dot{p}, ...) \hat{\psi}(t, r), \quad (4.D.4)$$

II') A deterministic law on the actual coordinates R(t) of the particles as a function of the *isoprobability current* J computed at R,

$$\frac{\partial R_k(t)}{\partial t} = \hat{1}_t \frac{dR_k(t)}{dt} = \frac{J_k(t, R)}{|\hat{\psi}(t, R)|^2}, \qquad (4.D.5a)$$

$$J = i\hbar(\hat{\psi} * \nabla\hat{\psi}^\dagger - \hat{\psi}^\dagger * \nabla\hat{\psi}) / 2m, \qquad (4.D.5a)$$

III') The assumption that the *isoprobability* for the particles to be located at R is equal to

$$|\hat{\psi}(t, R)|^2 = |\hat{\psi}(t, R) T(t, p, \dot{p}, ...) \hat{\psi}(t, R)|. \qquad (4.D.6)$$

It is easy to see that the above isotopies, being axiom-preserving, do preserve the original deterministic character of Bohm's mechanics and merely perform its transition to interior problems. The isodualities of Bohm's interior mechanics are then consequential.

APPENDIX 4.E: AXIOMATIC REFORMULATION OF SQUEEZED STATES UNCERTAINTIES

Yet another intriguing generalization of quantum mechanics known under the name of *squeezed states* has been the subject of considerable study (see, e.g., ref.s [30] and papers quoted therein. To illustrate the possibilities offered by hadronic mechanics in this additional field, in this appendix we present the axiomatic reformation of only one study, the generalized uncertainties under

squeezed states studied by McDermott and Solomon [31].

Consider the conventional Bosonic algebra

$$a\,a^\dagger - \hat{a}^\dagger a = I ,\tag{4.E.1}$$

With *coherent oscillator states* $|\lambda>$ on a Hilbert space $\mathcal{H}$

$$a\,|\lambda> = \lambda\,|\lambda>, \quad |\lambda> = e^{(|\lambda|^2)^{-\frac{1}{2}}}e^{\lambda\,a^\dagger}\,|0> .\tag{4.E.2}$$

Introduce the conventional quantities

$$x = 2^{-\frac{1}{2}}(a + a^\dagger), \quad p = i\,2^{-\frac{1}{2}}(a^\dagger - a).\tag{4.E.3}$$

Their variances are given by the familiar expressions

$$(\Delta x)^2 = <x^2> - <x>^2, \quad (\Delta p)^2 = <p^2> - <p>^2,\tag{4.E.4}$$

with uncertainties in the vacuum state

$$\Delta x|_0 = 2^{-\frac{1}{2}}, \quad \Delta p|_0 = 2^{-\frac{1}{2}}, \quad \Delta x|_0\,\Delta p|_0 = \tfrac{1}{2} .\tag{4.E.5}$$

McDermott and Solomon [31] have recently computed the corresponding generalized uncertainties for the q-deformed oscillator of the type

$$a\,a^\dagger - q\,a^\dagger a = I , \quad q \in R(n,+,\times), \quad q \neq 0 ,\tag{4.E.6}$$

as well as of more general types not considered here for brevity. In this case the *q-coherent states* can be defined by

$$a\,|\lambda>_q = \lambda\,|\lambda>_q, \quad |\lambda>_q = e_q^{(|\lambda|^2)^{-\frac{1}{2}}}e_q^{\lambda\,a^\dagger}\,|0>_q ,\tag{4.E.7}$$

where the *q-exponential* is given by [22]

$$e_q{}^x = \sum_{k=0,1,\dots,\infty} x^2 / [n]! ,\tag{4.E.8}$$

and the expression [n] is given by

$$[n+1] = I + n[n], \quad [n] = \sum_{k=0,1,\dots,n-1} (n-I)!\,/\,k! .\tag{4.E.9}$$

In this case the conventional derivative must be generalized into the *q-derivative* [32]

$$D_q = \frac{I}{x}[x\frac{d}{dx}], \quad D_q\,x^n = [n]\,x^{n-1}, \quad \text{etc.}\tag{4.E.10}$$

Consider then the x and p coordinates as in Eq.s (4.E.3) with variances as in Eq.s (4.E.4). Then McDermott and Solomon [loc. cit.] obtain in the q-deformed states

$$<x>_q = <\lambda\,|\,2^{-\frac{1}{2}}(\hat{a}^\dagger + a)|\lambda>_q = 2^{-\frac{1}{2}}(\lambda + \bar{\lambda}),\tag{4.E.11a}$$

$$< x^2 >_q \; = \; < \lambda \, | \tfrac{1}{2} \, (\, a^{\dagger 2} \, + \, a^2 \, + \, a^{\dagger} a \, + \, a \, a^{\dagger}) \lambda >_q \; =$$

$$= \; \tfrac{1}{2} \, [\, (\bar{\lambda} \, + \, \lambda \,)^2 \, + \, 1 \, - \, \epsilon_q \, | \lambda |^2 \,] \; , \tag{4.E.11b}$$

$$\epsilon_q \; = \; 1 \, - \, < \lambda \, | \, (q + 1) \, | \lambda >_q \, . \tag{4.E.11c}$$

As a result,

$$(\Delta x_q)^2 \; = \; (\Delta p_q)^2 \; = \; \tfrac{1}{2} \, (\, 1 \, - \, \epsilon_q | \lambda |^2) \; , \tag{4.E.12}$$

thus yielding the *q–uncertainties* on the ground state [31]

$$\Delta x_q \, \Delta p_q \; = \; \tfrac{1}{2} \, (\, 1 \, - \, \epsilon_q \, | \lambda |^2) \; < \; \tfrac{1}{2} \, , \tag{4.E.13}$$

which verify the general rule

$$\Delta x_q \, \Delta p_q \; = \; \tfrac{1}{2} < \lambda \, | \, (\, x \, p \, - \, p \, x \,) | \lambda >_q \; = \; \tfrac{1}{2} < \lambda \, | \, [\, x , p \,] | \lambda >_q \; = \; \tfrac{1}{2} \, (\, 1 \, - \, \epsilon_q \, | \lambda |^2) \, . \tag{4.E.14}$$

As one can see, the q–uncertainties (4.E.13) are *smaller* than the corresponding conventional value (4.E.5).

However, the above q–deformation are only valid at a fixed value of time because under time evolution they are afflicted by a number of problematic aspects for physical consistency studied in App. 1.7.9.A.1, 11.3.C and 11.4.A.

We therefore re-elaborate the above results with the Lie-admissible branch of hadronic mechanics (Sect.s 11.3.3 and 11.34.3). Let us begin by recalling the two possible axiomatically consistent forms, the Lie–admissible structure for motion forward to future time

$$\hat{a} < \hat{a}^{\dagger} \, - \, \hat{a} \dagger > \hat{a} \; := \; \hat{a} \, {}^{<}Q(q, ...) \, \hat{a}^{\dagger} \, - \, \hat{a}^{\dagger} \, Q^{>}(q, ...) \, \hat{a} \; = \; 1^{>} \; = \; (\, Q^{>} \,)^{-1}, \tag{4.E.15}$$

and that forward from past times

$$\hat{a} < \hat{a}^{\dagger} \, - \, \hat{a} \dagger > \hat{a} \; := \; \hat{a} \, {}^{<}Q(q, ...) \, \hat{a}^{\dagger} \, - \, \hat{a}^{\dagger} \, Q^{>}(q, ...) \, \hat{a} \; = \; {}^{<}1 \; = \; (\, {}^{<}Q \,)^{-1} \, . \tag{4.E.16}$$

where ${}^{<}Q(q, ...)$ and $Q^{>}(q, ...)$ are unrestricted integro–differential operators of Kadeisvili Class I (with a well behaved, nowhere null Hermitean component) depending on the parameter q and any needed additional quantity, under the restriction of admitting a conjugation (c)

$$Q^{>} \; = \; (\, {}^{<}Q \,)^{\dagger} \, . \tag{4.E.17}$$

As the reader will recall from Sect. 11.4.3, the latter condition is *necessary* for physical consistency, such as preservation of Hermiticity, causality and other properties under nonconservative conditions, axiomatization of irreversibility.

It is then easy to see that *McDermott–Solomon studies* [31] *deal with forward motion from past time* with genoaxiomatization

$$a\,a^{\top} - q\,a^{\top}\,a = a^{<}Q\,a^{\top} - a^{\top}\,Q^{>}\,a = [\,^{<}Q\,]^{-1} = 1\,, \quad {}^{<}Q = 1\,, \quad Q^{>} = q\,, \quad (4.E.18)$$

and we shall write

$$^{<}\hat{a}\,<\,^{<}\hat{a}^{\dagger} - \,^{<}\hat{a}^{\dagger}\,>\,^{<}\hat{a} = \,^{<}1 = 1\,. \tag{4.E.19}$$

In fact, to deal with forward motion to future time the correct axiomatization should read

$$a\,a^{\dagger} - q\,a^{\dagger}\,a = a^{<}Q\,a^{\dagger} - a^{\dagger}\,Q^{>}\,a = [\,Q^{>}\,]^{-1} = q^{-1}1\,, \quad {}^{<}Q = 1\,, \quad Q^{>} = q\,, \quad (4.E.20)$$

in which case we shall write

$$\hat{a}^{>}\,<\,\hat{a}^{>\dagger} - \hat{a}^{>\dagger}\,>\,\hat{a}^{>} = 1^{>} = q^{-1}\,1\,. \tag{4.E.21}$$

The fundamental genounit of studies [21] is therefore the conventional unit $^{<}1 = 1$. This implies that in the axiomatic Lie-admissible formulation of deformations (4.E.6) the formalism of quantum mechanics remains unchanged, including fields, Hilbert spaces, exponentials, derivatives, etc.

This implies that results [4.E.2) holds with *genobasis for forward motion from past time*

$$|\lambda\,>^{(<)} = e^{(\,|\lambda|^{2}\,)^{-\frac{1}{2}}}\,e^{\lambda\,^{<}\hat{a}^{\dagger}}\,|\,0\,>^{(<)}\,, \tag{4.E.22}$$

defined on a *conventional* Hilbert space, over *conventional* fields $R(n,+,\times)$, with *conventional* derivative, etc.

We then have the expressions on the ground genostate

$$< x\,>^{(<)} = <\lambda\,|\,<[\,2^{-\frac{1}{2}}\,(\,^{<}\hat{a}^{\dagger} + \,^{<}\hat{a}\,)\,]<|\lambda\,>^{(<)} = 2^{-\frac{1}{2}}\,(\lambda + \bar{\lambda})\,, \tag{4.E.23a}$$

$$< x^{2}\,>^{(<)} = <\lambda\,|\,<\{\tfrac{1}{2}\,[\,(\,^{<}\hat{a}^{\dagger})^{2} + (\,^{<}\hat{a})^{2} + \,^{<}\hat{a}^{\dagger}\,<\,\hat{a} + \,^{<}\hat{a}\,<\,^{<}\hat{a}^{\dagger})\,]\}<|\lambda\,>^{(<)} =$$

$$= <\lambda\,|\,<\{\tfrac{1}{2}\,[\,(\,^{<}\hat{a}^{\dagger} + \,^{<}\hat{a})^{2} + 1 - (\,q - 1)\,^{<}\hat{a}^{\dagger}\,<\,\hat{a}\,)<|\lambda\,>^{(<)} =$$

$$= \tfrac{1}{2}\,[\,(\bar{\lambda} + \lambda)^{2} + 1 + (q-1)\bar{\lambda}\lambda\,]\,, \tag{4.E.23b}$$

thus yielding the *genouncertainties for forward motion from past time* on the ground genostate

$$\Delta x^{(<)}\,\Delta p^{(<)} = \tfrac{1}{2}\,\big|\,[\,1 + (q-1)\bar{\lambda}\lambda\,]\,\big|\,. \tag{4.E.24}$$

The difference with result (4.E.13) is self-explanatory.

We now study the forward motion to future time with basic genoalgebras (4.E.21). In this case the fundamental unit of the theory is the genounit $1^{>} = q^{-1}\,1 \neq 1$. This implies the necessary lifting of the entire formalism of quantum mechanics into a form admitting of $1^{>}$ as the correct left and right unit, including: genofields $\hat{R}^{>}(\hat{n}^{>},+,>)$ with forward genonumbers $\hat{n}^{>} = n1^{>}$ and related

genomultiplication $\hat{n}^> > \hat{m}^> := \hat{n}^>$ q $\hat{m}^> = (nm)\hat{1}^>$; the genohilbert space and related genobasis

$$\mathcal{H}^>: \quad <\hat{\lambda}|\hat{\lambda}>^{(>)} = <\hat{\lambda}|>|\hat{\lambda}>\hat{1}^> = <\hat{\lambda}|q|\hat{\lambda}>^{(>)}q^{-1} \in \hat{R}^>(\hat{n}^>+,>), \quad (4.E.25a)$$

$$|\hat{\lambda}>^{(>)} = q^{-\frac{1}{2}}|\lambda>, \quad <\hat{\lambda}|>|\hat{\lambda}>^{(>)} = 1. \quad (4.E.25b)$$

Note that, since q is a constant, the genoinner product trivially coincides with the conventional product. Yet the genotopy is *necessary* to ensure the Hermiticity-observability of the *genooccupation number operator* $\hat{N}^> = \hat{a}^{>\dagger}>\hat{a}^> = $ q $\hat{a}^{>\dagger}$ $\hat{a}^>$ as well as its preservation at all future times (Sect.s 1.6.3, 11.3.3 and 11.4.3).

By defining the genocreation and genoannihilation algebras in the form

$$\hat{a}^> > |\hat{\lambda}>^{(>)} := \hat{a}^> q|\hat{\lambda}>^{(>)} = \hat{\lambda}^> > |\lambda>^{(>)} = \lambda|\hat{\lambda}>^{(>)}, \quad (4.E.26a)$$

$$\hat{a}^{>\dagger} > |\hat{\lambda}>^{(>)} = \bar{\lambda}|\hat{\lambda}>^{(>)}, \quad (4.E.26b)$$

it is easy to see that

$$<x>^{(>)} = <\hat{\lambda}|>[2^{-\frac{1}{2}}(\hat{a}^{>\dagger} + \hat{a}^>)]>|\hat{\lambda}>^{(>)} = 2^{-\frac{1}{2}}(\lambda + \bar{\lambda}), \quad (4.E.27a)$$

$$<x^2>^{(>)} = <\hat{\lambda}|>\{\tfrac{1}{2}[(\hat{a}^{>\dagger})^2 + (\hat{a}^>)^2 + \hat{a}^{>\dagger}>\hat{a}^> + \hat{a}^> > \hat{a}^{>\dagger})]>|\lambda>^{(>)} =$$

$$= <\hat{\lambda}|>\{\tfrac{1}{2}[(\hat{a}^{>\dagger} + \hat{a}^>)^2 + 1 + (q-1)\hat{a}^{>\dagger}>\hat{a})\}>|\hat{\lambda}>^{(>)} =$$

$$= \tfrac{1}{2}[(\bar{\lambda} + \lambda)^2 + 1 + (q-1)\lambda\bar{\lambda}]|, \quad (4.E.27b)$$

where absolute value is implied whenever needed and we have used the reinterpretation $\hat{a}^> < \hat{a}^{>\dagger} = q^{-1} \hat{a}^> > \hat{a}^{>\dagger}$. This yields the *genouncertainties for forward motion to future time* on the ground genostate

$$\Delta x^{(>)} \Delta p^{(>)} = \tfrac{1}{2}[1 + (q^{-1} - 1)\lambda\bar{\lambda}], \quad (4.E.28)$$

which evidently coincide with the preceding ones (4.E.24) for forward motion from past time, as expected for consistency.

As a further comment, the reader should keep in mind that, despite the achievement of the invariance under nonunitary time evolution, genodeformations (4.E.19) and (4.E.21) and related genouncertainties are still afflicted by problematic aspects of physical character because the Hamiltonian is not Hermitean–observable due to the lack of Hermitean conjugation between the products $\hat{a}>\hat{a}^\dagger = \hat{a}\hat{a}^\dagger$ and $\hat{a}<\hat{a}^\dagger = q\hat{a}\hat{a}^\dagger$.

REFERENCES

1. R. M. SANTILLI, Hadronic J. 1, 574 (1978)

2. R. M. SANTILLI, Hadronic J. **4**, 642 (1981)
3. R. M. SANTILLI, Comm. Theor. Phys. **3**, 24 (1994) (see also Hadronic J., Suppl. **4B**, issue no., 1 (1989))
4. R. M. SANTILLI, Lett. Nuovo Cimento **37**, 337 (1983)
5. R. M. SANTILLI, Hadronic J. **7**, 1680 (1984)
6. 23. A. JANNUSSIS et al., Lett. Nuovo Cimento **30**, 123 (1980); **31**, 177 (1981); **34**, 375 (1982); **37**, 119 (1983); J. Phys. **A24**, L775 (1991); **A25**, L329 (1992); **A26**, L233 (1993); **14**, 277 (1991); and in press; Hadronic J. **5**, 1923 (1982); **6**, 1653 (1983); **7**, 947 (1984); **14**, 257 (1991); and **16** (1993) in press
7. A. K. ARINGAZIN, Hadronic J. **13**, 183 (1990)
8. L. NTIBASHIRAKANDI and D. K. CALLEBAUT, Hadronic J. **17**, 1 (1994)
9. F. SELLERI, Editor, *Quantum Mechanics versus Local Realism: The Einstein–Podolsky–Rosen Paradox*, Plenum, New York (1988)
    D. HOME and F. SELLERI, Rivista Nuovo Cimento **14**, no. 9, p. 1 (1991)
10. A. EINSTEIN, B. PODOLSKY and N. ROSEN, Phys. Rev. **47**, 777 (1935)
11. R. M. SANTILLI, submitted to Ann. Fond. l. de. Broglie
12. E. MERZBACHER, *Quantum Mechanics*, Wiley and Sons, New York   (1961);
    P. ROMAN, *Advanced Quantum Theory*, Addison-Wessley, Reading   M A (1965)
13. R. W. HASSE, J. Math. Phys. **16**, 2005 (1975); I. BIALYNICKI–BIRULA, and J. MYCHIELSKI, Ann. Phys. **100**, 62 (1976); N. GISIN, J. Phys. A **14**, 2259 (1981); D. SCHUCH, K.–M.CHUNG and H. HARTMANN, J. Math. Phys. **24**, 1652 (1983); H.–D. DOEBNER and G. A. GOLDIN, Phys. Lett. A **162**, 397 (1992);
14. M. NISHIOKA, Hadronic J. **7**, 1158 (1984)
15. A. JANNUSSIS, G. BRODIMAS and R. MIGNANI, J. Phys. A: Math. Gen. **24**, L775 (1991)
16. B. VELJANOSKI and A. JANNUSSIS, Hadronic J. **10**, 53 (1987)
17. A. JANNUSSIS and R. MIGNANI, Physica A **152**, 569 (1988)
18. A. JANNUSSIS, L. C. PAPALOUCAS, P.I.TSILIMIGRAS and N. R. C. DEMOCRITOS, Lett. Nuovo Cimento **42**, 83 (1985)
19. R. M. SANTILLI, Lett. Nuovo Cimento **51**, 570 (1967)
20. T. L. CURTRIGHT, D. B. FAIRLIE and Z. K. ZACHOS Editors, *Quantum Groups,* World, Scientific (1991); MO–IN GE and BAO–HENG ZHAO, Editors, *Introduction to Quantum Groups and Integrable Massive Models of Quantum Field Theory,* World Scientific (1991); Yu. F. SMIRNOV and R. M. ASHEROVA, Editors, *Proceedings of the Fifth Workshop Symmetry Methods in Physics,* JINR, Dubna, Russia (1992)
21. D. F. LOPEZ, Hadronic J. **16**, 429 (1993)
22. A. JANNUSSIS et al., Hadronic J. **3**, 1622 (1982), **5**, 1923 (1982); **6**, 1653 (1983); **7**, 1515 (1984); **10**, 153 (1987); Lett. Nuovo Cimento **30**, 23 (1981); **34**, 375 (1982); Nuovo Cimento B **94**, 29 (1986); J. Phys. A **24**, L775 (1991); **25**, L329 (1992)
23. A, JANNUSSIS et al. Problematic aspects of q–deformed creation and annihilation operators and their axiomatic resolution<, Preprint, Univ. of Patras, Greece (1994)
24. E. WITTEN, Nucl. Phys. **B188**, 513 (1981); H. HI, Prof. Theor. Phys. **72**, 813 (1984); J. BECKERS and N. DEBERGH, Phys. Rev. **D42**, 1255 (1990)

25. J. VON NEUMANN, *The Mathematical Foundations of Quantum Mechanics*, Princeton Univ. Press, Princeton, N. J. (1955)

26. J. S. BELL, Physics 1, 195 (1965)

27. D. BOHM, *Quantum Theory*, Dover Publications, New York (1979)

28. D. BOHM, in *Quantum Theory and measurement*, J. A. Wheeler and W. H. Zurek, Editors, Princeton Univ. Press, Princeton, N.J. (1983)

29. D. Z. ALBERT, *Quantum Mechanics and Experience*, Harvard Univ. Press, Cambridge, MA (1992)

30. A. JANNUSSIS et al, Nuovo Cimento **103B**, 17 and 537 (1989); **104B**, 33 and 53 (1989); **108B**, 57 (1993); Phys. Lett. **A132**, 324 (1988); S. SEBAWE ABDALLA et al., Physica **A163**, 822 (1990); **A202**, 301 (1994); Phys. Rev. **A48**, 1526 and 3174 (1993); J. Modern Optics **39**, 771 and 1067 (1992); **40**, 441, 1351 and 1369 (1993); Phys. Lett. **A181**, 341 (1993)

31. R. J. McDERMOTT and A. I. SOLOMON, Double squeezing in generalized q-coherent states, J. Phys. A, in press (1994)

32. H. EXTON, *q-Hypergeometric Functions and Appplications*, Ellis Horwood, Chichester, Emngland (1983)

NOTE ADDED IN THIS SECOND EDITION

A presentation of the basic axioms and physical laws of hadronic mechanics with application and experimental verification in the Cooper pair in superconductivity has recently appeared in

A. O. E. ANIMALU and R. M. SANTILLI, Inter. J. Quantum Chem. **29**, 175 (1995)

5: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF SCHROEDINGER'S REPRESENTATION

5.1: STATEMENT OF THE PROBLEM

After having identified the basic axioms and laws of hadronic mechanics, the next logical step is a study of methods essential for applications, the isotopies, genotopies and isodualities Schrödinger's representation (see, e.g., ref.s [1] for the conventional case).

It is at this point where the functional isoanalysis of Ch. I.6 appears in its full light. In fact, the conventional spherical coordinates are inapplicable to the isoschrödinger representation, while the use of the conventional trigonometry leads to a number of generally undetected inconsistencies, the same occurrence holding for the use of the conventional special functions of quantum mechanics, such as Legendre polynomials, spherical harmonics, Bessel's functions, etc.

In a situation of this type, the reader should be aware that the terms "functional isoanalysis" have appeared only a few months ago in Kadeisvili paper [2]. As such, the study of the isoschrödinger representation is at its beginning and so much remains to be done.

In this chapter we shall study the isoscrödinger representation only for Kadeisvili's Classes I and II. The formalism for the remaining classes is intriguing (e.g., for gravitational collapse) but unknown at this writing.

We shall also study the subcase of hadronic mechanics with only one isotopic element T for the isofield $\hat{F}_T$, isoenvelopes $\hat{\xi}_T$ and isohilbert spaces $\mathcal{H}_T$, by omitting the subscript T for simplicity. To avoid gravitational profiles, we shall assume that all space isounits $\hat{1} = T^{-1}$ are independent of coordinates, $\partial\hat{1}/\partial r = 0$. All time isounits $\hat{1}_t$ are assumed to be independent of time, $\partial T_t/\partial t$.

As recalled in Ch. I.2, the isoschrödinger equation for conserved Hamiltonians was first proposed by Myung and Santilli in papers [3] of 1982 with the first treatment via the isohilbert spaces. The same equation was also independently submitted by Mignani [4] although on conventional Hilbert spaces.

This original form of the equation subsequently resulted to be incompatible

with the relativistic formulation of hadronic mechanics. This and other problematic aspects required a further generalization with the time isoderivative, which was reached in memoir [5] of 1989. The isotopic equation of the momentum was achieved for the first time in the same memoir [5] jointly with other aspects, such as the isospherical coordinates. The genoschrödinger equation for nonconserved Hamiltonian was also submitted in papers [3,4], although their formulation on genohilbert spaces appeared first in memoir [5].

No additional study has appeared in print to the author's best knowledge at this writing (early 1994), specifically, in the isotopies of Scrödinger's equations, that is, on *formulations based on the <generalization of the basic unit>* with consequential broadening of fields, metric spaces, Hilbert spaces, etc. Several *applications* of the isoschrödinger equations by a number of authors will be studied in Vol. III.

The generalizations of Schrödinger's equations *based on conventional functional analysis* (that is, based on the conventional unit +1) which have appeared in print throughout this century are so numerous to prevent an outline. We shall therefore quote them only when we are aware of some direct connections with the isoschrödinger or genoschrödinger forms. At any rate, as studied in Sect. 1.7.9, the isoscrödinger's (genoschrödinger's) equations are directly universal for all possible operator systems with conserved total energy (nonconserved energy) and they therefore include the *systems* represented by other generalizations, but not the generalizations themselves.

The author would be grateful to any colleague who cares to bring to his attention existing generalizations of Schrödinger's equations particularly relevant for these studies for proper quotation in future works.

5.2: ISOSCHROEDINGER'S EQUATIONS AND THEIR ISODUALS

As well known, the carrier space of nonrelativistic quantum mechanics is the Kronecker's product[44]

$$E(t,R_t) \times E(r,\delta,R): \quad r = \{r^k\} = \{r^1, r^2, r^3\} = \{x, y, z\}, \quad \delta = \text{diag.} \ (1, 1, 1) , \quad (5.2.1a)$$

$$r^2 = r^i \delta_{ij} r^j = xx + yy + zz \in R(n,+,\times) , \qquad (5.2.1b)$$

where R_t is the field of real numbers representing time, and $E(r,\delta,R)$ is the Euclidean space with space coordinates r and metric δ over the reals $R(n,+,\times)$. Space (5.2.1) originates in a number of ways, the most important one for this presentation being its derivation via a contraction of the Minkowski space for relativistic formulations (see, e.g., ref.s [1]).

The Schrödinger representation for one particle on $E(t,R_t) \times E(r,\delta,\Re)$ over a conventional Hilbert space $\mathcal{3C}$ with states $\psi(t, r)$ and inner product $\int dv \psi^\dagger \psi$ is based

---

[44] We shall use the usual convention on the summation of repeated indices.

on the following equations

$$i \frac{\partial}{\partial t} \psi(t, r) = H \psi(t, r) = \{ \frac{1}{2m} p_i p^j + V(r) \} \psi(t, r) =$$

$$= \{ -\frac{1}{2m} \Delta + V(r) \} \psi(t, r) = E_O \psi(t, r), \quad \hbar = 1, \quad (5.2.2a)$$

$$p_i \psi(t, r) = -i \frac{\partial}{\partial r^i} \psi(t, r) = k_i \psi(t, r), \quad (5.2.2b)$$

$$\Delta = \frac{\partial}{\partial r_i} \frac{\partial}{\partial r^i}. \quad (5.2.2c)$$

Numerous applications then follow for the *exterior particle problem*, that is, particles in point-like approximation while moving in the homogeneous and isotropic vacuum with no appreciable overlapping of their wavepackets.

In this chapter we shall first study the isotopies of Schrödinger's representation, also called *Schrödinger-isotopic* or *isoschrödinger's representation*, which we use for a quantitative representation of the *interior particle problem*, that is, extended, and therefore deformable, particles moving within an inhomogeneous and anisotropic physical medium called "hadronic medium" which is composed by the wavepackets of other particles. This implies conventional local-potential interactions, plus additional interactions that are generally nonlinear in all variables (including the derivatives of the wavefunctions $\partial_i \psi$, $\partial_i \psi^\dagger$), nonlocal (integral) and noncanonical (that is, not representable via the usual Hamiltonian operator $H = K + V$).

The fundamental carrier space is the following Kronecker's product of isospaces

$$\hat{E}(t.\hat{R}_t) \times \hat{E}(r,\hat{\delta},\hat{R}) : \quad \hat{1}_t = T_t^{-1} = b_4^{-2}, \quad b_4 = b_4(r, \dot{r}, \ddot{r}, ...) > 0, \quad (5.2.3a)$$

$$\hat{1} = T^{-1} > 0, \quad T = diag. (b_1^2, b_2^2, b_3^2), \quad b_i = b_i(t, \dot{r}, \ddot{r}, ...) > 0, \quad (5.2.3b)$$

$$\hat{\delta} = T \delta \equiv T, \quad r^2 = \{ r^i \delta_{ij} r^j \} \hat{1} = \{ r^i b_i^2 r^i \} \hat{1} \in \hat{R}(\hat{n},+,\times), \quad (5.2.3c)$$

where one should recall from Vol. I that: $\hat{R}_t$ is the *time isofield*; $\hat{1}_t$ (T_t) is the *time isounit* (*time isotopic element*); $\hat{E}(r,\hat{\delta},\hat{R})$ is the *isoeuclidean space*; $\hat{\delta}$ is the the *isometric*; $\hat{R}(\hat{n},+,\times)$ is the *isofield* of *isoreal numbers* $\hat{n} = n\hat{1}$; r^2 is the *isoseparation*; $\hat{1}$ is the *space isounit*; T is the *space isotopic element*; and the $b's$ are the *characteristic functions of the physical medium considered*.

We should also recall that, in case of an explicit dependence on the local coordinates via a nonlinearity in the wavefunctions and their derivatives, the characteristic functions can be averaged into constants $b^\circ_4 = < b_4 >$ and $b^\circ_i =$

$<b_i>$. This approximation is simple, yet effective because it permits a speedy appraisal of the physical implications of the nonlinearities here considered.

The extension of the theory to include an explicit coordinate–dependence, (thus an explicit dependence on the wavefunctions and their derivatives) is done first in Sect. II.5.4 and then, in a more systematic way, in the relativistic-gravitational treatment of Ch.s II.8 and II.9. As one can see, the formalism remains unchanged because of the structure of the isoderivatives.

Isospace (5.2.3) can also be derived in other ways, the most important one for this analysis being that as the nonrelativistic limit of the isominkowski space of Ch. II.8. In particular, isoschrödinger's formulations without a generalized unit of time $\hat{1}_t$ are incompatible with relativistic hadronic mechanics.

Note that the isoseparation r^2 is, rigorously speaking, an element of the isofield $\hat{R}(\hat{n},+,*)$ and this is the reason for the appearance of the isounit $\hat{1}$ as multiplier in Eq. (5.2.3c). However, such multiplicative term can be ignored in practical applications. Finally, since the isounits of time and space are assumed to be positive–definite, we have the local isomorphisms

$$\hat{R} \approx R_t, \qquad \hat{R}(\hat{n},+,*) \approx R(n,+,*), \qquad \hat{E}(r,\delta,\hat{R}) \approx E(r,\delta,R), \qquad (5.2.4)$$

In this way, the isotopies $E(t,R_t) \times E(r,\delta,\Re) \rightarrow \hat{E}(,t,\hat{R}_t) \times \hat{E}(r,\delta,\hat{R})$ permit nonlinear, nonlocal and noncanonical generalizations of the original space, but always in such a way to preserve the original geometric axioms. Recall finally that each conventional exterior problem admits an infinite number of geometrically equivalent, but physically different interior extensions, represented by the infinitely possible isounits $\hat{1}_t$ and 1.

The understanding of hadronic mechanics requires the knowledge of the need for these infinitely possible realizations of the isounit. One of the reasons for the physical effectiveness of the conventional Schrödinger equation is that *it admits infinitely possible masses* m. One of the reasons for the physical effectiveness of the isotopic covering of Schrödinger's equation is that *it admits infinitely possible isounits* $\hat{1}$ for <each> mass m.

This is due to the fact that each mass m can be realized in an infinite number of different ways because of different sizes, different densities, etc. The expectation that isotopies should restrict the value of the isounit $\hat{1}$ to a specific constant is therefore equivalent to the expectation that the conventional Schrödinger equation should restrict masses to only one single value.

In the physical reality, the value of the mass m is identified *from experimental measures* for <each> given particle. Along essentially the same lines, as we shall see in the applications and verifications of Vol. III, the explicit, numerical value of the isounit will also be identified for <each> given particle *after* the needed experimental knowledge of its physical characteristics and conditions.

We assume the reader is now familiar with the implication of the above

lines. In fact, they imply that the *unique constant* ħ *of quantum mechanics is not replaced by another constant, but with an infinite number of different integrodifferential quantities* ħ *depending on the local conditions at hand.*

The *isoschrödinger's equations* for one particle on $Ê(t,Â_t)×Ê(r,δ̂,Â)$ over the isohilbert space $ε\mathcal{H}$ with isoinner product $∫dv\hat{ψ}^†T̂ψ$ (Axiom I) are given by [3,4,5]

$$i \frac{∂}{∂t} \hat{ψ}(t, r) = i\hat{1}_t(t, p, \dot{p}, ...) \frac{∂}{∂t} \hat{ψ}(t, r) = H * \hat{ψ}(t, r) =$$

$$= H(t, r, p) T(t, p, \dot{p}, ...) \hat{ψ}(t, r) = Ê * \hat{ψ}(t, r) \equiv E \hat{ψ}(t, r), \qquad (5.2.5a)$$

$$p_i * \hat{ψ}(t, r) = p_i T(t, p, \dot{p}, ...) \hat{ψ}(t, r) = -i \hat{∇}_i \hat{ψ}(t, r) = i\hat{1}_{ij} \frac{∂}{∂r_j} \hat{ψ}(t, r) =$$

$$\hat{k}_i * \hat{ψ}(t, r) \equiv k_i \hat{ψ}(t, r), \qquad (5.2.5b)$$

$$H = H^† = H^{\hat{†}}, \quad p = p^† = p^{\hat{†}}, \quad \hat{1}_t = \hat{1}_t^† > 0, \quad \hat{1} = \hat{1}^† > 0, \qquad (5.3.5c)$$

where H is properly written in isoeuclidean space.

The *isoplane wave solution* of Eq.s (5.2.5) has already been indicated in Sect. II.2.4, and it is given by

$$\hat{ψ}(t, r) = \hat{N} * \hat{e}^{i\,k^i r_i} * \hat{e}_{\hat{1}_t}^{-iEt} = N\, e^{i(k^i δ̂_{ij} r^j - E b_4^2 t)} \qquad (5.2.6)$$

with isonormalization (in terms of *conventional* integrals)

$$∫ \hat{ψ}(t, r)^†T \hat{ψ}(t, r)\, dv = 1, \quad \hat{N} = \hat{1}\{(2π)^{-3/2} T^{-1/2}\} = (2π\,T)^{-3/2}. \qquad (5.2.7)$$

The isodual isoschrödinger equations are defined on the *isodual isoeuclidean spaces* [45]

$$Ê^d(t.Â_t^d)×Ê^d(r,δ̂^d,Â^d): \quad \hat{1}_t^d = -\hat{1}_t = -b_4^{-2} < 0, \quad b_4 = b_4(r, \dot{r}, \ddot{r}, ...) > 0, \qquad (5.2.8a)$$

$$\hat{1}^d = (T^d)^{-1} = -\hat{1} < 0, \quad T = -\text{diag.}(b_1^2, b_2^2, b_3^2), \quad \hat{t}_i = b_i(t, \dot{r}, \ddot{r}, ...) < 0, \qquad (5.2.8b)$$

$$δ̂^d = T^d δ \equiv -T, \quad r^{2d} = \{r^i δ̂^d_{ij} r^j\}\hat{1}^d = \{-r^i b_i^2 r^i\}\hat{1}^d \in Â^d(\hat{n}^d,+,*^d), \qquad (5.2.8c)$$

defined on the isofield $Â^d(\hat{n}^d,+,*^d)$ of *isodual isoreal numbers* $\hat{n}^d = n\hat{1}^d$ with negative–definite *isodual isonorm* $|\hat{n}^d|^d = -|n| < 0$ (for $n ≠ 0$), with *isodual isohilbert space* with *isodual isostates* $\hat{ψ}^d = -\hat{ψ}^†$ [46]

---

[45] As we shall see in next chapters, the *identity* of the isotopic and isodual isotopic separations $r^2 \equiv r^{2d}$ will permit a physically consistent representation of antiparticles with negative–definite energy while moving backward in time.

$$\mathfrak{IC}^d : \qquad 1^d \int dv \; \hat{\psi}^\dagger \; T^d \hat{\psi} \; \in \; \hat{C}^d(\hat{c}^d,+,\ast^d) , \qquad\qquad (5.2.9)$$

over the isofield $\hat{C}^d(\hat{c}^d,+,\ast^d)$ of *isocomplex numbers* $\hat{c}^d = c\hat{1}^d = -\hat{c}$ also with negative–definite isodual norm $|\hat{c}^d|^d = |c|\hat{1}^d < 0$, where $|c|$ is the conventional norm.

The *isodual isoschrödinger equations* are then given by

$$i \frac{\partial^d}{\partial t^d} \hat{\psi}^d(t, r) \; = \; i \hat{1}_t{}^d(t, p, \dot{p}, ...) \frac{\partial}{\partial t} \hat{\psi}^d(t, r) \; = \; H \ast^d \hat{\psi}^d(t, r) \; =$$

$$= \; H(t, r, p) \; T^d(t, p, \dot{p}, ...) \hat{\psi}^d(t, r) \; = \; \hat{E}^d \ast^d \hat{\psi}^d(t, r) \; \equiv \; - E \hat{\psi}^\dagger(t, r), \qquad (5.2.10a)$$

$$p_i \ast^d \hat{\psi}^d(t, r) \; = \; p_i \; T^d(t, p, \dot{p}, ...) \hat{\psi}^d(t, r) \; = - i \hat{\nabla}_i \hat{\psi}^d(t, r) \; = \; i \hat{1}_{ij} \frac{\partial}{\partial r_j} \hat{\psi}^\dagger(t, r) \; =$$

$$\hat{k}_i{}^d \ast^d \hat{\psi}^d(t, r) \quad = \; - k_i \hat{\psi}^\dagger(t, r) , \qquad\qquad (5.2.10b)$$

Note the natural emergence of negative–definite energies. Note also the *necessity* for their emergence of the negative definite time.

5.3: SIMPLE EXAMPLES

We shall now study a few representative examples of isoscrödinger equations derived via direct isoquantization of the corresponding classical examples [7].

The reader should be aware that the most important cases are closed–isolated *systems* of particles with nonhamiltonian internal forces. These systems require the study of the *many–body* isoschrödinger equations done later on in Ch. 11.7. In this chapter we are merely studying the *one–particle* case.

The first conceptually important example is that of the *free hadronic particle* in which all potential interactions are null, $V \equiv 0$, and the characteristic b–quantities are constant, resulting in the equations

$$i \hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) \; = \; \frac{1}{2m} p_k \ast p^k \ast \hat{\psi}(t, r) \; = - \; \frac{1}{2m} \hat{\Delta} \hat{\psi}(t, r) , \qquad (5.3.1a)$$

$$\hat{\delta} \; = \; T\delta \; = \; T \; = \; \text{diag.} \; (b^\circ_1{}^2, b^\circ_2{}^2, b^\circ_3{}^2) . \qquad\qquad (5.3.1b)$$

$$r^2 \; = \; (x \; b^\circ_1{}^2 \, x \; + \; y \; b^\circ_2{}^2 \, y \; + \; z \; b^\circ_3{}^2 \, z \,) \hat{1} . \qquad\qquad (5.3.1c)$$

As one can see, *in this case the isotopy* $\quad E(r,\delta,R) \rightarrow \hat{E}(r,\hat{\delta},\hat{R})$ *permits a direct*

---

[46] Note again the *identity* of the isoinner and of the isodual isoinner products.

representation of the actual nonspherical shape of the particle considered via the isotopic element (5.3.1b) *representing the semiaxes of all possible ellipsoidical deformations of the sphere.* [47]

By recalling that, jointly with deformations (5.3.1b) of the semiaxes, there is the inverse deformation of the unit,

$$I = \text{diag.} (1, 1, 1) \;\rightarrow\; \hat{1} \;=\; \text{diag.} (b°_1{}^{-2}, b°_2{}^{-2}, b°_3{}^{-2}), \qquad (5.3.2)$$

we have the following:

Lemma 5.3.1: *All infinitely possible nonspherical shapes of the charge distribution of hadrons are represented by the isosphere in isoeuclidean space (Sect. I.5.2).*

The actual nonspherical shape appears only when the isosphere is projected in our Euclidean space as in Eq. (5.3.1c). However, when considered in its own isoeuclidean space, the isosphere remains perfectly spherical.

To state it explicitly, the isoscrödinger equation has been conceived to reconstruct in isospace the perfect spherical character of particles which in the physical reality are nonspherical.

As we shall see later on in this volume, the above property has fundamental character because it permits the preservation of the rotational, and other space—time symmetries for *nonspherical* charge distributions although at the isotopic level. Note that such a representation exists, first, at the purely *classical* level of ref. [7], and then merely persists in first isoquantization.

It is also easy to verify that

$$\dot{r}_k = p_k / m, \qquad \dot{p}_k \equiv 0, \qquad (5.3.3)$$

thus confirming that the particle is indeed free.

As a specific case, one may think of an ordinary proton in free nonrelativistic motion in empty space. As it is the case for all spinning objects, the proton is not expected to possess a perfectly spherical charge distribution, but rather one of oblate spheroidal type.

In fact, specific calculations done within the context of hadronic mechanics by Nishioka and Santilli [8] suggest the following values for the characteristic b°—quantities of the proton

$$b°_1{}^2 = 1, \qquad b°_2{}^2 = 1, \qquad b°_3{}^2 = 0.6, \qquad (5.3.4)$$

which also permit one (evidently not unique) interpretation of the anomalous magnetic moment of the particle. The point is that the representation of ellipsoid

---

[47] Nondiagonal isotopic elements permit representations of more general shapes, but they are not considered here for brevity.

(5.3.4) via the isosphere restores the perfectly spherical character in isospace.

A second example is provided by an extended particle, such as a proton or a neutron, while still moving in vacuum but now under external *potential* interactions, say, of electromagnetic nature. In this case $V \neq 0$, and the equation can be written

$$i \hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) = = - \frac{1}{2m} \hat{\Delta} \hat{\psi}(t, r) + V(r) \hat{\psi}(t, r) = E \psi(t, r), \qquad (5.3.5a)$$

$$\hat{\delta} = T\hat{\delta} = T' = \text{diag.} [b_1^2(V_o), b_2^2(V_o), b_3^2(V_o)], \qquad (5.3.5b)$$

where the b–quantities can now depend, e.g., on the intensity V_o of the external field.   An important prediction of Eq.s (5.3.5) which is experimentally verifiable is that *hadrons such as protons and neutrons are expected to experience a deformation of the shape of their charge distributions under sufficiently intense external interactions (or collisions)* which is represented by the isotopy of the isotopy

$$\hat{\delta} = \text{diag.} (b^\circ{}_1{}^2, b^\circ{}_2{}^2, b^\circ{}_3{}^2) \quad \rightarrow \quad \hat{\delta} = \text{diag.} [b_1^2(V_o), b_2^2(V_o), b_3^2(V_o)]. \quad (5.3.6)$$

As we shall see in Vol. III, the above deformation of shape implies a necessary alteration of the intrinsic magnetic moment of the particle (without necessarily altering the spin $\frac{1}{2}$) which has been already experimentally measured, although in a preliminary way, via neutron interferometric techniques.

At any rate, perfectly rigid bodies do not exist in the physical reality. As a result, the *amount* of the deformation of the charge distribution of hadrons under given external conditions is open to experimental verifications, but its *existence* is beyond scientific doubts.

A comparison with conventional quantum mechanics is now in order. In essence, hadronic mechanics permits the direct representation at the discrete nonrelativistic level (and prior to any second isoquantization) of:

1) The extended character of the hadron considered;

2) Its actual nonspherical shape; and

3) All the infinitely possible deformations of the original nonspherical shape.

By comparison, for the case of quantum mechanics:

1') The particles are massive points without any extended character;

2') Remnants of the actual shape can be obtained, but only at the level of second quantization via the known *form factors*, with the understanding that nonspherical shapes are not permitted by the basic rotational symmetry; and

3') Deformations of the original perfectly spherical shape would be in conflict with the rotational symmetry because it is known to be a theory solely for *rigid bodies* (see next chapter).

The advances permitted by the transition from quantum to hadronic mechanics are then evident.

Numerous additional examples of *extended particles moving in vacuum under conventional potential interactions* can be worked out by the interested reader along the preceding lines. The relativistic formulation of extended nonspherical charge distributions will be presented in Ch. 11.10. A number of applications and experimental verifications will be considered in Vol. 111.

All the above examples deal with systems composed by *one hadronic particle with conserved energy on a stable orbit.*

We now study *one particle under external nonpotential interactions* which, as such, are necessarily in nonconservative conditions. The ideal methods for such systems are those of the genoschrödinger equations studied later on in this chapter. Nevertheless, it may be useful to study *nonconservative* systems via the *isoschrödinger* formalism as one way to illustrate the interplay between the two representations.

Consider the simplest possible nonconservative system, that of *one particle without potential interactions with* a *linear, velocity-dependent drag force in one dimension.* The classical equations of motion are given by

$$\dot{r} = p/m, \qquad \dot{p} = -\gamma p, \qquad m = 1, \qquad \gamma > 0. \qquad (5.3.7)$$

Numerous *Hamiltonian* representations of this system have been identified in the literature, but they are *not* relevant for this section because they lead to conventional quantum treatments. The treatment significant for this section is the *Hamilton–isotopic* representation computed in ref. [9], p. 102 with isotopic Hamilton–Jacobi equations

$$\partial \hat{A}^\circ / \partial t + H = 0, \qquad H = = \tfrac{1}{2} p e^{\gamma t} p, \qquad (5.3.8a)$$

$$\partial \hat{A}^\circ / \partial r = p e^{\gamma t}. \qquad (5.3.8b)$$

The isoquantization techniques of Ch. 11.2 then lead to the unique operator system with isotopic element $T = \exp\{\gamma t\}$ and isounit $\hat{1} = \exp(-\gamma t)$

$$p * \hat{\psi}(t, r) = p e^{\gamma t} \hat{\psi}(t, r) = -i \hat{\nabla} \hat{\psi}(t, r) = -i \hat{1} \nabla \hat{\psi} = -i e^{-\gamma t} \partial_r \hat{\psi}(t, r), \quad (5.3.9a)$$

$$i \hat{1}_t \partial_t \hat{\psi}(t, r) = H * \hat{\psi}(t, r) = \tfrac{1}{2} p_l * p^l * \hat{\psi}(t, r) = \tfrac{1}{2} e^{-\gamma t} p e^{\gamma t} p e^{\gamma t} \hat{\psi}(t, r) =$$

$$= \tfrac{1}{2} e^{-\gamma t} \hat{1} \partial_r \hat{1} \partial_r \hat{\psi}(t, r) = \tfrac{1}{2} e^{-3\gamma t} \frac{\partial^2}{\partial r^2} \hat{\psi}(t, r) = E(t) \hat{\psi}(t, r), \quad (5.3.9b)$$

where we have taken into account the contraction in isospace

$$p^k * p_k = \delta^{ij} p_i * p_j, \quad (\delta^{ij}) = (\delta_{ij})^{-1}. \qquad (5.3.10)$$

for the one-dimensional case which implies the appearance of the third factor exp{−γt} in Eq.s (5.3.9) (see next section for a detailed treatment).

The verification that the above system is indeed damped requires the calculation of the isoexpectation value of H (Axiom V). We then introduce the isoplanewaves and related isonormalization

$$\hat{\psi}(t, r) = N e^{i(kTr - ET_t t)}, \quad N = (2\pi)^{-3/2} \gamma^{\frac{1}{2}}, \quad \int dv \, \hat{\psi}^\dagger * \hat{\psi} = 1. \quad (5.3.11)$$

The desired isoexpectation value is then given by

$$\hat{<} H \hat{>} = | \tfrac{1}{2} \int dv \, \hat{\psi}^\dagger * e^{-\gamma t} p * p * \psi | = | -\tfrac{1}{2} \, \gamma^3 \, k^2 \, T^2 \int \hat{\psi}^\dagger * \hat{\psi} | = \tfrac{1}{2} e^{-\gamma t} k^2,$$
$$(5.3.12)$$

which confirms the damped character of the Hamiltonian exactly as occurring in classical mechanics [9]. Note also that the isoexpectation value coincides with the isoeigenvalue, as expected.

Note that the time isounit remains undefined in Eq. (5.3.9b). The latter can be identified in the corresponding isoheisenberg representation, in which the system considered is characterized by the equation

$$i \, \partial H / \partial t = [H, H] + \partial H / \partial t = H T H - H T H + \hat{1}_t \partial H / \partial t =$$

$$= \hat{1}_t \partial H / \partial t = \gamma \hat{1}_t H = \gamma \hat{1}_t e^{\gamma t} \tfrac{1}{2} \, p \, p, \quad (5.3.13)$$

thus implying the value

$$\hat{1}_t = -e^{-\gamma t}. \quad (5.3.14)$$

The above simple example illustrates the very essence of hadronic mechanics, that the isotopy of quantum mechanics is a direct representative of nonpotential interactions. In fact, the isotopy of the free particle with space isounit $\hat{1} = exp\{−γt\}$ represents the particle under the *nonpotential* force $F = −γp$.

An important aspect of the above example is that *one extended particle moving within a hadronic medium in the absence of potential interactions, when treated with hadronic mechanics, behaves geometrically like an ordinary free particle.* In fact, the two systems coincide at the abstract axiomatic level.

The linearly damped particle can be extended by the interested reader to the *damped oscillator* provided that, again, one uses the Hamilton−isotopic representation of ref. [9], p. 102,

$$\partial_t \hat{A}^\circ + H = 0, \quad H = \tfrac{1}{2} p \, e^{\gamma t} p + \tfrac{1}{2} r \, e^{\gamma t} r, \quad \partial_r \hat{A}^\circ = p e^{\gamma t}. \quad (5.3.15)$$

The reader can then work out the hadronic representation of all other nonconservative systems represented in ref. [9] via the Birkhoffian formalism

fact, all these systems are of Hamilton-isotopic type.[48]

All these examples deal with nonpotential interactions approximated via the *local* forces. The extension of the results to *nonlocal-integral nonpotential forces* requires the full use of the Hamilton-isotopic mechanics, that is, the transition of the local-differential classical treatment of ref. [9] to the covering integral treatment of ref. [7].[49]

A generalization to an extended particle with semiaxes T_0 = diag. (b_1^2, b_2^2, b_3^2) moving within a hadronic medium with nonlinear, nonlocal and nonpotential interactions, is classically described by the equations in three dimension with motion along the r-direction (ref. [7], p. 86)

$$\dot{r} = p, \qquad \dot{p} = -\gamma\, p^2 \int d\sigma \mathcal{F}(\sigma) \ , \ \ m = 1, \quad \gamma > 0, \tag{5.3.16}$$

where p and $\dot{p}$ refer to the center of mass, the particle experiences a local nonlinear drag force $F = -\gamma\, p^2$, with a nonlocal correction $\int d\sigma \mathcal{F}(\sigma)$ due to its shape σ, where $\mathcal{F}$ is a suitable functional depending on the shape of the body.

A classical representation of the particle considered within the context of the Hamilton-isotopic mechanics and underlying isosymplectic geometry (Sect. II.1.4) is characterized by the isotopic element

$$T = \hat{T}_0 \exp\{\gamma\, r \int d\sigma\, \mathcal{F}(\sigma)\}\, , \quad T_0 = \text{diag.}\,(\,b_1^{\circ\,2}, b_2^{\circ\,2}, b_3^{\circ\,2}\,)\,, \tag{5.3.17}$$

and Hamiltonian

$$H = \tfrac{1}{2}p^2 = \tfrac{1}{2}p\, T\, p, \tag{5.3.18}$$

with corresponding isotopic Hamilton-Jacobi equations of type (II.5.3.8). The operator version is therefore given by Eq.s (5.3.9) with the above integral realization of the isotopic element.

The latter example illustrates that hadronic mechanics is naturally set for the representation of nonlocal-integral and nonpotential interactions. The interested reader can construct a virtually endless variety of additional examples via any desired combination of local-potential interactions represented via the Hamiltonian H, as well as nonlinear-nonlocal-noncanonical interactions represented via the isotopic element T, including the desired realization of

---

[48] The operator form of the general Birkhoffian representations of ref. [9] with $R = \{R_\mu\}$ = $\{pT(t, r, p), rQ(t, r, p)\}$ is presented in App. II.2.C. Note that the reduction $R = \{pT, rQ\} \to = R'$ = $\{p''T', 0\}$ is *singular* and, as such, it cannot be accomplished via the Birkhoffian transforms of gauge and other type.

[49] As the reader may recall, this is due to the fact that monograph [9] is based on the *conventional* symplectic geometry, although in its most general possible exact realization, thus permitting only local-differential systems. Monograph [7] is based instead on the covering isosymplectic geometry by therefore admitting integral systems *ab initio*.

nonlocality among the several possible forms.

In particular, the reader should keep in mind the *direct universality* of hadronic mechanics studied in Ch.s II.I and II.2, that is, the capability of achieving an isotopic operator image of any given (well behaved) classical Newtonian system.

Note the <inapplicability> (and not the <violation>) of Galilei's relativity for the open–nonconservative particle (5.3.16) on numerous independent topological, analytic, algebraic and other counts. The *classical isogalilean relativity* of monographs [7] establish that systems (5.3.16) are invariant under the isogalilean symmetry which is locally isomorphic to the conventional Galilean symmetry. In Ch.II.7 we shall show that this property persists at the operator level in its entirety.

5.4: ISOLAPLACIAN AND ITS ISODUAL

We shall now identify the explicit form of the Hamiltonian operator in hadronic mechanics for the most general possible case of diagonal isometrics with an explicit dependence on the coordinates, $\hat{\delta}_{ij}(t, r, \dot{r}, \ddot{r}, ...) = b_i^2(t, r, \dot{r}, \ddot{r},...) \delta_{ij}$.

The *isomeasure* on $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$ in the case here considered is given by an isotopy of the conventional form (for the latter see, e.g., ref. [6], p. 493)

$$\hat{d}s^2 = \hat{d}r^i \hat{\delta}_{ij}(t, r, \dot{r}, ...) \hat{d}r^j = \sum_k \hat{d}r^k b_k^2(t, r, \dot{r}, ...) \hat{d}r^k , \qquad (5.4.1)$$

where $\hat{d}r^k$ are the *isodifferentials* with related *isoderivatives* (Sect. I.6.7)

$$\hat{d} r^k = T^k_i d r^i \qquad \partial / \hat{\partial} r^k = \hat{1}_k^i \partial / \partial r^i . \qquad (5.4.2)$$

where the reader should keep in mind the geometric differences between the isometric $\hat{\delta}_{ij}$ and the isotopic element T^i_j or isounit $\hat{1}_i^j = [(T^r_s)^{-1}]_i^j$.

The *isolaplacian* on a isoriemannian space then assumes the form

$$\hat{\Delta} = \hat{\delta}^{-1/2} \frac{\partial}{\partial r^i} \hat{\delta}^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial r^j} , \qquad (5.4.3)$$

where

$$\hat{\delta}^{1/2} = (\det \hat{\delta})^{1/2} \quad , \quad (\hat{\delta}^{ij}) = (b_i^{-2} \delta^{ij}) = (\hat{\delta}_{rs})^{-1} . \qquad (5.4.4)$$

When the isometric is of Class I, independent from local coordinates and diagonal, the isolaplacian assumes the simpler form

$$\hat{\Delta} = \hat{\delta}^{-1/2} \frac{\partial}{\partial r^i} \hat{\delta}^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial r^j} = \frac{\partial}{\partial r^k} b_k^{-6} \frac{\partial}{\partial r^k}, \qquad (5.4.5)$$

which can be used in practical calculations.

We must now verify that the above form, as derived from a geometric viewpoint, is indeed compatible with the basic axioms of hadronic mechanics as they have been formulated in Ch. II.3 (for isometrics independent of the local coordinates). For this purpose, recall expression (II.3.1.10) for the linear momentum. Then the *isooperator form of the kinetic energy* for one particle of mass m can be written

$$\hat{K} * \hat{\psi}(t, r) = \frac{1}{2m} p_k * p^k * \hat{\psi}(t, r) = \frac{1}{2m} \delta^{ij} p_i * p_j * \hat{\psi}(t, r) = \quad (5.4.6)$$

$$= - \frac{1}{2m} \delta^{ij} \hat{1}_i{}^p \frac{\partial}{\partial r^p} \hat{1}_j{}^q \frac{\partial}{\partial r^q} \hat{\psi}(t, r) = - \frac{1}{2m} b_k^{-6} \frac{\partial}{\partial r^k} \frac{\partial}{\partial r^k} = - \frac{1}{2m} \hat{\Delta}.$$

The above results confirm that expression (5.4.3) is indeed the correct form of the isolaplacian. The results also illustrate the unavoidability of the isoderivatives for a rigorous treatment of hadronic mechanics.

For the case of one interior particle, Eq.s (5.2.5a) and (5.2.5b) can then be unified into the single expression

$$i \hat{1}_t \frac{\partial}{\partial t} \hat{\psi}(t, r) = H * \hat{\psi}(t, r) = (\hat{K} + \hat{V}) T \hat{\psi}(t, r) =$$

$$= \{ - \frac{1}{2m} \hat{\Delta} + \hat{V}(r) \} \hat{\psi}(t, r) = \hat{E} * \hat{\psi}(t, r) = E \hat{\psi}(t, r) , \quad (5.4.7)$$

where $\hat{V} = V \hat{1}$.

The *isodual isolaplacian* is the image of structure (5.4.3) under the antiautomorphic map $\hat{1} \rightarrow \hat{1}^d = -\hat{1}$, $\hat{\delta} \rightarrow \hat{\delta}^d = -\hat{\delta}$, and can be readily computed by the interested reader.

5.5: ISOSPHERICAL COORDINATES

Recall that the conventional spherical coordinates in Euclidean space $E(r,\delta,R)$

$$x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta, \quad (5.5.1)$$

have the measure [1]

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \quad (5.5.2)$$

It is an instructive exercise for the interested reader to verify that *the use of the conventional spherical coordinates for generalized space* $\hat{E}(r,\hat{\delta},\hat{R})$ *does not permit a separation of the isolaplacian* (5.4.3) *into a radial and an angular part.* It then follows that the isowavefunction $\hat{\psi}(t, r)$ cannot be factorized into a radial and an angular part, $\hat{\psi}(t, r) \neq \hat{U}(t, r)\hat{\ast}\hat{Y}(\theta, \phi)$, under a suitable isoproduct $\hat{\ast}$, thus confirming the need for a generalization of the spherical coordinates themselves.

It is also recommendable to the interested reader to verify that *isolaplacian* (5.4.3) *is not separable into radial and angular component for other conventional coordinate systems, such as parabolic, elliptic, etc.*

The only separations known to this author are those via the images of conventional coordinate systems based *the isotopic generalization of the unit* $1 = $ diag. $(1, 1, 1) \rightarrow \hat{1} = $ diag. $(b_1^{-2}, b_2^{-2}, b_3^{-2})$, as expected for consistency with the mathematical methods underlying structure (5.4.3).

The new coordinates were called *isospherical* [5] to stress the fact that they are not defined for the sphere, but rather for the "isosphere" (Sect. I.5.2) which represents all infinitely possible isotopic deformations of the sphere. In particular, they are such to coincide with the conventional coordinates at the abstract, realization–free level, as it must be the case for all isotopies.

We shall present first the simplest possible derivation of the isospherical coordinates, and then its more general form as needed for the isorepresentation theory of the isotopic $\hat{SO}(3)$ symmetry studied in the next chapter. Introduce the redefinitions of the Cartesian coordinates

$$\bar{x}_1 = x\, b_1, \qquad \bar{y} = y\, b_2, \qquad \bar{z} = z\, b_3, \tag{5.5.3}$$

which are evidently such to reduce the isoinvariant in $\hat{E}(r,\hat{\delta},\hat{R})$ into an identical form in a conventional Euclidean space $E(\bar{r},\delta,R) \neq E(r,\delta,R)$,

$$r^2 = x\, b_1^2\, x + y\, b_2^2\, y + z\, b_3^2\, z \equiv \bar{x}\,\bar{x} + \bar{y}\,\bar{y} + \bar{z}\,\bar{z} = \bar{r}^2. \tag{5.5.4}$$

Next, we introduce the *isospherical angles* (I.6.4.11),

$$\hat{\theta} = b_3\, \theta, \qquad \hat{\phi} = b_1\, b_2\, \phi, \tag{5.5.5}$$

defined to coincide with the angles measured in quantum mechanics θ_{QM} and ϕ_{QM} according to the rules

$$\theta_{QM} = \hat{\theta} = b_3\, \theta_{HM}, \qquad \phi_{QM} = \hat{\theta} = b_1\, b_2\, \phi_{HM}. \tag{5.5.6}$$

As we shall see better in the next chapter, the above isoangles are derivable from the representation theory of the isorotational group and, as such, contain a considerable geometrical significance[50].

---

[50] The full understanding of the isospherical coordinates can be achieved at the level of

The experimental relevance of the isoangles (5.5.4) is indicated in Vol. III where we show that *the deviations of the quantum mechanical quantities* θ_{QM} *and* ϕ_{HM} *from the values needed for exact symmetries are replaced by the quantities of hadronic mechanics* $\hat{\theta}_{HM}$ *and* $\hat{\phi}_{HM}$ *which restore the exact character of the symmetry considered.* For instance, if neutron interferometric techniques for the two spin flips need the value 720° for the exact SU(2) symmetry, but the measured angle is $\theta_{QM} = 716°$, hadronic mechanics yields the angle $\hat{\theta}_{HM} = 720°$ which restores the exact symmetry.

Under these assumptions, the *isospherical coordinates* can be first written in the form [5] (see Fig. 5.5.1)

$$x = r \, b_1^{-1} \sin(b_3 \theta) \cos(b_1 b_2 \phi) \qquad (5.5.7a)$$

$$y = r \, b_2^{-1} \sin(b_3 \theta) \sin(b_1 b_2 \phi), \qquad (5.5.7b)$$

$$z = r \, b_3^{-1} \cos(b_3 \theta), \qquad (5.5.7c)$$

The use of identity (5.5.4) then yields the simplest possible form of the *isomeasure* on $\hat{E}(r, \hat{\delta}, \hat{R})$, that in terms of conventional differentials

$$ds^{\hat{2}} = d\bar{x} \, d\bar{x} + d\bar{y} \, d\bar{y} + d\bar{z} \, d\bar{z} =$$

$$= dx \, b_1^2 \, dx + dy \, b_2^2 \, dy + dz \, b_3^2 \, dz =$$

$$= dr^2 + r^2 \, [\, b_3^2 \, d\theta^2 + b_1^2 \, b_2^2 \sin^2(b_3 \theta) \, d\phi^2 \,] \equiv$$

$$\equiv dr^2 + r^2 \, [\, d\hat{\theta}^2 + (\sin^2 \hat{\theta}) \, d\hat{\phi}^2 \,]. \qquad (5.5.8)$$

The isospherical coordinates in form (5.5.7) are useful for practical calculations, although they are not in their most general possible form because conventional trigonometric functions admit isotopic images. Their formulation in terms of the isotrigonometric functions then permits deeper insights.

Recall from Appendix 1.6.A that the *isopolar coordinates* expressed in terms of the *isotrigonometric functions* in the *isogauss* (x, y)-*plane* with isotopic element $T = \text{diag.} \, (b_1^2, b_2^2)$ are given by

$$x = r \, \hat{\cos} \, \hat{\phi} = r \, b_1^{-1} \cos[\,(b_1 b_2)\phi\,], \qquad (5.5.9a)$$

$$y = r \, \hat{\sin} \, \hat{\phi} = r \, b_2^{-1} \sin[\,(b_1 b_2)\phi\,], \qquad (5.5.9b)$$

and verify the *isopythagorean theorem*

---

the isoriemannian geometry (Sect. 1.5.5) from which one can see that the ellipsoids are indeed geodesics trajectories for the rotational symmetry, of course, at the isotopic level.

$$x \, b_1^2 \, x \; + \; y \, b_2^2 \, y \; = \; r^2 (\, b_1^2 \, \hat{\cos}^2 \, \hat{\phi} \; + \; b_2^2 \, \hat{\sin}^2 \, \hat{\phi} \,) \; = \; r^2 . \qquad (5.5.10)$$

In particular, the isotopic element of the above isotrigonometric functions is *not* that of the isogauss plane, but rather the element T in the isoexponentiation (I.6.A.21)

$$\hat{e}_{\phi}^{\, i \phi} = \hat{1}_{\phi} \, e^{i \, T \, \phi} = \hat{1}_{\phi} \, e^{i \, b_1 \, b_2 \, \phi} = \hat{1}_{\phi} \, (\, \cos (\, b_1 \, b_2 \, \phi) \; + \; i \sin (\, b_1 \, b_2 \, \phi) \,] \; =$$

$$= \; b_2^{-1} \, \hat{\cos} \, \hat{\phi} \; + \; i \, b_1^{-1} \, \hat{\sin} \, \hat{\phi} , \qquad (5.5.11a)$$

$$T \; = \; b_1 \, b_2 , \qquad\qquad \hat{1}_{\phi} \; = \; b_1^{-1} \, b_2^{-1} . \qquad (5.5.11b)$$

The next issue is the appropriate isotrigonometric formulation of the remaining terms in θ. At this point there is the emergence of a further degree of freedom which is "hidden" in the *isotopic* theory itself and completely absent in quantum mechanics.

By inspecting structure (5.5.7) one could conclude that the isogauss plane for the polar angle has the isotopic element $T = \text{diag.} \, (b_3^2, \, 1)$. However, one can also introduce the following redefinition of the isotopic element in three-dimensional space

$$b_1 \; = \; B_{22} \, B_{11} , \quad b_2 \; = \; B_{22} \, B_{12} , \quad b_3 \; = \; B_{21} , \qquad (5.5.12a)$$

$$B_{21} \, B_{22} \; = \; B_{11} \, B_{12} . \qquad (5.5.12b)$$

with solution

$$B_{11}^{\;3} \; = \; b_1^2 \, b_3 \, / \, b_2 , \quad B_{12}^{\;3} = \; b_2^2 \, b_3 \, / \, b_1 , \quad B_{22}^{\;3} \; = \; b_1 \, b_2 \, / \, b_3 , \; B_{21} = b_3, \quad (5.5.13a)$$

$$B_{11}^{\;3} \, B_{12}^{\;3} \; = \; B_{22}^{\;3} \, B_{21}^{\;3} \; = \; b_1 \, b_2 \, b_3^2 , \qquad (5.5.13b)$$

under which we can introduce the *general isospherical coordinates*

$$x \; = \; r \, \hat{\sin} \, \theta \; \hat{\cos} \, \hat{\phi} \; = \; r \, [\, B_{22}^{-1} \, \sin (\, B_{21} \, B_{22} \, \theta \,) \,] [\, B_{11}^{-1} \, \cos (\, B_{11} \, B_{12} \, \phi) \,] , \; (5.5.14a)$$

$$y \; = \; r \, \hat{\sin} \, \theta \; \hat{\sin} \, \hat{\phi} \; = \; r \, [\, B_{22}^{-1} \, \sin \; (\, B_{21} \, B_{22} \, \theta \,) [\, B_{12}^{-1} \, \sin (\, B_{11} \, B_{12} \, \phi) \,] , \quad (5.5.14b)$$

$$z \; = \; r \, \hat{\cos} \, \theta \; = \; r \, B_{21}^{-1} \, \cos \; (\, B_{21} \, B_{22} \, \theta \,) , \qquad (5.5.14c)$$

with isoidentity

$$x \, b_1^2 \, x \; + \; y \, b_2^2 \, y \; + \; z \, b_3^2 \, z \; = \qquad\qquad (5.5.15)$$

$$= \; r^2 (\, B_{22}^2 \, B_{11}^2 \, \hat{\sin}^2 \, \theta \, \hat{\cos}^2 \, \hat{\phi} \; + \; B_{22}^2 \, B_{12}^2 \, \hat{\sin}^2 \, \theta \; \hat{\sin}^2 \, \hat{\phi} \; + \; B_{21}^2 \, \hat{\cos}^2 \, \theta \,) \; = \; r^2 ,$$

(see Fig. 5.5.1 for more details).

Redefinitions (5.5.14) are important for the isorepresentation theory studied in the next chapter because they permit the identification of the values of the *isospherical isotopic elements* and *isounits* for the θ and ϕ angles

$$T_\theta = B_{21} B_{22} \equiv T_\phi = B_{11} B_{12} , \qquad (5.5.16a)$$

$$1_\theta = B_{21}^{-1} B_{22}^{-1} \equiv T_\phi = B_{11}^{-1} B_{12}^{-1} , \qquad (5.5.16b)$$

with evident computational advances.

The "hidden" isotopic degree of freedom in the transition from the decomposition

$$T = \text{diag} (b_1^2, b_2^2, b_3^2) = \text{diag} (b_1^2, b_2^2) \times \text{diag} (b_3^2, 1) , \qquad (5.5.17)$$

ISOSPHERICAL COORDINATES

$$\hat{\theta} = b_3 \theta$$

$$\hat{\phi} = b_1 b_2 \phi$$
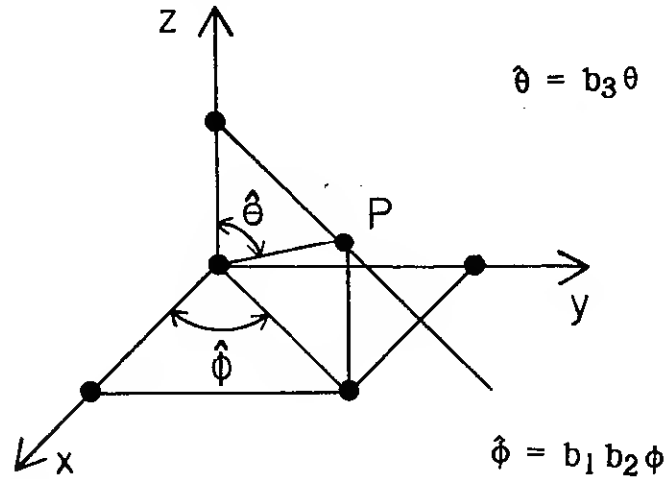
FIGURE 5.5.1: A schematic view of the coordinates of a point on the isosphere in isoeuclidean space in three–dimension, Eq.s (5.5.14). As one can see, the representation coincides at the abstract level with the conventional one in Euclidean space with coordinates (5.5.1). However, the projection of the former in the space of the latter exhibits the ellipsoidical character of the isosphere.

to the more general form underlying structure (5.5.14)

$$T = \text{diag} (b_1^2, b_2^2, b_3^2) = \text{diag.} (B_{11}^2, B_{12}^2) \times \text{diag} (B_{21}^2, B_{22}^2) , \qquad (5.5.18a)$$

$$b_1^2 = B_{22}^2 B_{11}^2 , \qquad b_2^2 = B_{22}^2 B_{12}^2 , \qquad b_3^2 = B_{21}^2 , \qquad (5.5.18b)$$

is also important for the isotopies of partial wave expansions and of the scattering theory, as we shall see later on in this volume.

5.6: SEPARATION OF THE ISOLAPLACIAN AND HADRONIC ANGULAR MOMENTUM

In quantum mechanics, the conventional spherical coordinates (5.5.1) permit the separation of the Laplacian into the familiar radial and spherical/angular parts

$$\Delta = \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z} =$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{1}{r^2}L^2(\theta,\phi), \qquad (5.6.1)$$

where the angular momentum has the familiar components $(\hbar = 1)$

$$L_x = x\,p_y - y\,p_x = i\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right), \qquad (5.6.2a)$$

$$L_y = z\,p_x - x\,p_z = -i\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right), \qquad (5.6.2b)$$

$$L_z = x\,p_y - y\,p_x = -i\frac{\partial}{\partial\phi}, \qquad (5.6.2c)$$

and magnitude

$$L^2 = L_xL_x + L_yL_y + L_zL_z = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \qquad (5.6.3)$$

In this section we study the separation of the isolaplacian (5.4.3) in isospherical coordinates on isospaces $\hat{E}(r,\hat{\delta},\hat{R})$ first studied in ref. [5]. The analysis will then permit the introduction of the *hadronic angular momentum* [5], that is, the image of the conventional angular momentum under isotopies.

In this section we are merely interested in presenting the separation of the isolaplacian and in the ensuing identification of the components of the hadronic angular momentum in isospherical coordinates. Physical aspects will be studied in the next chapter from the viewpoint of the Lie–isotopic theory. The hadronic angular momentum will then be subjected to applications and experimental

verifications in Vol. III.

Let us rewrite isolaplacian (5.4.3) in the conventional Euclidean space $E(\bar{r},\delta,R)$ with local coordinates

$$\bar{r} = \{\,\bar{x}, \bar{y}, \bar{z}\,\} = \{\, b_1 x , b_2 y , b_3 z \,\}, \qquad (5.6.4)$$

in which case we have

$$\hat{\Delta} = \delta^{-1/2} \frac{\partial}{\partial r^i} \delta^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial r^j} = \delta^{-1/2} \frac{\partial}{\partial \bar{r}^i} \delta^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial \bar{r}^j}, \qquad (5.6.5)$$

where one should note in the last identity the *isoderivatives* in the $\bar{r}$ coordinates and the *conventional* metric δ^{ij}.

Next we note the identity

$$d\bar{x}\, d\bar{y}\, d\bar{z} = b_1 b_2 b_3\, dx\, dy\, dz = dr\, d\hat{\theta}\, d\hat{\phi} = dr\,(\, b_3\, d\theta\,)(\, b_1 b_2 d\phi\,), \quad (5.6.6)$$

from which we infer that the isounits for isospherical coordinates and related isoderivatives are

$$\hat{1}_r = 1, \qquad \hat{\partial}/\hat{\partial}r = \hat{1}_r \partial/\partial r \equiv \partial/\partial r, \qquad (5.6.7a)$$

$$\hat{1}_\theta = b_3^{-1} = B_{21}^{-1} B_{22}^{-1}, \qquad \hat{\partial}/\hat{\partial}\theta = \hat{1}_\theta\, \partial/\partial\theta, \qquad (5.6.7b)$$

$$\hat{1}_\phi = b_1^{-1} b_2^{-1} = B_{11}^{-1} B_{12}^{-1}, \qquad \hat{\partial}/\hat{\partial}\phi = \hat{1}_\phi\, \partial/\partial\phi. \qquad (5.6.7c)$$

The use of the isospherical coordinates in the form

$$\bar{x} = r\sin\theta \cos\hat{\phi}, \quad \bar{y} = r\sin\theta \sin\hat{\phi}, \quad \bar{z} = r\cos\theta, \qquad (5.6.8)$$

and the conventional procedure [1] then yields the result

$$\hat{\Delta} = \delta^{-1/2} \frac{\partial}{\partial r^i} \delta^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial r^j} = \delta^{-1/2} \frac{\partial}{\partial \bar{r}^i} \delta^{1/2} \hat{\delta}^{ij} \frac{\partial}{\partial \bar{r}^j} =$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right\}. (5.6.9)$$

We reach in this way a first expression of the separation of the isolaplacian which coincides with the conventional expression at the abstract level, precisely as expected (and desired) under isotopies.

In correspondence with the above separation, we can now introduce the separation of the isowavefunction into a radial and an isospherical part

$$\hat{\psi}(t, \mathbf{r}) \;=\; U(r)\,\hat{\times}\,\hat{Y}(\theta, \hat{\phi}) \;=\; U(r)\,\hat{Y}(\theta, \hat{\phi})\,, \qquad (5.6.10)$$

where the quantity $\hat{Y}(\theta,\hat{\phi})$, called *isospherical harmonics,* will be studied in the next chapter and the isotopic product is the conventional one because $\hat{1}_r = 1$.

The isoeigenvalue problem (5.4.7) for one particle can then be written

$$i\,\hat{1}_t\frac{\partial}{\partial t}\,\hat{\psi}(t, r) \;=\; H * \hat{\psi}(t, r) \;=\; (\hat{K} + \hat{V})\,T\,\hat{\psi}(t, r) \;=\; \Big\{ - \frac{1}{2m}\,\hat{\Delta} + \hat{V}(r) \Big\}\,\hat{\psi}(t, r) \;=$$

$$=\Big\{ -\frac{1}{2m}\,\frac{1}{r^2}\,\frac{\partial}{\partial r}\,r^2\,\frac{\partial}{\partial r} \;-\; \frac{\alpha}{r^2} \;+\; V(r) \Big\}\,U(r)\,\hat{Y}(\theta,\hat{\phi}) \;=\; E\,U(r)\,\hat{Y}(\theta, \hat{\phi}) \quad (5.6.11)$$

where

$$\hat{L}^2 * \hat{Y}(\theta, \hat{\phi}) \;=\; \delta^{ij}\,\hat{L}_i * \hat{L}_j * \hat{Y}(\theta, \hat{\phi}) \;=\; \qquad (5.6.12)$$

$$= -\Big\{ \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} \;+\; \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2} \Big\}\,\hat{Y}(\theta, \hat{\phi}) = \hat{a}\,\hat{x}_{\theta,\phi}\,\hat{Y}(\theta, \hat{\phi}) = a\,\hat{Y}(\theta, \hat{\phi}),$$

is the *isoeigenvalue equation of the hadronic angular momentum* to be studied in the next chapter.

Result (5.6.9) can be verified via the isotopy of other conventional separations, for instance, the method reviewed by Pauling and Wilson in [1], p. 104, in which case we have

$$\hat{\Delta} \;=\; \frac{\partial}{\partial x}\,b_1^{-6}\,\frac{\partial}{\partial x} \;+\; \frac{\partial}{\partial y}\,b_2^{-6}\,\frac{\partial}{\partial y} \;+\; \frac{\partial}{\partial z}\,b_3^{-6}\,\frac{\partial}{\partial z} \;=$$

$$=\; \frac{\partial}{\partial\bar{x}}\,\frac{\partial}{\partial\bar{x}} \;+\; \frac{\partial}{\partial\bar{y}}\,\frac{\partial}{\partial\bar{y}} \;+\; \frac{\partial}{\partial\bar{z}}\,\frac{\partial}{\partial\bar{z}} \;=$$

$$=\; \frac{1}{\hat{A}_r\,\hat{B}_\theta\,\hat{C}_\phi}\,\Big\{ \frac{\partial}{\partial r}\Big(\frac{\hat{B}_\theta\,\hat{C}_\phi}{\hat{A}_r}\,\frac{\partial}{\partial r} \Big) \;+\; \frac{\partial}{\partial\theta}\Big(\frac{\hat{A}_r\,\hat{C}_\phi}{\hat{B}_\theta}\,\frac{\partial}{\partial\theta} \Big) \;+$$

$$+\; \frac{\partial}{\partial\phi}\Big(\frac{\hat{A}_r\,\hat{B}_\theta}{\hat{C}_\phi}\,\frac{\partial}{\partial\phi} \Big) \Big\}\,, \qquad (5.6.13)$$

where

$$\hat{A}_r^2 \;=\; \partial_r x\,B_{22}^{\;2}\,B_{11}^{\;2}\,\partial_r x \;+\; \partial_r y\,B_{22}^{\;2}\,B_{12}^{\;2}\,\partial_r y \;+\; \partial_r z\,B_{21}^{\;2}\,\partial_r z\,, \qquad (5.6.14a)$$

$$\hat{B}_\theta^2 \;=\; \partial_\theta x\,B_{22}^{\;2}\,B_{11}^{\;2}\,\partial_\theta x \;+\; \partial_\theta y\,B_{22}^{\;2}\,B_{12}^{\;2}\,\partial_\theta y \;+\; \partial_\theta z\,B_{21}^{\;2}\,\partial_\theta z\,, \qquad (5.6.14b)$$

$$\hat{C}_\phi^2 \;=\; \partial_\phi x\,B_{22}^{\;2}\,B_{11}^{\;2}\,\partial_\phi x \;+\; \partial_\phi y\,B_{22}^{\;2}\,B_{12}^{\;2}\,\partial_\phi y \;+\; \partial_\phi z\,B_{21}^{\;2}\,\partial_\phi z\,, \qquad (5.6.14c)$$

and the partial derivatives in the latter expressions can be easily rewritten in their isotopic form.

The use of isospherical coordinates (5.5.7) then yields the values

$$\hat{A}_r = 1, \qquad \hat{B}_\theta = 1, \qquad \hat{C}_\phi = r\sin\theta, \qquad (5.5.15)$$

and the desired results

$$\hat{\Delta} = \frac{\partial}{\partial x} b_1^{-6} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} b_2^{-6} \frac{\partial}{\partial y} + \frac{\partial}{\partial z} b_3^{-6} \frac{\partial}{\partial z} =$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial}{\partial\hat{\phi}^2} \;, \quad (5.6.16)$$

which evidently coincides with (5.6.9).

For use in the next chapter, we now derive another realization of the hadronic angular momentum. Let us rewrite the isoeigenvalue equations of the linear momentum in isospace $\hat{E}(r,\hat{\delta},\hat{R})$ in term of conventional equations in $E(\bar{r},\delta,R)$ according to the rule

$$p_k * \hat{\psi} = -i \hat{\nabla}_k \hat{\psi} = -i 1_{ki} \frac{\partial}{\partial r^i} \hat{\psi} = = -i b_k^{-2} \frac{\partial}{\partial r^k} \hat{\psi} = -i b_k^{-1} \frac{\partial}{\partial \bar{r}^k} \hat{\psi}$$

(5.6.17)

Then the *hadronic angular momentum components* can be written

$$\hat{L}_x * \hat{Y} = (y\,P_z - z\,p_y) * \hat{Y} = -i\,b_2^{-1} b_3^{-1} (\bar{y}\,\partial_{\bar{z}} - \bar{z}\,\partial_{\bar{y}}) \hat{Y} =$$

$$= i\,b_2^{-1} b_3^{-1} L_{\bar{x}} \hat{Y} = i\,b_2^{-1} b_3^{-1} (\sin\hat{\phi} \frac{\partial}{\partial\theta} + \cot\theta\cos\hat{\phi}\frac{\partial}{\partial\hat{\phi}}) \hat{Y}, \quad (5.6.18a)$$

$$\hat{L}_y * \hat{Y} = (z\,p_x - x\,p_z) * \hat{Y} = -i\,b_1^{-1} b_3^{-1} (\bar{z}\,\partial_{\bar{x}} - \bar{x}\,\partial_{\bar{z}}\hat{Y} =$$

$$= -i\,b_1^{-1} b_3^{-1} L_{\bar{y}} \hat{Y} = -i\,b_1^{-1} b_3^{-1} (\cos\hat{\phi}\frac{\partial}{\partial\theta} - \cot\theta\sin\hat{\phi}\frac{\partial}{\partial\hat{\phi}}) \hat{Y}, \quad (5.6.18b)$$

$$\hat{L}_z * \hat{Y} = (x\,p_y - y\,p_x) * \hat{Y} = -i\,b_1^{-1} b_2^{-1} (\bar{x}\,\partial_{\bar{y}} - \bar{y}\,\partial_{\bar{x}}) \hat{Y} =$$

$$= -i\,b_1^{-1} b_2^{-1} L_{\bar{z}} \hat{Y} = -i\,b_1^{-1} b_2^{-1} \frac{\partial}{\partial\hat{\phi}}, \quad (5.6.18c)$$

with *magnitude of hadronic angular momentum*

$$\hat{L}^2 * \hat{Y}(\theta,\hat{\phi}) = \delta^{ij} \hat{L}_k * \hat{L}_j \hat{Y}(\hat{q},\hat{\phi}) =$$

$$= (\, b_1^{-2} \; \hat{L}_x * \hat{L}_x \; + \; b_2^{-2} \hat{L}_y * \hat{L}_y \; + \; b_3^{-2} \hat{L}_z * \hat{L}_z \,) * \Upsilon(\theta, \hat{\phi}) =$$

$$= b_1^{-2} b_2^{-2} b_3^{-2} (\, L_{\bar{x}} \, L_{\bar{x}} \; + \; L_{\bar{y}} \, L_{\bar{y}} \; + \; L_{\bar{z}} \, L_{\bar{z}} \,) \, \Upsilon(\theta, \hat{\phi}) =$$

$$= - D^{-2} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right\} \Upsilon(\theta, \hat{\phi}) \; = \; \tilde{a} \; \Upsilon(\theta, \hat{\phi}) \; (5.6.19)$$

Where $D = b_1 b_2 b_3$. The difference between realization (5.6.19) and expression (5.6.12) should be noted. The latter is in term of *conventional* derivatives, while the former is in terms of the *isoderivatives*. As a result, the corresponding isoeigenvalues are expected to be different, $\tilde{a} \neq a$.

5.7: GENOSCROEDINGER EQUATIONS AND THEIR ISODUALS

We now study nonconservative systems represented via genotopic methods. First, it is important to understand the differences between the isochrödinger and genoschrödinger representations.

In Sect. 11.5.3 we have seen that nonconservative systems can be well represented via the isoscrödinger equation. However, they do not provide an axiomatization of irreversibility, that is, the emergence of irreversibility even for reversible Hamiltonians.

Also, a main objective of the genoschrödinger representation is that of achieving a *Hermitean* and therefore observable representation of the energy under the condition that it is not conserved because of external interactions,

$$H = K + V = H^\dagger, \qquad dH / dt \neq 0. \tag{5.7.1}$$

One should keep in mind that these features are not generally possible in conventional quantum mechanics where nonconservation is represented, e.g., via the addition of an "imaginary potential" to the Hamiltonian.

To outline the genoschrödinger representation, one must recall the four possible directions of time: motion forward to future time $<$, motion forward from past time $>$, motion backward from future time $<^d = -<$ and motion backward to past time $>^d = - >$, which are denoted for brevity with the unified symbols $<>$ and $<>^d = - <>$. We therefore have four *time genounits* $^<1_t{}^>$ and $^<1_t{}^{>d}$ and four corresponding genofields

$$^<\hat{1}^> = t^<\hat{1}^> = t\,(\,^<T_t{}^>\,)^{-1} \;\in\; {}^<\hat{R}^>(^<\hat{1}^>,+,<>), \tag{5.7.2a}$$

$$^<\hat{1}^{>d} = t^<\hat{1}_t{}^{>d} = (\,^<T_t{}^{>d}\,)^{-1} \;\in\; {}^<\hat{R}^{>d}(^<\hat{1}^{>d},+,<>^d). \tag{5.7.2b}$$

In addition, we have *four space genounits* $^<\tilde{1}^>$ and $^<\tilde{1}^{>d}$ ands four corresponding genofields.

The background carrier spaces are the following *genoeuclidean spaces* (Sect. I.7.5)

$$\hat{E}^>(t,\hat{R}_t^>)\times\hat{E}^>(r,\hat{\delta}^>,\hat{R}^>): \quad \hat{1}^> = (\,^<T^>)^{-1} > 0, \quad \hat{\delta}^> = \,^<\hat{1}^> \delta, \quad (5.7.3a)$$

$$r^{<2>} = \{\, r^i \,^<\delta^>_{ij}\, r^j\,\}\,^<\hat{1}^> \in\, ^<\hat{R}^>(^<\hat{n}^>,+,<>), \quad (5.7.3b)$$

with corresponding isodual genospaces here omitted for brevity, where the genoproduct <> is characterized by the *space genotopic element* $^<T^> \neq\, ^<T_t^>$ in each direction of motion, only one of them being usable at the time.

The *genohilbert space* is characterized by genonormalization

$$\hat{\mathfrak{IC}}^>: \quad <\hat{\psi}\,^<\hat{1}^>\,\hat{\psi}> = \,^<\hat{1}^> \int dv\, ^<\hat{\psi}(t,r)\,^<T^>\,\hat{\psi}^>(t,r) = \,^<\hat{1}^>, \quad (5.7.4)$$

and it is evidently defined in general over a genofield of complex numbers.

The *genoschrödinger's equations for forward motion to future time* on $\hat{E}^>(t,\hat{R}_t^>)\times\hat{E}^>(r,\hat{\delta}^>,\hat{R}^>)$ over $\hat{\mathfrak{IC}}^>$ are given by [3,4,5]

$$i\frac{\partial^>}{\partial t^>}\,\hat{\psi}^>(t,r) = i\hat{1}_t^>\frac{\partial}{\partial t}\,\hat{\psi}^>(t,r) = H > \hat{\psi}^>(t,r) =$$

$$= H(t,r,p)\,T^>(t,p,\dot{p},...)\,\hat{\psi}^>(t,r) = \hat{E}^> > \hat{\psi}^>(t,r) \equiv E\,\hat{\psi}^>(t,r), \quad (5.7.5a)$$

$$p_i > \hat{\psi}^>(t,r) = p_i\,T^>(t,p,\dot{p},...)\,\hat{\psi}^>(t,r) = -i\,\hat{\nabla}_i\,\hat{\psi}(t,r) = i\hat{1}_{ij}\frac{\partial}{\partial r_j}\,\hat{\psi}(t,r) =$$

$$= \hat{k}^>_i > \hat{\psi}^>(t,r) \equiv k_i\,\hat{\psi}^>(t,r), \quad (5.7.5b)$$

while the *genoschrödinger equations for forward motion from past time* on the genospace $^<\hat{E}(t,^<\hat{R}_t)\times{}^<\hat{E}(r,^<\hat{\delta},^<\hat{R})$ over $^<\hat{\mathfrak{IC}}$ are given by [3,4,5]

$$-i\,^<\hat{\psi}(t,r)\frac{^<\partial}{^<\partial t} = -i\,^<\hat{\psi}(t,r)\frac{\partial}{\partial t}\,^<\hat{1}_t = \,^<\hat{\psi}(t,r) < H =$$

$$= \,^<\hat{\psi}^\dagger(t,r)\,^<T(t,p,\dot{p},...)\,H(t,r,p) = \,^<\hat{\psi}(t,r) < \hat{E} \equiv \,^<\hat{\psi}(t,r)\,E = -\hat{\psi}^\dagger\,E, \quad (5.7.6a)$$

$$^<\hat{\psi} < p_i = \,^<\hat{\psi}(t,r)\,^<T(t,p,\dot{p},...)\,p_i = i\,^<\hat{\psi}(t,r)^<\hat{\nabla}_i =$$

$$= i\,^<\hat{\psi}^\dagger\frac{\partial}{\partial r_j}\,^<\hat{1}_{ji} = \,^<\hat{\psi}^\dagger\,k_i = -\hat{\psi}^\dagger\,k_i, \quad (5.7.6b)$$

$$H = H^\dagger, \quad \hat{T}^> = (^<T)^\dagger, \quad p = p^\dagger, \quad ^<\hat{\psi} = (\hat{\psi}^>)^\dagger, \quad (5.7.6c)$$

where the latter Hermitean conjugation is *necessary* for the Hermiticity, causality, etc. (Sect.s II.3.3 and II.4.3).

We now illustrate the above equations with the same linearly damped

particle studied in Sect. 11.5.3. For this purpose we assume the Hamiltonian and genotopic elements

$$H = \tfrac{1}{2} p > p = \tfrac{1}{2} p \, T^> p , \qquad T^> = i e^{\gamma t} , \qquad {}^<T = - i e^{\gamma t} , \qquad (5.7.7)$$

whose genoexpectation values (Axiom $^<V^>$) yield exactly the same result as in Eq. (5.3.12),

$$<H> = |\int dv \, \hat{\psi}^\dagger > \tfrac{1}{2} \ p > p > \hat{\psi}| = \tfrac{1}{2} e^{-\gamma t} k \, k. \qquad (5.7.8)$$

To study the internal compatibility of the genorepresentation *for the selected genotopic elements*, we consider now the Lie–admissible form of the equation of motion

$$i \hat{1}_t \dot{H} = H_0 ({}^<T - T^>) H , \qquad (5.7.9)$$

where H_0 is the value of $H(t)$ at $t = 0$.

It is then easy to see that the time genounit

$$\hat{1}_t = - \tfrac{1}{2} \gamma \, e^{-\gamma t} H_0 , \qquad (5.7.10)$$

yields the correct evolution in time

$$dH / dt = - \gamma H . \qquad (5.7.11)$$

The reader should be aware that genoschrödinger representations have considerable degrees of freedom in the selection of the Hamiltonian H and the isotopic elements $T^>$ and $T_t^>$ for each given system. The above example has been mainly selected as a genotopy of the corresponding example in Sect. 11.5.3.

Numerous other genorepresentations of the same system are therefore possible. Their study is left to the interested reader for brevity jointly with other cases.

APPENDIX 5.A: MAKHALDIANI REPRESENTATION

By no means the isoscrödinger and genoschrödinger representations exhaust all possible representations of hadronic mechanics. Among various other possible representations, we here select one due to Makhaldiani [10].

Let us consider first conventional quantum mechanics. Another treatment known since the early states of the theory is the factorization of the wavefunction

$$\psi(t) = v(t) \, \psi(0) , \qquad (5.A.1)$$

and the equations $(\hbar = I)$

$$i \, \partial_t \, v(t) \; = \; H \, v(t) \, , \qquad\qquad v(o) \; = \; I \, , \qquad\qquad\qquad (5.A.2)$$

where I is the unit matrix of the same dimension as that of the representation.

The resolvent can be written

$$v(t) = e^{\,i\,t\,H} = (2\pi i)^{-1} \oint dz \, e^{-\,i\,t\,z} \, / \, (z \, I - H) \; = (2\pi i)^{-1} \oint dz \, R(z) \, e^{-\,i\,t\,z}, \quad (5.A.3)$$

where the integration path is counterclockwise on the real axis.

The reader can prove that resolvent (5.A.3) does indeed verify both conditions(5.A.2)

Consider now the familiar structure of the Hamiltonian

$$H \; = \; K \, + \, \lambda \, V \, , \qquad\qquad\qquad (5.A.4)$$

where $K = \tfrac{1}{2} \, pp$ is the kinetic energy, V is a potential and λ is a (real) coupling constant.

Then, *Makhaldiani's quantum representation* is given by the equation

$$\frac{\partial R}{\partial \lambda} \; = \; R \, v \, R \, , \qquad\qquad\qquad (5.A.5)$$

under the values

$$R(z) \; = \; (\, z \, I \, - H \,)^{-1} , \quad R_0 \; = \; (\, z \, I \, - \, H_0 \,)^{-1} . \qquad\qquad (5.A.6)$$

Equation (5.A.5) is fully acceptable for the description of a quantum mechanical system. In particular, it permits novel treatments of conventional problems, such as the perturbation theory. In fact, the equation can be easily discretized in the form

$$R_{k+1} \; = \; R_k \, + \, R_k \, V \, R_k \, (\lambda_{k+1} - \lambda_k) \, , \qquad\qquad (5.A.7)$$

with *perturbative series*

$$R(\lambda) \; = \; \sum_{k \geq 0} \lambda^k \, R_k \, , \quad R_{k+1} \; = \; (\, k + 1 \,)^{-1} \sum_{k=0,..,n} R_n \, v \, R_{k-n} \, , \qquad (5.A.8a)$$

$$R_0 = (\mathrm{x} I - H_0)^{-1} , \quad R_1 \; = \; R_0 \, v \, R_0 \, , \, R_2 = (\, R_0 \, v \, R_1 \, + \, R_1 \, v \, R_0 \,) \, / \, 2 \, , \, ... \qquad (5.A.8b)$$

Another interesting property of the Makhaldiani representation is that it admits simple yet effective isotopies into hadronic mechanics. Consider the isoenveloping operator algebra ξ with generic operators A, B, etc., isotopic product $A{*}B := ATB$ where T_t is the time isotopic element and $\hat{I} = T^{-1}$ is the time isounit.

Then, the time component of the isowavefunction solution of the isoschrödinger equations admits the factorization

$$\hat{\psi}(t) = \hat{V}(t) * \hat{\psi}(0) , \qquad (5.A.9)$$

and the equations expressed in terms of the timer isoderivative $(\hbar = 1)$

$$i \, \partial_t \, \hat{v}(t) = = i \hat{1}_t \, \hat{v}(t) = H * \hat{v}(t) , \qquad \hat{v}(o) = \hat{1} . \qquad (5.A.10)$$

where the isounit $\hat{1}$ evidently has the dimension of the representation.

The *isoresolvent* can be written in terms of the isoexponentiation

$$\hat{v}(t) = \hat{e}^{\,i\,t\,H} = \hat{1}_t \{ e^{\,i\,t\,T_t\,H} \} = \{ e^{\,i\,H\,T_t\,t} \} \hat{1} =$$

$$= (2\pi i)^{-1} \oint dz \, \hat{e}^{-i\,t\,z} / (z\hat{1} - H) = (2\pi i)^{-1} \oint dz \, \hat{R}(z) \, \hat{e}^{-i\,t\,z} , \qquad (5.A.11)$$

where the integration path is the same as before. Thus, Eq. (5.A.11) recovers exactly the time component of the isowavefunction.

It is instructive to prove that isoresolvent (5.A.11) does indeed verify both conditions(5.A.10)

Consider again the familiar structure (5.A.4). Then, *Makhaldiani's isorepresentation* is characterized by the equation

$$\frac{\partial \hat{R}}{\partial \lambda} = \hat{1}_t \frac{\partial R}{\partial \lambda} = \hat{R} * \hat{v} * \hat{R} , \qquad (5.A.5)$$

under the values

$$\hat{R}(z) = (z\hat{1} - H)^{-1} , \quad \hat{R}_0 = (z\hat{1} - \hat{H}_0)^{-1} . \qquad (5.A.6)$$

We have presented above the simplest possible case based only on the time isotopic element and isounit. More general cases are possible via the use of different isotopic elements for time and space.

We shall consider again Makhaldiani isorepresentation later on when studying the regaining of convergence for divergent perturbative expansions.

REFERENCES

1. D. BOHM, *Quantum Theory*, Dover Publications, New York (1979); P. ROMAN, *Advanced Quantum Theory*, Addison-Wessley, Reading MA (1965); L.PAULING

and E. B. WILSON, *Quantum Mechanics, with Applications to Chemistry,
Dover Publication, New York (1963)*

1. J. V. KADEISVILI, Algebras, Groups and Geometries 9, 283 (1992)
3. H. C. MYUNG and R. M. SANTILLI, Hadronic J. 5, 1277 and 1367 (1983)
4. R. MIGNANI, Hadronic J. 5, 1120 (1982)
5. R. M. SANTILLI, Hadronic J. Suppl. 4B, issue no. 1 (1989)
6. N. Ja. VILENKIN, *Special Functions and the Theory of Group Representations*
   Amer. Math. Soc., Providence R. I. (1968)
7. R. M. SANTILLI, *Isotopic Generalizations of Galilei's and Einstein's Relati-
   vities*, Vol. II, second edition, Academy of Sciences of the Ukraine,   Kiev
   (1994)
8. M. NISHIOKA and R. M. SANTILLI, Physics Essays 5, 44 (1992)
9. R. M. SANTILLI, *Foundations of Theoretical Mechanics*, Vol. II,
   Springer-Verlag, Heidelberg/new York (1983)
10. N. V. MAKHALDIANI, JINR, Dubna, private communication

6: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF ANGULAR MOMENTUM AND SPIN

6.1: STATEMENT OF THE PROBLEM

It is generally assumed (see, e.g., ref.s [1]) that the SU(2)–spin symmetry characterizes in a unique and final way the eigenvalues

$$J^2|b> = j(j+1)|b>, \tag{6.1.1a}$$

$$J_3|b> = m|b>, \tag{6.1.1b}$$

which solely admit the quantum numbers

$$j = 0, \tfrac{1}{2}, 1, ..., \qquad m = j, j - 1, ..., - j. \tag{6.1.2}$$

The isotopies of SU(2)[51] have proved this assumption to be erroneous because, while $\hat{SO}(2)$ is locally isomorphic to SU(2) for all isotopies of Class I, its isorepresentations admit, among others, the eigenvalues

$$\hat{J}^2 * |\hat{b}> = \hat{j}[\hat{j} + 1]|\hat{b}>, \tag{6.1.3a}$$

$$\hat{J}_3 * |\hat{b}> = \tilde{m}|\hat{b}>, \tag{6.1.3b}$$

with generalized numbers

$$\hat{j} = f(b) j, \quad j = 0, \tfrac{1}{2}, 1, ..., \quad \tilde{m} = f(b) m, \quad \tilde{m} = \tilde{j}, \tilde{j} - 1, ..., -\tilde{j}, \tag{6.1.4}$$

where f(b) is a (smooth) function of the characteristic quantities b_k of the underlying isospace such that f(1) = 1.

---

[51] We shall use hereon the notation SO(3) and SU(2) for *groups* and so(3) and su(2) for *algebras* with corresponding isotopies $\hat{SO}(3)$, $\hat{SU}(2)$, $\hat{so}(3)$ and $\hat{su}(2)$.

In this chapter we shall review the (rather limited) theoretical knowledge of the isotopies SÔ(2) of SU(2) spin or isospin available at this writing (early 1994). In particular, we shall study first the isorepresentations (or isoreps for short) of the hadronic angular momentum Ĺ, and then pass to a study of the isoreps of the hadronic spin or isospin Ĵ.

The primary difficulties in the understanding and appraisal of the content of this chapter are of *conceptual*, rather than of technical nature. Because of protracted use by generations of physicists during this century, we are generally inclined to assume the universal validity of the *quantum mechanical angular momentum and spin* under whatever physical conditions, without the awareness that these notions were specifically conceived and experimentally verified for the *exterior dynamical problem* (i.e., point–like particles moving in vacuum under action–at–a–distance interactions).

The objective of the covering notions of *hadronic angular momentum and spin* is to study the orbital and intrinsic angular momentum for a particle in *interior dynamical conditions* (i.e., an extended particle moving within the hadronic medium composed by the wavepackets/charge distributions of other particles), under the condition of recovering the conventional notions whenever the particle exits the medium and returns to move in vacuum.

As an example, while the conventional notion describes the angular momentum L of an electron moving in vacuum in an atomic cloud, or the spin J of the proton as the nucleus of the hydrogen atom, the objective of the hadronic angular momentum Ĺ is the study of the orbital motion of the same electron or the spin of the same proton but when the particles are immersed in the medium in the core of a star.

The *mathematical differences* between L and Ĺ or J and Ĵ are then conceived to provide a quantitative representation of the *physical differences* of the exterior and interior conditions considered. Equivalently, we can say that the isotopies L → Ĺ and J → Ĵ are conceived to provide a quantitative representation of *the deviations from conventional angular momentum and spin caused by physical media.*

When first confronted with this problem, a rather general tendency to preserve existing notions is to pass to second quantization and treat the problem via the related methodology including particle exchanges, Feynman diagrams, etc.

While the validity of such an approach as a *first approximation* of interior conditions is undeniable, its lack of final character is beyond scientific doubts. In fact, the treatment of the angular momentum and spin via second quantization for a particle in interior conditions eliminates the very physical conditions to be represented.

The primary emphasis in the conventional angular momentum is to represent the *stability* of the orbits of the atomic electrons. On the contrary, one of the primary problems of the covering hadronic notions is to represent the *instability* of the orbits of an electron when in the core of a star to avoid a quantum mechanical version of the "perpetual motion" whereby an electron can

orbit in the core of a star with a conserved angular momentum. The transition to second quantization has the effect of eliminating precisely the *continuous* instability of the orbital motion, and reduce the problem to an infinite sequence of locally stable trajectory, that is, approximating a continuously varying angular momentum within the hyperdense medium with a sequence of conserved angular momenta in vacuum.

At any rate, in this chapter we shall strictly a*void* second (iso)quantization. In fact the instability of interior orbital motion must be fully represented at the level of first (iso)quantization. Only *after* acquiring such a consistent representation, the passage to second (iso)quantization may be physically meaningful.

But there are deeper geometric reasons to prevent an effective treatment of interior problems via the transition to second quantization. As well known, the geometric pillars of quantum mechanics are the *homogeneity and isotropy of empty space,* as emerging in the structure of the Laplacian, the formulation of SU(2), etc.

Some of the geometric pillars of interior dynamical problems are the *inhomogeneity and anisotropy of physical media,* which have been embedded in the structure of the isolaplacian of the preceding chapter and will be embedded in the structure of SÔ(3) and SU(2) of this chapter beginning at the *classical* level, and then persisting at the level of first isoquantization. The treatment of interior problems of angular momentum and spin via conventional quantum methods, whether of first or second quantization, implies the evident, complete elimination of such basic characteristics of the medium in which motion occurs.

Additional more subtle reasons to construct a covering theory of angular momentum and spin for interior conditions emerge after the reader has acquired a technical knowledge of the topic and, particularly, in the novel physical applications of Vol. III.

In summary, throughout the analysis of this chapter the reader is encouraged to abandon the traditional notion of angular momentum and spin for a point-particle moving in vacuum, and consider instead the motion of an extended particle within a hyperdense medium.

As now familiar from the preceding studies, we represent the transition from exterior to interior conditions via the isotopies of the unit $I \rightarrow \hat{I}$ which is restricted to be of class I $(\hat{I} > 0)$ to ensure the local isomorphisms $S\hat{O}(3) \approx SO(3)$ and $S\hat{U}(2) \approx SU(2)$, as we shall see. The emerging structures $S\hat{O}(3)$ and $S\hat{U}(2)$ are a *covering* of the conventional ones, in the sense that they admit the entire theory of conventional angular momentum and spin for the particular cases

$$S\hat{O}(3)_{\hat{I}=I} \equiv SO(3), \qquad S\hat{U}(2)_{\hat{I}=I} \equiv SU(2), \qquad (6.I.5)$$

but admit in addition hitherto unknown novel properties for novel physical conditions which simply cannot be studied via the conventional theory.

The *mathematical* results of this chapter can be anticipated from the Lie–isotopic theory of Ch. I.4, particularly Sect. I.4.7 on isorepresentations. In fact, all possible isoreps of $S\hat{O}(3)$ or $S\hat{U}(2)$ can be divided into the following three primary classes:

I) Standard isorepresentations. They are the simplest possible isoreps with *no deviation in the quantum eigenvalues*, in which the structure of the theory is generalized with general rules

$$\mathfrak{J}^2 * | \mathfrak{b} > = \hat{\mathfrak{j}}(\hat{\mathfrak{j}} + 1) | \mathfrak{b} >, \quad \hat{\mathfrak{j}} = 0, \tfrac{1}{2}, 1, \dots , \tag{6.1.6a}$$

$$\mathfrak{J}_3 * | \mathfrak{b} > = \tilde{m} | \mathfrak{b} >, \quad m = \hat{\mathfrak{j}}, \hat{\mathfrak{j}} - 1, \dots, -\hat{\mathfrak{j}}. \tag{6.1.6b}$$

The standard isoreps can be constructed via the simplest possible realization of *Klimyk's rule* (Lemma I.4.7.5) which essentially states the following. Let $\mathfrak{R}$ be the adjoint representation of a (finite–dimensional) Lie algebra A and let $\hat{A}$ be its isotopes of Class I. Then, up to degrees of freedom studied later on, the regular adjoint isorepresentation $\hat{\mathfrak{R}}$ of $\hat{A}$ is given by the rule

$$\hat{\mathfrak{R}} = \mathfrak{R}\hat{1}, \quad \hat{1} = T^{-1}, \det. T = 1. \tag{6.1.7}$$

In fact, under the above rule we have the isotopy of the $su(2)$ into the $s\hat{u}(2)$ algebra

$$[J_i, J_j] = J_i J_j - J_j J_i = (\mathfrak{J}_i T \mathfrak{J}_j - \mathfrak{J}_j T \mathfrak{J}_i) T = \epsilon_{ijk} J_k = \epsilon_{ijk} \mathfrak{J}_k T, \tag{6.1.8}$$

that is,

$$[\mathfrak{J}_i \hat{,} \mathfrak{J}_j] := \mathfrak{J}_i * \mathfrak{J}_j - \mathfrak{J}_j * \mathfrak{J}_i = \mathfrak{J}_i T \mathfrak{J}_j - \mathfrak{J}_j T \mathfrak{J}_i = \epsilon_{ijk} \mathfrak{J}_k, \tag{6.1.9}$$

from which the local isomorphism $s\hat{u}(2) \approx su(2)$ is evident. The identity of the isoeigenvalues with the conventional eigenvalues follows from the condition det T = 1, as we shall see.

Far from being trivial, the standard isorepresentations of $s\hat{o}(3)$ and $s\hat{u}(2)$ are intriguing indeed because, while coinciding with conventional representations for all conceivable practical purposes, they admit novel, hitherto unknown degrees of freedom which constitute a specific realization of the "hidden variables" λ, as anticipated in App. II. 4.C. As an example, the standard isorepresentation for spin occur for isospaces $\hat{E}(z, \hat{\delta}, \hat{C})$, $\hat{\delta} = T\delta$, T = diag. (g_{11}, g_{22}) with $g_{11} = g_{22}^{-1} = \lambda$.

To illustrate the nontriviality of such hidden degree of freedom, we shall show in Vol. III that, contrary to a rather popular belief throughout this century, *the isospin symmetry is indeed exact under weak and electromagnetic interactions.* In fact, the mass differences between protons and neutrons can be represented via the hidden degree of freedom λ, while the symmetry remains exact and the physical masses are recovered identically under isoexpectation values.

Additional applications which are inconceivable with the quantum theory of angular momentum and spin, but quantitatively treatable with the new one will be presented in Vol. III.

II) Regular isorepresentations. They constitute a second level of generalization with the first significant departures from quantum eigenvalues, in which the spectrum of isoeigenvalues can be completely factorized into that of conventional eigenvalues, e.g.,

$$J^2 * | \hat{b} > = K^{-2}(b) [\hat{j}(\hat{j} + 1)] | \hat{b} >, \quad \hat{j} = 0, 1, \dots \tag{6.1.10a}$$

$$J_3 * | \hat{b} > = K^{-1}(b) \tilde{m} | \hat{b} >, \quad \tilde{m} = \hat{j}, \hat{j} - 1, \dots, -\hat{j}. \tag{6.1.10b}$$

The above isoreps can also be constructed via Klimyk's rule although in its most general form

$$\hat{\mathfrak{R}} = \mathfrak{R} P, \quad P = K\hat{1}, \quad \hat{1} = T^{-1}, \quad \det T \neq 0, 1, \tag{6.1.11}$$

where K is a nonzero element of the field of real numbers $R(n,+,\times)$.

In fact, under the above rule we have the isotopy of the su(2) into the $\hat{su}(2)$ algebra

$$[J_i, J_j] = J_i J_j - J_j J_i = K^{-2}(J_i T J_j - J_j T J_i) T = \epsilon_{ijk} J_k = \epsilon_{ijk} J_k T \tag{6.1.12}$$

that is,

$$[J_i \hat{,} J_j] := J_i * J_j - J_j * J_i = J_i T J_j - J_j T J_i = K^2 \epsilon_{ijk} J_k, \tag{6.1.13}$$

from which the complete factorization of the isoeigenvalues as in Eq.s (6.1.10) follows, as we shall see.

In Vol. III we shall sow that physical applications of the regular isoreps require genuine interior conditions. As an illustration, we can easily represent continuously decaying orbits in first isoquantization via realizations of the type K = exp(γt). This illustrates the transition from the "quantum" structure of the conventional formalism to the "continuous" structure of the covering isotopic formalism.

III) Irregular isorepresentations. They constitute the broadest known generalization implying the maximal possible deviations from quantum eigenvalues and representing the most general possible interior conditions. They occur when the isoeigenvalues are not completely factorizable into conventional eigenvalues, i.e., are of type (5.1.10)

The irregular reps a*re not* derivable from Klimyk's rule because the isotopic algebra is obtained from the original one via nonunitary transforms

$$U U^\dagger = \hat{1} = \hat{1}^\dagger \neq 1, \quad T = (U U^\dagger)^{-1} = T^\dagger, \quad J_k = U J_k U^\dagger, \tag{6.1.14a}$$

$$U [J_i , J_j] U^| = U (J_i J_j - J_j J_i) U^| =$$

$$= U J_i U^\dagger (U U^\dagger)^{-1} U J_j U^\dagger = - U J_j U^\dagger (U U^\dagger)^{-1} U J_i U^\dagger =$$

$$= \hat{J}_i \, T \, \hat{J}_j - \hat{J}_j \, T \, \hat{J}_i = [\hat{J}_i , \hat{J}_j] = \epsilon_{ijk} \, U J_k U^\dagger = \epsilon_{ijk} \, \hat{J}_k . \qquad (6.1.14b)$$

Note the Hermiticity of the emerging isotopic element T as well as the preservation of the original structure constants (which ensure the local isomorphism $\hat{su}(2) \approx su(2)$. Yet, as we hall see, as expected from the nonunitary character of the transform, the spectrum of eigenvalues is structurally altered.

The studies under consideration originated with the proposal of the Lie-isotopic symmetries of 1978, ref. [2], which also contains the first explicit examples of the isotopic $\hat{O}(3)$ symmetries. The first detailed studies of the $\hat{O}(3)$ symmetry were presented in papers [3] of 1985[52]. The isotopic $\hat{SU}(2)$ symmetry was first identified in memoir [4] and then studied in detail in papers [5] which presented some of the $\hat{SU}(2)$ theory reviewed in this chapter. The relation of $\hat{SU}(2)$ with q-deformations was studied in paper [6], where the Klimyk rule first appeared in print. The classical formulation was presented in monographs [7] under the name of *isorotational theory*. Important studies by G. Eder on $\hat{SU}(2)$ will be reviewed in Vol. III. Independent reviews are available in monographs [8,9]. No additional contribution on the isotopies of SO(3) or SU(2) is available at this writing to the author's best knowledge.

We are referring, specifically, to studies of the rotational and/or spin symmetry which are *essentially dependent on the generalization of the conventional unit*.[53] Other types of generalizations cannot be studied for brevity, with the sole exception of the q-deformations which we shall briefly consider in App. 11.6.C.

The isodualities of SO(3) were first identified in papers [3], while the

---

[52] Articles [3] were written in 1982, but published only in 1985 in the Hadronic Journal because of truly unreasonable editorial obstructions by a number of journals (for a detailed report, see p. 26 of ref.s [3]).

[53] In regard to the paucity of independent papers in the isotopies of SU(2), despite its manifestly fundamental character and numerous solicitations by this author, it is appropriate here to quote J. V. Kadeisvili who, in page 247 of his review [9] states:

"We felt obliged to mention in the preceding sections the scarcity of independent investigations on Santilli's isotopies. In fact, Santilli is the sole originator (in 1978) and sole contributor in both the $\hat{O}(3)$ and $\hat{U}(2)$ isosymmetries to this writing (1992) without any contribution by independent researchers.

At first, this author could not believe such an occurrence because the rotational symmetry is the most fundamental part of all of contemporary theoretical physics. The birth of a structural generalization of the rotational symmetry with fundamentally novel capabilities was not expected to remain ignored. But that's as it has been. In fact, library searches and consultations with experts in the field have confirmed the lack of independent contributions on the isosymmetries $\hat{O}(3)$ and $\hat{U}(2)$ up to this writing."

isodualities of $S\hat{O}(2)$ were briefly indicated in papers [5]. No study on the isotopies of the Legendre transforms, the spherical harmonics, the matrix representation of the hadronic angular momentum, the genodualities of the rotational symmetry and other topics has appeared in print prior to this presentation, to our best knowledge.

6.2: ABSTRACT ISOTOPIES AND ISODUALITIES OF THE ROTATIONAL SYMMETRY

Let us begin by reviewing the isotopies of the conventional rotational theory[54] at the abstract level, that defined as the symmetry of the isosphere in three-dimensional isoeuclidean space (Ch. I.3).

The analysis of this section is not restricted to physical applications and it will be therefore conducted for isotopies of Class III (characterized by sufficiently smooth, bounded, nowhere degenerate, Hermitean isotopic elements T which can be either positive- or negative-definite), with an unrestricted functional dependence $T = T(t, r, \dot{r}, \ddot{r}, ...)$, including that in the local coordinates r.

This implies that the isosphere unifies all infinitely possible compact and noncompact deformations of the conventional sphere, including all spheroidal ellipsoids and all elliptic or hyperbolic paraboloids. In turn, the isotopies of the rotational symmetry studied in this section will unify in one single, abstract notion the symmetries of all these quadrics, as clarified in more details below.

The theory studied in this section will be restricted to Class I in the physical studies of the subsequent sections and used as the foundation of the operator studies of the subsequent sections.

Definition 6.2.1 [3,7]: *The "rotational isotopic group" of Class I hereon denoted $\hat{O}(3)$, is the largest possible nonlinear, nonlocal and noncanonical, simple invariance group of the three-dimensional isoeuclidean spaces of Class III (Sect. I.3.3)*

$$\hat{E}(r,\delta R): \qquad \hat{\delta} = T(t, r, \dot{r}, \ddot{r}, ...)\delta \equiv \hat{\delta}, \qquad \hat{1} = T^{-1}, \qquad (6.2.1a)$$

$$\delta = diag. (1, 1, 1), \qquad det. T \neq 0, \qquad T = T^\dagger, \qquad \hat{\delta}^\dagger = \hat{\delta}, \qquad (6.2.1b)$$

$$r^2 = (r \hat{,} r) = (r, \hat{\delta}r)\hat{1} = (\hat{\delta}r, r)\hat{1} = \hat{1}(r, \hat{\delta}r) = (r^i \hat{\delta}_{ij} r^j)\hat{1}, \qquad (6.2.1c)$$

characterized by: the right, modular-isotopic transformations

[54] The literature on the conventional rotational theory is truly vast indeed. One may consult, e.g., ref.s [1] for introductory presentations and ref.s [10–12] for more advanced treatments.

$$r' = \hat{\mathfrak{R}}(\theta) * r = \hat{\mathfrak{R}}(\theta)\, T\, r, \qquad T = \text{fixed}, \qquad (6.2.2)$$

under which they are isolinear, isolocal and isocanonical (App. II.4.C), where the θ's are the conventional Euler's angles, with elements $\hat{\mathfrak{R}}(\theta)$ verifying the isoorthogonality conditions

$$\hat{\mathfrak{R}} * \hat{\mathfrak{R}}^t = \hat{\mathfrak{R}}^t * \hat{\mathfrak{R}} = \hat{1}, \qquad (6.2.3)$$

or, equivalently, $\hat{\mathfrak{R}}^t = \hat{\mathfrak{R}}^{-1}$, and verify the rules for a Lie–isotopic group

$$\hat{\mathfrak{R}}(\theta) * \hat{\mathfrak{R}}(\theta') = \hat{\mathfrak{R}}(\theta') * \hat{\mathfrak{R}}(\theta) = \hat{\mathfrak{R}}(\theta + \theta'), \quad \hat{\mathfrak{R}}(\theta) * \hat{\mathfrak{R}}(-\theta) = \hat{\mathfrak{R}}(0) = \hat{1} = T^{-1}. \quad (6.2.4)$$

Equivalently, the isorotational group $\hat{O}(3)$ can be defined as the isosymmetry of the isosphere in $\hat{E}(r,\hat{\delta},\hat{\mathfrak{R}})$, e.g., in the diagonal realization

$$\hat{\delta} = \text{diag.}\,(\,g_{11}, g_{22}, g_{33}\,) \qquad (6.2.5a)$$

$$r^2 = (\,r_1\, g_{11}\, r_1 + r_2\, g_{22}\, r_2 + r_3\, g_{33}\, r_3\,)\hat{1} = \text{inv.} \qquad (6.2.5b)$$

where the g_{kk} quantities can be either positive or negative.

By using the general rules of the Lie–isotopic theory (Ch. 1.4), the isogroups[55] $\hat{O}(3)$ results to be a tridimensional simple Lie group which can be constructed from the sole knowledge of the original group $\hat{O}$ and the isometric $\hat{\delta}$, i.e., via the original parameters and generators of $O(3)$ and the isotopic element T.

From Eq. (6.2.3) it is easy to see that isorotations satisfy the conditions

$$\det(\hat{\mathfrak{R}}\, T) = \pm\, 1. \qquad (6.2.6)$$

Therefore, $\hat{O}(3)$ is characterized by a continuous semisimple subgroup denoted $S\hat{O}(3)$ for the case $\det(\hat{\mathfrak{R}}T) = +1$, and a discrete invariant part for the case $\det(\hat{\mathfrak{R}}T) = -1$ representing isoinversions (see below).

The realization of the infinitely possible $S\hat{O}(3)$ subgroups is the first application of the Lie–isotopic theory we encounter in this volume. First, the liftings $SO(3) \to S\hat{O}(3)$ imply the preservation of the original parameters, the Euler's angles,

---

[55] The rotational "group" is unique, whether in abstract or in a specific physical application. On the contrary, the isorotational group is unique at the abstract level, while admitting infinitely different realizations evidently depending on the explicit form of the isometric. The situation is similar to the Minkowski "space", which is unique, versus the Riemannian "spaces" which are infinite in number owing to their infinitely possible, different metrics.

$$\theta = \{\theta_1, \theta_2, \theta_3\}, \tag{6.2.7}$$

with the understanding that they are now elements of the isofield $\hat{R}(\hat{\theta},+,*)$ of isoreal numbers $\hat{n} = n\hat{1}$ (Ch. 1.2) and must therefore be written $\hat{\theta} = \theta\hat{1}$.

Nevertheless, their product with an arbitrary quantity Q coincides with the conventional product, $\hat{\theta}*Q \equiv \theta Q$. Whenever the Euler angles *multiply* other quantities, we can therefore continue to use the ordinary form θ. Their range is also remains the original one for isotopies of Class I, but not of Class III (see below).

Second, the lifting SO(3) $\to$ SÔ(3) implies the preservation of the original normalized basis $|b>, <b|b> = 1$ (Proposition I.3.2.1), although expressed in the form $|\hat{b}> = T^{-\frac{1}{2}}|\hat{b}>$ requested by the isonormalization $<\hat{b}|*|\hat{b}> = <\hat{b}|T|\hat{b}> = 1$. We shall then assume as generators of SÔ(3) the conventional adjoint generators of SO(3) in the familiar form

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\tag{6.2.8}$$

The abstract enveloping associative algebra ξ of the sô(3) algebra is now lifted into the isoform $\hat{\xi}$ (Sect. I.4.3) characterized by the isounit $\hat{1}$, the conventional (ordered set of) generators (6.2.9), and all their possible polynomials, resulting in the infinite dimensional basis (isotopic Poincaré–Birkhoff–Witt Theorem I.4.3.1)

$$\hat{\xi}(so(3)): \quad \hat{1}, \quad J_k, \quad J_i * J_j \ (i \leq j), \quad J_i * J_j * J_k \ (i \leq j \leq k), \quad \dots . \tag{6.2.9}$$

The *isocommutation rules* of the *isorotational algebra* sô(3), first studied in ref. [3], are expressible in a variety of forms. The realization given in the original derivation is given by

$$[J_i \hat{,} J_j] = J_i * J_j - J_j * J_i = J_i T J_j - J_j T J_i =$$

$$= \hat{C}_{ijk}(t, r, \dot{r}, \dots) * J_k = \tilde{C}_{ijk}(t, r, \dot{r}, \dots) J_k \tag{6.2.10}$$

where

$$\tilde{C}_{ijk} = \epsilon_{ijk} g_{kk}(t, r, \dot{r}, \dots) \tag{6.2.11}$$

are the *structure functions* (see Sect. I.4.5 and the *isocartan tensor* of Eq.s (I.4.4.24)). The tensor ϵ_{ijk} is the conventional totally antisymmetric tensor characterizing the structure constants of so(3). Additional realizations via redefinition of the generators will be provided below.

The *connected isorotational groups* SÔ(3) are obtained via an isoexponentiation in $\hat{\xi}$ (Corollary I.3.1A), resulting in the expression

$$\hat{\Re}(\theta) = \{ e_{|\xi}^{J_1 * \theta_1} \} * \{ e_{|\xi}^{J_2 * \theta_2} \} * \{ e_{|\xi}^{J_3 * \theta_3} \}, \qquad (6.2.12)$$

whose explicit form will be studied in the next section. The preceding expression can be rewritten in the conventional associative envelope ξ of $so(3)$ for computational simplicity

$$\hat{\Re}(\theta) = (\prod_{k=1,2,3} e_{|\xi}^{J_k \hat{T} \theta_k}) \hat{1} = \hat{1} (\prod_{k=1,2,3} e_{|\xi}^{\theta_k \hat{T} J_k}) =$$

$$\stackrel{\text{def}}{=} [\hat{S}(\theta)] \hat{1} = \hat{1} [\hat{s}^t(\theta)] . \qquad (6.2.13)$$

Note that in Eq. (6.2.12) we have the identities $J_k * \theta_k \equiv J_k T \theta_k \hat{1} \equiv J_k \theta_k$. Thus, the unrestricted, integro–differential element T appearing in the exponent of Eq. (6.2.12) originates from the isoexponentiation and *not* from the isoangles θ_k.

The isorotations can then be written in the simpler form

$$r' = \hat{\Re}(\theta) * r \equiv \hat{S}(\theta) r, \qquad (6.2.14)$$

which is used in practical calculations. The understanding is that the mathematically correct form remains the isotopic form (6.2.13) to prevent the violation of linearity, transitivity, etc.

Note that the sole unknown in structures (6.2.12) or (6.2.13) is the explicit form of the isotopic element T, i.e., the explicit functional dependence of the diagonal elements g_{kk} in $T = \text{diag.} (g_{11} , g_{22} , g_{33})$. Note also that expression (6.2.12) holds for an arbitrary functional dependence of g_{kk}, including a dependence in the local coordinates, as well as an integral dependence on all needed quantities. Note finally that the convergence of isoexponentials (6.2.12) into finite, explicitly computable transforms is ensured by the convergence of the original exponentials under the topological conditions of Class III. Thus, all isorotational transformations of Class III are explicitly computable in a finite form (see below for examples).

The discrete part of $\hat{O}(3)$ is characterized by the *isoinversions*

$$\hat{\pi} * r = \pi r = - r, \qquad (6.2.15)$$

where π characterizes the conventional discrete components of $O(3)$. It is easy to verify that the *isocasimir invariants* of $\hat{O}(3)$ (Sect. 1.4.3) are given by

$$\hat{C}^{(1)} = \hat{J}^2 = J_k * J_k , \qquad \hat{C}^{(2)} = \hat{\pi} , \qquad (6.2.16)$$

because, as one can see

$$[J_k \hat{,} J^2] = [J_k \hat{,} \hat{\pi}] \equiv 0 , \quad k = 1, 2, 3 . \tag{6.2.17}$$

This confirms that the isoinversions constitute a discrete invariant isosubgroup of $\hat{O}(3)$, as in the conventional case.

The notion of isorotation groups is turned into that of *isorotational symmetries* by noting that isotransforms (6.2.2) leave invariant, by construction, the separation in $\hat{E}(r,\hat{\delta},\hat{\Re})$ (see Theorem I.4.6.1)

$$r'^2 = r'^i \hat{\delta}_{ij} r'^j \equiv r^i \hat{\delta}_{ij} r^j = r^2, \tag{6.2.18}$$

owing to the property (Theorem I.4.6.1)

$$\tilde{S}^t \hat{\delta} \tilde{S} \equiv \hat{\delta}, \tag{6.2.19}$$

which is identically verified for all possible metrics $\hat{\delta}$ of the class admitted, plus similar identities for the isoinversions.

The capability for the isorotational symmetries $\hat{O}(3)$ to leave invariant all possible deformations of the sphere then trivially follows from invariance (6.2.18). The understanding is that, on more rigorous grounds, the isorotations are the symmetries of the isosphere because defined on the isoeuclidean space.

Isorotations are easily computable from Eq.s (6.2.12) via the use of the conventional parameters (6.2.7), the conventional, regular representation of the angular momentum components, Eq.s (6.2.9), and realization (6.2.5a) of the isometric, $\hat{\delta}$ = diag. (g_{11}, g_{22}, g_{33}). As an example, an isorotation around the third axis is given by [3]

$$r' = \hat{\Re}(\theta_3) * r = \tilde{S}(\theta_3) r =$$

$$= \begin{pmatrix} \cos [\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & g_{22}(g_{11}g_{22})^{-\frac{1}{2}} \sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ -g_{11}(g_{11}g_{22})^{-\frac{1}{2}}\sin[\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & \cos [\theta_3(g_{11}g_{22})^{\frac{1}{2}}] & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{6.2.20}$$

Note the characterization of the *isoangle*

$$\hat{\theta}_3 = \theta_3 \Delta^{\frac{1}{2}} , \quad \Delta = \det T = g_{11} g_{22} . \tag{6.2.21}$$

Note also the possibility of reformulating isotransforms (.6.2.20) in terms of *isotrigonometric functions*

$$\hat{\cos} \hat{\theta}_3 = g_{11}^{-\frac{1}{2}} \cos [\theta_3 (g_{11} , g_{22})^{\frac{1}{2}}], \tag{9.2.22a}$$

$$\hat{\sin} \hat{\theta}_3 = g_{22}^{-\frac{1}{2}} \sin [\theta_3 (g_{11} , g_{22})^{\frac{1}{2}}], \tag{6.2.22b}$$

(see App. I.6.A for details).

We learn from the results of this section that the crucial dependence (6.2.21)

of the isoangle originates from the very structure of the isotopic lifting of the rotational symmetry.

In turn, lifting (6.2.21) is predictably at the origin of the isotopic generalization of the eigenvalues of the quantum mechanical angular momentum and spin, as we shall see.

The verification that isotransformations (6.2.20) do indeed leave invariant the isoseparation (6.2.18) is a simple but instructive exercise for the interested reader.

The isodualities were first studied in ref.s [3] via the the classification of all possible isotopes $\hat{O}(3)$ of Class III. Such classification can be done in essentially two ways. First, we can classify all possible deformations (6.2.18) of the sphere and classify them into compact and noncompact hypersurfaces. Alternatively, we can classify directly all simple groups $\hat{O}(3)$ into compact and noncompact structures. To reach a full classification within the context of the isotopic methods, we need the following

Definition 6.2.2 [3]: *Let $\hat{M}(x,\hat{g},\hat{R})$ be a (finite-dimensional) isospace on the isoreal field $\hat{R}(\hat{n},+,\hat{*})$. Recall that the corresponding isodual isospace (Sect. I.3.2) is characterized by*

$$\hat{M}^d(x,\hat{g}^d,\hat{R}^d): \quad \hat{g}^d = -\hat{g}, \quad \hat{1} \to \hat{1}^d = -\hat{1} \qquad (6.2.23)$$

Let $\hat{G}(n)$ be an n-dimensional Lie-isotopic isosymmetry of the isometric $\hat{g}$ of $\hat{M}(x,\hat{g},\hat{R})$. Then, the corresponding "isodual isosymmetry" $\hat{G}^d(n)$ of $\hat{g}^d$ is the symmetry of the isodual space $\hat{M}^d(x,\hat{g}^d,\hat{R}^d)$, i.e., the isosymmetry constructed with respect the isodual isounit $\hat{1}^d = -\hat{1}$,

$$\hat{G}^d(n), \quad \hat{1}^d = -\hat{1}. \qquad (6.2.24)$$

Similarly, the conventional group of rotations $O(3)$ on $E(r,\delta,R)$ admits an isodual image called *isodual rotational symmetry* $O^d(3)$ which is the group of isometries of $E^d(r,\delta,R^d)$, first identified in ref. [3].

It is evident that $O^d(3)$ is antiisomorphic to $O(3)$. Note however the essential and necessary role of the isotopies of Lie's theory for the very conception and construction of isodual groups. In fact the isodual structure $O^d(3)$ demands *bona fide* isotopic commutators with a nontrivial isotopic element $T = (-1)$ which, as such, cannot be possibly formulated with the conventional Lie theory.

Finally, to avoid misinterpretation of the isodual symmetry $O^d(3)$ one should have a knowledge of its underlying isodual space $E^d(r,\delta^d,R^d)$, that is, a *space whose basic unit is negative* (Ch. 1.2). This implies the inversion of all quantities, including their modulus i.e.

$$O(3): \quad |r|_R > 0 \quad \rightarrow \quad \hat{O}^d(r): \quad |r|_{R^d} = - |r| < 0 \tag{6.2.25}$$

To understand the classification of $\hat{O}(3)$ one should further recall from Sect. 1.3.4 the unifying power of the notion of the isospaces of Class III. In fact, the three-dimensional isometrics $\hat{\delta}$ of Class III include in their classification all possible conventional (Hermitean) metrics of the same dimension, such as:

1) the conventional Euclidean δ = diag. (1, 1, 1)

2) the Riemannian metric $\hat{\delta}$ = g(r) = T(r)δ,

3) the (2+1)-dimensional Minkowski metric δ = Tδ = diag (1, 1, -1),

4) the Finslerian metric $\hat{\delta}$ = $\hat{\delta}$(r, ṙ),

as well as all their infinitely possible isotopies and all their infinitely possible isodualities.

Thus, the isotopic methods of Class III unify the Euclidean, Minkowski, Riemannian, Finslerian and all other possible spaces of the same dimension into one, single, abstract, geometric structure, the isospace $\hat{E}(r,\hat{\delta},\hat{R})$. Their separation must be <imposed> via suitable restrictions, such as that for the isounit to be positive-definite (Class I), or of being negative definite (Class II), etc.

With the above notions in mind, we are now in a position to present the following results

Theorem 6.2.1 (Classification of $\hat{O}(3)$ [3]): *All possible isogroups $\hat{O}(3)$ of Class III (characterized by sufficiently smooth, bounded, nowhere degenerate and Hermitean , but not necessarily positive-definite isounits $\hat{1}$ are given by:*

1) The conventional rotational group O(3) with unit 1 = diag. (1, 1, 1),

2) an infinite class of nonlinear, nonlocal and noncanonical "isorotational groups" properly speaking, those isomorphic to O(3) occurring for sig. $\hat{1}$ = (+, +, +),

3) the conventional Lorentz group in (2+1)-dimension O(2.1) with isounit $\hat{1}$ = diag. (1, 1, -1),

4) an infinite class of nonlinear, nonlocal and noncanonical isolorentz groups in (2+1)-dimensions $\hat{O}(2.1)$ isomorphic to O(2.1), and occurring for sig $\hat{1}$ = (+, +, -),

5) The isodual rotational group $O^d(3)$ with isounit 1^d = diag. (-1, -1, -1),

6) an infinite class of isodual isorotational groups $\hat{O}^d(3) \approx O^d(3)$ characterized by sig $\hat{1}$ = (-, -, -),

7) the isodual (2+1)-dimensional Lorentz group $O^d(2.1)$ with isounit $\hat{1}^d$ = diag. (-1, -1, 1), and

8) the infinite class of isodual isolorentz groups $\hat{O}^d(2,1) \approx O^d(2.1)$ characterized by sig. $\hat{1}_d$ = (-, -, +)

We have recalled before the unifying power of the isotopic techniques for

metric or pseudo–metric spaces. The corresponding unifying power for Lie groups is expressed by the following:

Lemma 6.2.1 [3]: *The simple abstract isogroup $\hat{O}(3)$ of Class III unifies all compact and noncompact three–dimensional simple Lie groups of Cartan's classification*

A similar property holds for the isotope $\hat{O}(4)$ (see Ch. 11.8) which unifies all simple six–dimensional Lie groups of Cartan's classification. This unifying property has been *conjectured* to exist for all simple Lie–groups of a given dimension on a field of characteristic zero, but it has not yet been proved until now.[56]

The above classification can also be reached via the use of the isocartan form of Definition 1.4.4.3 which identifies the compact and noncompact elements of $\hat{O}(3)$ which, in turn, can then be classified into isogroups and isodual isogroups.

But perhaps the most suggestive classification is that of the basic invariant (6.2.4b) given by all possible isotopic elements

$$T = \text{diag.} = \text{diag.}(g_{11}, g_{22}, g_{33}) = (\pm b_1^2, \pm b_2^2, \pm b_3^2), \qquad (6.2.26a)$$

$$b_k = b_k(t, r, \dot{r}, \ddot{r}, ...) > 0, \qquad k = 1, 2, 3, \qquad (6.2.26b)$$

which implies the following classification of hypersurfaces

$$\hat{r} = \pm r_1 b_1^2 r_1 \pm r_2 b_2^2 r_2 \pm r_3 b_3^2 r_3 > 0. \qquad (6.2.27)$$

One then recognizes the following geometric structure which evidently corresponds to the classification of Theorem 6.2.1:

1) the sphere in $E(r,\delta,R)$ with $T = \text{diag.}(+1, +1, +1)$,

2) the infinitely possible ellipsoidical deformations of the sphere with $T = \text{diag.}(+b_1^2, +b_2^2, + b_3^2)$,

3) the hyperboloid with $T = \text{diag.}(+1, +1, -1)$ in two–dimensional Minkowski space $M(x,\eta,R)$,

4) the hyperboloids of one sheet (elliptic paraboloids) with $T = \text{diag.}(+b_1^2, +b_2^2, -b_3^2)$,

5) the isodual sphere in isodual space $E^d(r,\delta^d R^d)$ with $T = \text{diag.}(-1, -1, -1)$,

6) the isodual ellipsoidical deformations of the sphere with $T = \text{diag.}(-b_1^2, -b_2^2, - b_3^2)$,

7) the isodual hyperboloid with $T = \text{diag.}(-1, -1, +1)$ in isodual Minkowski

[56] The reader interested in this problem should keep in mind that Theorem 1.2.7.1 on the isotopic unification of all possible fields has been worked out and presented in Ch. 1.2 precisely as a foundation for this possible isotopic unification and, more specifically, for the possibility of including the exceptional, simple Lie groups.

space M^d (x, η^d, R^d), and

    8) the hyperboloids with two sheets (hyperbolic paraboloids) with

$T = \text{diag. } (-b_1{}^2, -b_2{}^2, +b_3{}^2).$

THE ISOSPHERE
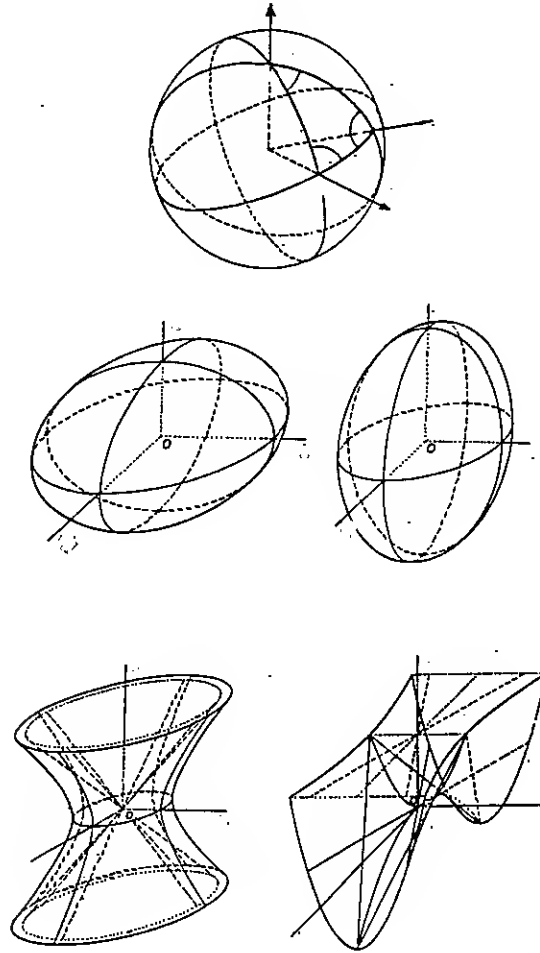
Figure 6.2.1: A schematic view of all possible hypersurfaces in three dimensional Euclidean space which are unified by the isosphere of Class III in isoeuclidean space and left invariant by the isogroup Ô(3) of the same class, as per their original derivation by the author in ref. [3]. The classification includes the conventional sphere, the oblate and prolate spheroidal ellipsoids, the elliptic paraboloids, and the hyperbolic paraboloids. When formulated in an isoeuclidean space $E(r, \hat{\delta}, \hat{R})$ of Class III, all these figures become perfectly spherical. By no means, such a unification exhausts the notion of isosphere. In fact, we remain with the isosphere of Class IV which is a novel geometric notion because the semiaxes tend to zero but, jointly,

their related isounits tend to infinity. Note that the isosphere of Class IV contains all possible compact and noncompact cones. Finally, we have the isosphere of Class V which encompasses all preceding notions and provides additional novel geometric concepts, such as the isosphere whose unit is a step-functions or a lattice or a distribution. The notion of the isosphere used in these volumes is rather limited and restricted for simplicity to that of Class I, which includes only the ellipsoidical deformations of the sphere. Nevertheless, more general realizations of the isosphere will emerge rather naturally when studying more advanced geometric notions, such as the isodual gravitation for antimatter and its possible connection with ordinary gravitation of matter.

What is remarkable is that all these surfaces are unified into one abstract notion, that of isosphere of Class III (see Fig. 6.2.1 for additional details).

It is an instructive exercise for the interested reader to verify that isotransform (6.2.20) apply for *all* the above quadrics. One can begin by verifying that isotransforms (6.6.20) characterize, first, *compact* SO(3) transforms for T = diag. (1, 1, 1) and *noncompact* SO(2.1) transforms for T = diag. (1, 1, −1). In the latter case the trigonometric functions become hyperbolic and the range of the parameters becomes evidently infinite.

It is then equally instructive to verify that the same isotransform (6.2.20) characterizes *isodual compact* and *noncompact* structures $SO^d(3)$ and $SO^d(2.1)$ for T^d = diag. (−1, −1, −1) and T^d = (−1, −1, 1).

It is finally instructive to verify that the *same* isotransform (6.2.20) characterizes the isotopic structures SÔ(3) and SÔ(2.1) for trT = (+, +, +) and trT = (+, +, −), respectively, as well as their isoduals $SÔ^d(3)$ and $SÔ^d(2.1)$ for trT = (−, −, −) and trT = (−, −, +), respectively. This illustrates the unifying power of isotopic transformation theory.

As we shall see throughout this volume, the isotopies of the rotational symmetry have fundamental implications in various branches of physics. Their use in this chapter will be quite restricted and limited to the ellipsoidical deformations of the sphere. Nevertheless it may be useful to point out an example of application for the explicit construction of the space–time symmetries of *conventional* Riemannian separations, such as those of the familiar Schwartzschild's line element in three–dimension (4.2.16), i.e.,

$$ds^2 = (1 - 2M/r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \tag{6.2.28}$$

It is important for the reader to verify that the isorotational symmetry Ô(3) constructed with isotopic element

$$T = \text{diag.} \{ (1 - 2M/r)^{-1}, r^2, r^2 \sin^2 \theta \}, \tag{6.2.29}$$

i.e., for the isometric elements

$$g_{11} = (1 - 2 M / r)^{-1}, g_{22} = r^2, g_{33} = r^2 \sin^2 \theta, \qquad (6.2.30)$$

leaves invariant line element (6.2.28).

The explicit form of the symmetry transformations has already been computed and is given by Eq.s (6.2.20) for all infinitely possible Riemannian metrics in three space–dimensions.One must merely plot the Riemannian g_{kk} elements in isotransforms (6.2.20) without any additional calculation or verification, because the invariance of the Riemannian line element is ensured by the Fundamental Theorem on Isosymmetries I.4.6.1. In Ch. II.9 we shall then study the invariance of all corresponding Riemannian metrics in (3+1)-dimensions.

What we have proved is therefore the following:

Lemma 6.2.2 [3]: *All infinitely possible Riemannian line elements with metric* g(r) *in three space–dimensions are invariant under the rotational symmetry.*

The results of this section are important to illustrate a property indicated in Ch. I.4, that the isotopies of Lie's symmetries do not create new Lie symmetries (because all Lie symmetries over a field of characteristic zero are known), but provide instead new *nonlinear, nonlocal* and *noncanonical realizations* of known Lie symmetries.

In fact, the historical difficulties in computing the symmetry transformations of the Schwartzschild line element in the needed explicit form are due precisely to its nonlinearity in the coordinates. These difficulties should then be compared with the simplicity of the explicit form (6.2.20) produced by the isotopic methods, as well as the universality of the solution for all possible Riemannian metrics g(r).

But the restriction of the applications of the isotopic methods only to Riemannian spaces is grossly unjustified, owing to the arbitrariness of the isometric. We reach in this way the following

Lemma 6.2.3 [3]: *The isorotational symmetry also constitutes the symmetry of the isoriemannian spaces of the interior gravitational problem in three space dimensions (Sect. 1.3.6) with the most general possible isometrics* $\hat{g} = \hat{g}(t, r, \dot{r}, \ddot{r}, \mu \tau, n, ...)$ *with a nonlinear, nonlocal-integral and noncanonical dependence on time* t, *coordinates* r *, velocities* $\dot{r}$, *accelerations* $\ddot{r}$, *density* μ, *temperature* τ, *index of refraction* n *and any needed additional variable.*

As we shall see in Ch. 9, the introduction, apparently for the first time, of a "universal symmetry" for gravitation has rather profound implications for the entire theory.

6.3: ISOTOPIES AND ISODUALITIES OF CLASSICAL ANGULAR MOMENTUM

From now on, unless otherwise stated, we restrict our attention to the isotopes $\hat{O}(3)$ of $O(3)$ properly speaking, those of Class 1 with sig. $\hat{\delta} = (+1, +1, +1), \hat{\delta} > 0$, as well as to the isoduals $\hat{O}^d(3)$ of Class 11 with sig. $\hat{\delta} = (-1, -1, -1)$, $\delta^d < 0$. The terms *isorotations* and *isorotational symmetries* shall therefore be restricted to isotopes locally isomorphic to the conventional rotations (see below for a characterization).

By recalling that all nonsingular Hermitean and positive–definite matrices can be diagonalized, all isometrics of the class considered can be written in the diagonal form

$$\hat{\delta} = T\delta \equiv \hat{T} = \text{diag.}(b_1^2, b_2^2, b_3^2), \quad b_k = b_k(t, r, \dot{r}, \ddot{r}, \ldots) > 0, \quad (6.3.1)$$

which is assumed hereon as our basic form. In essence, the isotopies we are studying characterize the deformations of the sphere

$$r^2 = r^1 r^1 + r^2 r^2 + r^3 r^3 > 0, \quad (6.3.2)$$

into all infinitely possible compact (ellipsoidical) forms

$$\hat{r}^2 = r^k b_k^2 r^k = + r^1 b_1^2 r^1 + r^2 b_2^2 r^2 + r^3 b_3^2 r^3 > 0, \quad (6.3.3)$$

which are the only one achievable in the physical reality.

Some of the main properties of isorotations can then be expressed as follows:

Theorem 6.3.1 [3]: *The groups of (compact) isosymmetries $\hat{O}(3)$ of all infinitely possible ellipsoidical deformations of the sphere on the isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{R})$, verify the following properties:*

1) The groups $\hat{O}(3)$ consist of infinitely many different simple groups corresponding to the infinitely many possible deformations of the sphere (explicit forms of the isometric), Eq. (6.3.3);

2) All isosymmetries $\hat{O}(3)$ are locally isomorphic to $O(3)$ because of the positive–definiteness of the isounits; and

3) The groups $\hat{O}(3)$ constitute "isotopic coverings" of the conventional group $O(3)$ in the sense that:

3.a) The groups $\hat{O}(3)$ are constructed via methods (the Lie–isotopic theory) structurally more general than those of $O(3)$ (the conventional Lie's theory);

3.b) The groups Ô(3) represent physical conditions (deformations of the sphere; inhomogeneous and anisotropic interior physical media; etc.) which are broader than those of the conventional symmetry (perfectly rigid sphere; homogeneous and isotropic space; etc.); and

3.c) All groups Ô(3) recover O(3) identically whenever Î = 1 and they can approximate the latter as close as desired for Î ≈ 1.

It is generally believed in both the mathematical and physical literature that the rotational symmetry is broken by ellipsoidical deformations of the sphere. This is yet another belief disproved by hadronic mechanics. In fact, we have the following:

Corollary 6.3.1A [3]: *The rotational symmetry is not broken by ellipsoidical deformations of the sphere, but it is instead exact, provided that it is realized at the covering Lie–isotopic level with respect to the isounit* $Î = δ^{-1}$.

Note that the conventional <rotations> are indeed no longer a symmetry of the deformed sphere. Corollary 6.3.1.A therefore focuses the attention on the difference between the violation of a symmetry in conventional spaces and its exact validity for the corresponding isospace.

We encounter in this way another example of *reconstruction of an exact space-time symmetry when believed to be broken.* The same reconstruction holds in a variety of cases for all remaining space–time symmetries, e.g., for the Galilei and the Lorentz symmetries, as we shall see in the next chapters.

Note that the isorotations can be explicitly written in E(r,δ,R)

$$r' = \Re(\theta) * r = \Re(\theta) \, δ(t, r, \dot{r}, \ddot{r},...) \, r = \tilde{S}(t, r, \dot{r}, \ddot{r}, ...) \, r \qquad (6.3.4)$$

and therefore result to be intrinsically nonlinear. This is due to the fact that the functional dependence of the isotopic elements is completely unrestricted by the isotopies, thus having the most general possible functional dependence $g_{kk} = b_k^2(t, r, \dot{r}, \ddot{r}, ...)$ which enters in the arguments and coefficients of the trigonometric function in Eq.s (6.2.20). We therefore have the following

Corollary 6.3.1B [3]: *While conventional rotations are linear, local and canonical transformations in* E(r,δ,R), *isorotations are isolinear, isolocal and isocanonical in* Ê(r,δ,R̂), *but nonlinear, nonlocal and noncanonical when projected into* E(r,δ,R)

A further important result is the isotopic generalization of the conventional *Euler's theorem* on the general displacement of a rigid body with one point fixed which we can express via the following:

Theorem 6.3.2 [3]: *The general displacement of an elastic body with one fixed point is an isorotation* Ô(3) *of Class I around an axis through the fixed point.*

The above theorem illustrates the use of the classical isorotational symmetry for the characterization of deformable bodies [7].

ISOROTATIONS AS A THEORY OF DEFORMABLE BODIES
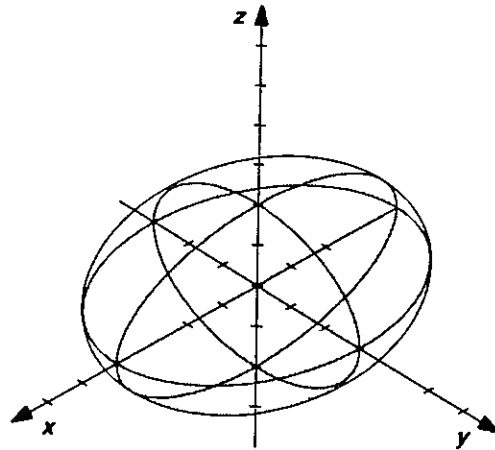
FIGURE 6.3.1: A schematic view of the central features of the isorotational theory Ô(3) of Class I: the capability of providing the invariance, first, of spheroidal ellipsoids and, then, of all their infinitely possible deformations, by always preserving the local isomorphism with the conventional rotational symmetry O(3). Thus, *the conventional theory of rotations is a theory of rigid bodies, as well known from undergraduate courses in physics, while the theory of isorotations is a theory of elastic and deformable bodies. This feature is at the foundation of the notion of <isoparticle> we shall study later on in Ch.s II.8 and. II.9.*

For comparative purposes with the following operator treatment, we now briefly outline the *classical realizations of isorotations.* First, let us consider the three–dimensional isoeuclidean space Ê(r,$\hat{\delta}$,R̂). Its local coordinates are usually assumed to be *contravariant* and we shall write r = {r^k}, k = 1, 2, 3. The simplest possible case is evidently that with the characteristic b–quantities independent on the local coordinates, but dependent on time t, velocities ṙ, accelerations r̈ and any needed additional quantity. In this case we have the isometric with

covariant indices

$$\delta = (\hat{\delta}_{ij}) = T^\circ \delta = \text{diag.} (b_1{}^2, b_2{}^2, b_3{}^2), \quad \partial b_k / \partial r = 0, \quad b_k > 0, \qquad (6.3.5)$$

with *contravariant* form

$$(\hat{\delta}^{ij}) = (\hat{\delta}_{ij})^{-1} = \text{diag.} (b_1{}^{-2}, b_2{}^{-2}, b_3{}^{-2}) , \quad \hat{\delta}_{ik} \hat{\delta}^{kj} = \delta^i{}_j . \qquad (6.3.6)$$

We consider now the *isocotangent bundle* (*isophase space*) T*Ê(r,δ̂,R̂) with local coordinates $a = \{a^\mu\} = \{r, p\} = \{k, p_k\}$, $\mu = 1, 2, ..., 6$, where the linear momentum p_k is *contravariant*, as usual. The raising and lowering of the indices therefore follows the rules

$$r_k = \hat{\delta}_{ki} r^i = b^\circ{}_k{}^2 r^k , \qquad p^k = \hat{\delta}^{ki} p_i = b_k{}^{-2} p_k . \qquad (6.3.7)$$

The Lie–isotopic brackets (11.1.4.13) then assume the simple form

$$[A \,\hat{,}\, B] = \frac{\partial A}{\partial r^p} \, 1^p{}_q \, \frac{\partial B}{\partial p_q} - \frac{\partial B}{\partial r^p} 1^p{}_q \frac{\partial A}{\partial p_q} =$$

$$= \frac{\partial A}{\partial r^k} \, b_k{}^{-2} \frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k} b_k{}^{-2} \frac{\partial A}{\partial p_k} , \qquad (6.3.8)$$

To identify the Lie–isotopic algebra sô(3), let us compute first the *classical fundamental isocommutation rules* which are readily given by

$$([a^\mu \,\hat{,}\, a^\nu]) = \begin{pmatrix} [r^i \,\hat{,}\, r^j] & [r^i \,\hat{,}\, p_j] \\ [p_i \,\hat{,}\, r^j] & [p_i \,\hat{,}\, p_j] \end{pmatrix} = (\hat{\omega}^{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -\hat{1} & 0 \end{pmatrix}, \qquad (6.3.9)$$

showing the isotopy $1 \to \hat{1}$ as in Eq.s (11.3.1.11). However, when considering the isocommutation rules between r_i and p_j we have

$$\begin{pmatrix} [r_i \,\hat{,}\, r_j] & [r_i \,\hat{,}\, p_j] \\ [p_i \,\hat{,}\, r_j] & [p_i \,\hat{,}\, p_j] \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -I & 0 \end{pmatrix}, \qquad (6.3.10)$$

Lemma 6.3.1: *The classical isocommutation rules between* r_i *and* p_j *coincide with the conventional, canonical ones.*

Next, we introduce the generators of the Lie–isotopic algebra sô(3) which, by central assumption, are given by the *conventional, contravariant generators* of

O(3), [57]

$$J^k \;=\; \epsilon^{kij} r_i \, p_j \;=\; \epsilon^{kij} b_i^{\,2} \, r^i \, p_j \,. \tag{6.3.11}$$

The above quantities are called the components of the *Birkhoffian angular momentum* [7] to emphasize the fact that they characterize a generalized notion defined on $T^*\hat{E}(r,\hat{s},\hat{R})$ rather than on $T^*E_2(r,\delta,R)$.

In particular, the magnitude of the Hamiltonian angular momentum is given by the familiar expression $J^2 = J^k J_k = \delta^{ij} J_k J_k$, while the magnitude of the Birkhoffian angular momentum is given by[58]

$$J^{\,\hat{2}} = \; J * J = \; J^k * J_k \;=\; \hat{\delta}^{ij} J_i * J_j \;=\; \hat{\delta}^{ij} J_i \, T \, J_j \,. \tag{6.3.12}$$

Next, the following isocommutation rules are readily computed

$$[\, J^k \,\hat{,}\, r_i \,] \;=\; \epsilon^{kij} r_j \,, \tag{6.3.13a}$$

$$[\, J^k \,\hat{,}\, p_i \,] \;=\; \epsilon^{kij} p_j \,, \tag{6.3.13a}$$

and evidently coincide with the conventional ones.

The desired *classical isorotational algebras* $\hat{so}(3)$ are then given by [3][59]

$$\hat{so}(3): \qquad [\, J^i \,\hat{,}\, J^j \,] \;=\; \epsilon^{ijk} J^k , \tag{6.3.14}$$

namely, the isocommutation rules of $\hat{so}(3)$ have the same structure constants as for the conventional $so(3)$. This establishes the local isomorphism $\hat{so}(3) \approx so(3)$ *ab initio*. For different classical realizations one may consult ref.s [3,7].

The *isocenter* of the enveloping algebra $\hat{\xi}$ is given by the isomagnitude of the Birkhoffian angular momentum, $C^{(2)} = J^{\hat{2}}$, as expected. In fact,

---

[57] Unlike the operator case to be considered soon, note that the quantities r^i and p_j here are ordinary functions and, thus, they do not require the isotopic product $r^i * p_j$. Note also the subtle but important differences of the indices of $\hat{\delta} = (\hat{\delta}_{ij})$, $I = (1^i{}_j)$ and $T = (T_i{}^j)$. Thus, only the tensor $\hat{\delta}_{ij}$ or its inverse $\hat{\delta}^{ij}$ used for lowering or raising indices.

[58] $J^{\hat{2}}$ is in this case an *isoscalar*, that is, a scalar quantity in isospace. For this reason it must be contracted in the form $J^{\hat{2}} = J^k * J_k$. In other cases encountered later on, this condition is not required, and the isocasimir is $J^{\hat{2}} = J^k * J^k$.

[59] Isocommutation rules (6.1.14) disprove another popular belief in Lie's theory, that the compactness or noncompactness of an algebra can be ascertained from the structure constants. In fact, the structure constants ϵ^{ijk} are those of the *compact* SO(3) algebra, yet isoalgebra (6.1.14) can represent the *noncompact* so(2.1) algebra for $T = $ diag. $(1, 1, -1)$. The latter possibility has been *excluded* from the physical studies of this and of the following sections by restricted the isotopic element to be of Class I.

$$[J^2 \,\hat{}\, J^i] = 2\,\epsilon^{kij}\,J^k\,J^j \equiv 0. \qquad (6.3.15)$$

The desired *classical isorotational group* $S\hat{O}(3)$ can then be expressed [3,6]

$$\hat{\mathfrak{R}}(\theta) = \{ \prod_{k=1,2,3} e^{\theta_k \omega^{\mu\alpha} 1_{2\alpha}{}^\nu (\partial J_k / \partial a^\nu)(\partial / \partial a^\mu)}|_\xi \}\hat{1} == \hat{S}(\theta)\hat{1}, \qquad (6.3.16)$$

where the exponentials are expanded in the conventional associative envelope ξ for simplicity.

Note the true realization in structure (6.3.16) of the notion of isotopic lifting of a Lie symmetry, consisting of the preservation of the original generators and parameters of the symmetry, and the isotopic generalization of the structure of the Lie group itself via the liftings
$1 \Rightarrow \hat{1}.$

The computation of examples is straightforward. For instance, a (classical) *isorotation around the third axis* is given by [3]

$$r' = \hat{\mathfrak{R}}(\theta_3) * r = \hat{S}(\theta_3)\, r = \qquad (6.3.17)$$

$$= \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos(\theta_3 b_1 b_2) - y\, b_2\, b_1^{-1}\sin(\theta_3 b_1 b_2) \\ x\, b_1\, b_2^{-1}\sin(\theta_3 b_1 b_2) + y\cos(\theta_3 b_1 b_2) \\ z \end{pmatrix}.$$

The invariance of the deformed sphere under the above transformation is then trivial.

We now compute a general isorotation which brings a point P on the isosphere to an arbitrary point Q. Its projection in Euclidean space is the transformation of a point P on an *ellipsoid* into another arbitrary point Q of the same ellipsoid (Figure 6.3.1). As in the conventional case (see, e.g., ref. [12], Sect. 1.2), such a rotation can be computed via three successive isorotations:

1) An isorotation $\hat{\mathfrak{R}}(\phi_1)$ of an angle $\phi_1 = \theta_1$ in the (x, y)-plane such that $0 \leq \hat{\phi}_1 \leq 2\pi,\ \hat{\phi}_1 = b_1 b_2 \phi,$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\phi_1 b_1 b_2) & -b_2\, b_1^{-1}\sin(\phi_1 b_1 b_2) & 0 \\ b_1\, b_2^{-1}\sin(\phi_1 b_1 b_2) & \cos(\phi_1 b_1 b_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad (6.3.18)$$

2) An isorotation $\hat{\mathfrak{R}}(\theta)$ around the polar axis z of an angle $\theta = \theta_2,$ such that $0 \leq \hat{\theta} \leq \pi,\ \hat{\theta} = b_3 \theta,$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta\, b_3) & -b_2\, b_3^{-1}\sin(\theta\, b_3) \\ 0 & b_3\, b_2^{-1}\sin(\theta\, b_3) & \cos(\theta\, b_3) \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad (6.3.19)$$

3) An isorotation $\hat{\Re}(\phi_2)$ in the (x, y)-plane with angle $\phi_2 = \theta_3$ such that $0 \leqq \hat{\phi}_2 \leqq 2\pi$, $\hat{\phi}_2 = b_1 b_2 \phi_2$,

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos(\phi_2 b_1 b_2) & -b_2 b_1^{-1}\sin(\phi_2 b_1 b_2) & 0 \\ b_1 b_2^{-1}\sin(\phi_2 b_1 b_2) & \cos(\phi_2 b_1 b_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \quad (6.3.20)
$$

We therefore have the *general isorotation* on the isosphere

$$
r' = \hat{\Re}(\phi_2, \theta, \phi_1) * r = \hat{\Re}(\phi_1) * \hat{\Re}(\theta) * \hat{\Re}(\phi_2) * r = \hat{S}(\phi_2, \theta, \phi_1) r, \quad (6.3.21a)
$$

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{cc} [\cos\hat{\phi}_1 \cos\hat{\phi}_2 - \sin\hat{\phi}_1 \cos\theta \sin\hat{\phi}_2] \\ [b_1 b_2^{-1}(\sin\hat{\phi}_1 \cos\hat{\phi}_2 + \cos\hat{\phi}_1 \cos\theta \sin b\hat{\phi}_1)] \\ [\sin\theta \sin\hat{\phi}_2] \end{array} \right.
$$

$$
\begin{array}{cc}
[-b_1^{-1}b_2(\cos\hat{\phi}_1 \sin\hat{\phi}_2 - \sin\hat{\phi}_1 \cos\theta \cos\hat{\phi}_2)] & [b_1^{-1}b_3 \sin\hat{\phi}_1 \sin\theta] \\
[-b_2^{-2}b_3(\sin\hat{\phi}_1 \sin\hat{\phi}_2 + \cos\hat{\phi}_1 \cos\theta \cos\hat{\phi}_2)] & [-b_2^{-1}b_3 \cos\hat{\phi}_1 \sin\theta] \\
[b_1 b_2^{-1} \sin\theta \cos\hat{\phi}_2] & [\cos\theta]
\end{array} \left. \begin{array}{c} \\ \\ \\ \end{array} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix}
$$

$$
\hat{\phi}_1 = b_1 b_2 \phi_1, \quad \hat{\theta} = b_3 \theta, \quad \hat{\phi}_2 = b_1 b_2 \phi_1. \quad (6.3.21c)
$$

The *inverse general isorotation* is then given, as in the conventional case [12], by

$$
r = \hat{\Re}(\pi - \phi_2, \theta, \pi - \phi_2) * r'. \quad (6.3.22)
$$

Note that for $0 \leqq \hat{\phi}_1 \leqq 2\pi$, $0 \leqq \theta \leqq \pi$ and $0 \leqq \hat{\phi}_2 \leqq 2\pi$, the corresponding angles ϕ_1, θ and ϕ_2 have different values. This is due to the fact that the angles ϕ_1, θ and ϕ_2 are those measured in our Euclidean space, while the isoangles $\hat{\phi}_1$, $\hat{\theta}$ and $\hat{\phi}_2$ are those measured on the isosphere.

The above relationship between isoangles and angles has been derived from the general rule studied in ref. [7]

$$
\hat{\phi}|_{Birk} = \phi|_{Ham} b_1 b_2, \quad (6.3.23)
$$

according to which the isoangle $\hat{\phi} = b_1 b_2 \phi$ for a deformed sphere is such to restore the conventional angle of the perfect sphere.

In different terms, *the rotation ϕ on a deformed sphere, when represented by the isotopic Euler's theorem via the b-quantities, is compensated by the isorotation in such a way to reproduce the angle $\hat{\phi}$ of rigid rotation of the perfect sphere.* We shall encounter the same rule several times in the remaining studies and will have opportunity to study it in more details.

Note also that for all isosphere of class I, the isoangles are real (not so for Class III as studied in the preceding section).

The representation of a continuously decaying angular momentum is notoriously not possible with conventional rotations, but it is readily achieved by

the isorotations with a functional dependence of the type $b_k = \exp(-\gamma t)$, under which we have

$$J^k = e^{-\gamma t} \epsilon^{kij} r^i p_j. \tag{6.3.24}$$

This illustrates the type of argument of the trigonometric functions of the isorotations one may encounter in practical applications.

The isodualities of the classical rotational symmetry have been studied in ref.s [7] and they will be assumed as known.

The $S\hat{O}(3)$ invariant isomeasure, invariant isointegration and other aspects will be studied in the subsequent operator parts of this chapter.

The study of the conditions for a classical Hamiltonian to be isoinvariant under $\hat{O}(3)$ is instructive [7]. Its operator counterpart will be studied in the next chapter. The extension of the theory to b-functions with an arbitrarily nonlinear and nonlocal dependence also on the coordinates is instructive.

6.4: ISOTOPIES AND ISODUALITIES OF QUANTUM ANGULAR MOMENTUM

The *isoquantization* of the preceding classical isotopies is straightforward via the techniques of Ch. 11.2, as originally done in ref. [4]. Assume for simplicity that the isotopic element G of the Hilbert space $\mathcal{H}$ coincides with the element T of the isoenvelope $\hat{\xi}$, and assume that it is of Class I.

The carrier space is chain the isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ which is here assumed with isotopic elements in the diagonal form $T = \text{diag.} (b_1^2, b_2^2, b_3^2)$ independent on the local coordinates. The local coordinates are the usual contravariant components $r = \{r^k\}$, $k = 1, 2, 3$. As in the preceding section, the coordinates in their covariant form are then given by $r_k = \hat{\delta}_{ki} r^i = b_k^2 r^i$.

Consider then the isooperator form of the momentum, Eq.s (11.3.1.10a)

$$P_k * | \hat{\psi} > = - i \hat{1}_k{}^i \nabla_i | \hat{\psi} > = - i b_k^{-2} \nabla_k | \hat{\psi} >, \quad k=1, 2, 3, \quad \hbar = 1. \tag{6.4.1}$$

The *fundamental operator isocommutation rules* are then given by (Eq.s (11.3.1.11), i.e.,

$$[r^i \hat{,} r^j] * | \hat{\psi} > = [p_i \hat{,} p_j] * | \hat{\psi} > = 0, \ [r^i, p_j] * | \hat{\psi} > = i b_i^{-2} \delta^i{}_j * | \hat{\psi} >, \tag{6.4.2a}$$

$$[r_i \hat{,} r_j] * | \hat{\psi} > = [p_i \hat{,} p_j] * | \hat{\psi} > = 0, \ [r_i, p_j] * | \hat{\psi} > = i \delta^i{}_j * | \hat{\psi} >, \tag{6.4.2b}$$

whose equivalence to the classical forms (11.6.2.9) is manifest. Note also that the isocommutation rules for r_i and p_i coincide with the conventional ones, in a way

fully parallel to the corresponding classical occurrence.

The *hadronic angular momentum* components are then given by

$$\jmath^k = \epsilon^{kij} r_i \, p_j = \epsilon^{kij} b_i^2 \, r^i \, p_j, \qquad (6.4.3)$$

where the coordinates are assumed as ordinary numbers and the momenta as isooperators (or viceversa in the momentum representation) [60].

We therefore have the isoquantization of isocommutation rules (II.6.3.13)

$$[\jmath^k \, \hat{,} \, r_i] * |\hat{\psi} > = \epsilon^{kij} r_j * |\hat{\psi} >, \qquad (6.4.4a)$$

$$[\jmath^k \, \hat{,} \, p_i] * |\hat{\psi} > = \epsilon^{kij} p_j * |\hat{\psi} >, \qquad (6.4.4a)$$

which evidently coincide with the conventional ones.   The

The desired *operator isorotational algebras* are then given by [3,4]

$$\hat{so}(3): \qquad [\jmath^i \, \hat{,} \, \jmath^j] * |\hat{\psi} > = \epsilon^{ijk} \jmath^k * |\hat{\psi} >, \qquad (6.4.5)$$

and their structure constants again coincide with the quantum mechanical form, thus establishing the local isomorphism $\hat{so}(3) \approx so(3)$.

The *operator isocasimirs* are given by

$$\hat{C}^{(0)} = 1, \qquad \hat{C}^{(2)} = \jmath^k * \jmath_k = \delta_{ij} \jmath^i * \jmath^j, \qquad (6.4.6)$$

which are manifestly equivalent to the classical ones.

The above results therefore establish the algebraic equivalence of classical and operator isorotations. In particular, the explicit form of the isorotations coincides with the classical ones (II.6.3.17), and the same result holds for other aspects.

Most importantly, the emerging operator rotational theory is that for deformable spheres (Fig. II.6.3.1). As we shall see, this property is ultimately at the foundations of the new notion of particle characterized by hadronic mechanics.

As indicated in Sect. II.6.1, the most important difference between quantum and hadronic angular momenta is of *conceptual* nature, and given by the fact that the former occurs for a particle moving in empty space, while the latter occurs, by conception, for an extended particle/wavepacket moving within a hadronic medium.

While the conventional theory is local, and results in the known discrete spectrum of admissible stable orbits, the hadronic theory is nonlocal, and monotonically varying orbits are now allowed, as we shall see momentarily. To put it differently, one of the objectives of the hadronic angular momentum is to

---

[60]Recall that the assumption of r as an isoscalar $\hat{r}$ would not alter the definition of hadronic angular momentum because $\hat{r} * p \equiv r \, p$.

avoid the perpetual motion implied by quantum mechanics whereby an electron can orbit in the core of a collapsing star in a locally stable orbit.

6.5: ISOREPRESENTATIONS OF SÔ(2) AND THEIR ISODUALS

It is best to initiate the study of the isorepresentation theory with the case of the one–dimensional SÔ(2) isogroup. To proceed in stages, in this section we shall be as elementary as possible. A more rigorous definition of isoreps is given in the next section, jointly with the notion of isounitarity and other topics, when studying SÔ(3). No contribution in the topic has appeared as of this writing (early 1994) and the isoreps of SÔ(2) are apparently introduced here for the first time.

Let us recall from Ch. I.6 the notions of *isodifferential, isoderivative, isointegral* and *isoexponentiation* with respect to an independent variable ϕ given respectively by

$$\hat{d}\phi \;=\; T_\phi \, d\phi, \qquad \hat{\partial}/\hat{\partial}\phi \;=\; \hat{1}_\phi \, \partial/\partial\phi, \qquad \hat{1}_\phi \;=\; T_\phi^{-1}, \tag{6.5.1a}$$

$$\int \hat{d}\phi \;=\; \hat{1}_\phi \!\int d\theta \;=\; \theta, \qquad \hat{e}^\phi \;=\; \{\, e^{\phi\, T_\phi} \,\}\hat{1}_\phi, \tag{6.5.1}$$

where the independence of the isounits from ϕ is hereon assumed.[61]

The fundamental problem in the construction of the irreducible isoreps of SÔ(2) is the identification of the underlying isotopic element T_ϕ or isounit $\hat{1}_\phi = T_\phi^{-1}$. In fact, the knowledge of T_ϕ permits the identification of the *isomeasure* which, in turn, permits the rigorous treatment of the spectrum of eigenvalues.

The correct expression of the isounit $\hat{1}_\phi$ of SÔ(2) in the isogauss plane can be identified as follows. Consider the two–dimensional isoeuclidean space

$$\hat{E}(r,\hat{\delta},\hat{R}): \quad \hat{\delta} = T\delta, \quad T = \mathrm{diag.}\,(\,b_1^2, b_2^2\,), \quad \hat{1} = T^{-1} = \mathrm{diag.}\,(\,b_1^{-2}, b_2^{-2}\,), \tag{6.5.2a}$$

$$r^2 \;=\; (\, x\, b_1^2\, x \;+\; y\, b_2^2\, y\,)\hat{1} \;\in\; \hat{R}(\hat{n},+,*). \tag{6.5.2b}$$

As studied in Ch. II.5, the *isogauss plane* is the image of the above space under the *isopolar coordinates*

$$x \;=\; r\, b_1^{-1} \cos\hat{\phi}, \qquad y \;=\; r\, b_2^{-1} \sin\hat{\phi}, \tag{6.5.3a}$$

---

[61] As we shall see in Vol. III, drag effects generally depend on velocities, rather than coordinates. Thus, the dependence of the isounit $\hat{1}_\phi$ which is most important for physical applications is that on the derivative (angular velocity) $\phi = d\phi/dt$ which is admitted in expressions (6.5.1) and in the rest of this section. The case with a functional dependence of $\hat{1}_\phi$ on ϕ is left to the interested researcher.

$$r = (x \, b_1{}^2 \, x + y \, b_2{}^2 \, y)^{1/2}, \quad \hat{\phi} = b_1 \, b_2 \, \phi \, . \tag{6.5.3b}$$

From the above expressions we see that, as known from App. I.6.A, the isotopic element and isounit of the isogauss plane are given by

$$T_\phi = b_1 \, b_2 = (\det T)^{1/2}, \quad \hat{1}_\phi = T_\phi{}^{-1} = (b_1 \, b_2)^{1/2} . \tag{6.5.4}$$

The construction of the isoreps of $S\hat{O}(2)$ is then consequential and it is given by a simple isotopy of the corresponding reps of $SO(2)$ (see, e.g., ref. [10], Ch. II).

Recall that in the conventional case the irreducible reps are one-dimensional and are given by $\Re(\phi) = \exp(iM\phi)$ where the exponentiation is that in the enveloping algebra of $\xi[so(2)]$ of the Lie algebra $so(2)$. In particular, $SO(2)$ is defined on the unit circle in the conventional Gauss plane.

Along similar lines, the covering $S\hat{O}(2()$ symmetry can be constructed as the invariant of the *isocircle* in the isogauss plane. By recalling from Sect. II.4.7 that the isotopies do not alter the dimensionality of the representation, it is easy to prove the following

Lemma 6.5.1: *The "irreducible isoreps" of $S\hat{O}(2)$ are given by the isoexponentiation in the isoenvelope of the $s\hat{o}(2)$ isoalgebra*

$$\hat{\Re}(\phi) = \hat{e}^{\, i \, M \, \phi} = \{ e^{\, i \, M \, b_1 \, b_2 \, \phi} \} \hat{1}_\phi , \tag{6.5.5}$$

which *do not* verify the group laws, but rather the covering isogroup laws

$$\hat{\Re}(\phi) * \hat{\Re}(\phi') = \hat{\Re}(\phi') * \hat{\Re}(\phi) = \hat{\Re}(\phi + \phi'), \quad \hat{\Re}(\phi) * \hat{\Re}(-\phi) = \hat{\Re}(0) = \hat{1}_\phi . \tag{6.5.6}$$

Lemma 6.5.2: *The isoinvariant isomeasure of $S\hat{O}(2)$ in the isogauss plane is given by*

$$\hat{d}\phi := T_\phi \, d\phi = (b_1 \, b_2) \, d\phi . \tag{6.5.7}$$

Proof. Let ϕ be a point in the isocircle and let ϕ' be its image under the action of $S\hat{O}(2)$. Then, the necessary and sufficient conditions for an isomeasure $\hat{d}\phi$ to be isoinvariant under $S\hat{O}(2)$,

$$\hat{d}\phi = T_\phi \, d\phi \equiv \hat{d}\phi' = T_{\phi'} \, d\phi' , \tag{6.5.8}$$

are met iff $T_\phi = b_1 \, b_2$. **q.e.d.**

It is then easy to verify the following *isoorthogonality property*

$$(2\pi)^{-1} \int \hat{d}\phi * \hat{e}^{\, i \, M \, \phi} * \hat{e}^{\, i \, N \, \phi} =$$

$$= (2\pi)^{-1} \hat{1}_\phi \int T_\phi \, d\phi \, T_\phi \{ e^{i M T_\phi \Phi} \} \hat{1}_\phi \, T_\phi \{ e^{i N T_\phi \Phi} \} \hat{1}_\phi =$$

$$= (2\pi)^{-1} \int d\phi \{ e^{i M \phi} \} \{ e^{i N \phi} \} \hat{1}_\phi = \hat{\delta}_{MN} = \hat{1}_\phi \, \delta_{MN} . \qquad (6.5.9)$$

Recall that the conventional irreducible reps of SO(2) must be normalized to 1 and be such that $\mathfrak{R}(2\pi) = \mathfrak{R}(0) = 1$ for various reasons of consistency [10]. By the same token, the irreducible isoreps of $S\hat{O}(2)$ must be isonormalized to $\hat{1}_\phi$, as in isogroup laws (6.5.6) and be such that $\hat{\mathfrak{R}}(2\pi) = \hat{\mathfrak{R}}(0) = \hat{1}_\phi$.

Lemma 6.5.3: *The admissible value of the hadronic number* M *in the irreducible isoreps* $\hat{\mathfrak{R}}(\phi) = \hat{\exp}\{iM\phi\}$ *of* $S\hat{O}(2)$ *is no longer discrete, and are characterized by the hadronic spectrum*

$$\tilde{M} = T_\phi \, M = b_1 \, b_2 \, M = 0, 1, 2, \dots . \qquad (6.5.10)$$

Proof. The isorep $\hat{\mathfrak{R}}(2\pi)$ can be explicitly written

$$\hat{\mathfrak{R}}(2\pi) = \hat{e}^{\, i M 2\pi} = [\cos(M \, b_1 \, b_2 \, 2\pi) + i \sin(M \, b_1 \, b_2 \, 2\pi)] \hat{1}_\phi , \qquad (6.5.11)$$

and they can verify isonormalization condition $\hat{\mathfrak{R}}(2\pi) = \hat{1}_\phi$ iff the hadronic spectrum (6.5.10) holds. **q.e.d.**

We now study the isoeigenvalues of the hadronic angular momentum. Consider an isohilbert space with isonormalized isostates

$$\hat{\mathcal{H}}: \quad <\hat{\phi} | \hat{\phi}> = <\phi_{\hat{M}} | T_\phi | \phi_{\hat{N}}> \hat{1}_\phi = \hat{1}_\phi \, \delta_{M\hat{N}} , \quad | \phi_{\hat{M}}> = \hat{e}^{\, i M \phi} . \qquad (6.5.12)$$

As we learned in Ch. II.5, the hadronic angular momentum in the (x, y)-plane in polar coordinates admits the realization

$$\hat{L}_z * | \phi_{\hat{M}}> = -i \hat{1}_\phi \frac{\partial}{\partial \hat{\phi}} | \phi_M> = -i \frac{\partial}{\partial \phi} | \phi_M> , \qquad (6.5.13)$$

Lemma 6.5.4: *The isoeigenvalues of the hadronic angular momentum in the isogaussian plane are given by*

$$\hat{L}_z * | \phi_M> = \hat{1}_\phi \, M | \phi_M> = \frac{M}{b_1 b_2} | \phi_M> = \frac{\tilde{M}}{D} | \phi_M> . \qquad (6.5.14a)$$

$$\tilde{M} = 0, 1, 2, \dots , \qquad D = (\det T)^{1/2} = b_1 \, b_2 . \qquad (6.5.14b)$$

We encounter in this way the fundamental property according to which the discrete quantum mechanical number $M_{QM} = 0, 1, 2, \dots$ of SO(2) is mapped under isotopies into a *continuously varying hadronic number* M_{HM}

$$M_{QM} = 0, 1, 2, \ldots \quad \rightarrow \quad M_{HM} = H_{QM} / (\det T)^{1/2}. \tag{6.5.15}$$

As a simple example, the isoeigenvalues $M_{HM} = n/2$, $n = 1, 3, 5, \ldots$, are admitted in hadronic mechanics under the full preservation of the irreducible character of the isoreps as well as of the isosquare integrability and isounitarity (see next section). By comparison, the value $M = n/2$ are prohibited in quantum mechanics (see, e.g., Biedenharn and Louck [11], the Note in p. 319)

As we shall see in Vol. III, this *departure* from quantum mechanics permits fundamentally *novel* predictions for interior conditions such as a novel structure model of hadrons with ordinary massive particles as constituents, the prediction of a novel source of "hadronic energy", and other advances which are not even conceivable in terms of quantum mechanics, let alone treatable in the quantitative form needed for experimental verifications.

We close this section with the following

Lemma 6.5.5: *The regular and irregular irreducible isoreps of* $S\hat{O}(2)$ *coincide (the isoreps being one-dimensional), while the standard irreducible isoreps are characterized by the "hidden variable"*

$$b_1 = b_2^{-1} = \lambda. \tag{6.5.16}$$

In fact, under the above value we have $T_\phi = 1_\phi = 1$ and the hadronic spectrum coincides with the conventional one. Nevertheless, the isotopic theory permits an explicit and concrete realization of the hidden variable λ which simply cannot be identified with quantum mechanics. As we shall see in Vol. III, hadronic mechanics therefore permits novel applications even in the case when the conventional spectrum of $SO(2)$ holds.

We now outline the *isodual irreducible isoreps* of $S\hat{O}^d(2)$. To understand them, the reader should keep in mind that conventional positive angles and positive eigenvalues have no meaning under isoduality. The basic carrier space is now the *isodual isoeuclidean space*

$$\hat{E}^d(r, \delta^d, \hat{R}^d): \quad \delta^d = T^d \delta, \quad T^d = -\text{diag.}(b_1^2, b_2^2), \quad 1^d = (T^d)^{-1} = -1, \tag{6.5.17a}$$

$$r^{2d} = (-x\, b_1^2\, x - y\, b_2^2\, y)1^d \equiv r^2 \in \hat{R}^d(\hat{n}^d, +, *^d). \tag{6.5.17b}$$

The *isodual isogauss plane* is the image of the above space under the *isodual isopolar coordinates*

$$x = r\, b_1^{-1} \cos \hat{\phi}^d, \qquad y = r\, b_2^{-1} \sin \hat{\phi}^d, \tag{6.5.18a}$$

$$r = (x\, b_1^2\, x + y\, b_2^2\, y)^{1/2}, \quad \phi^d = -b_1\, b_2\, \phi. \tag{6.5.18b}$$

Note that in the above isodualities the local coordinates remain formally unchanged, although they are now defined with respect to a negative–definite unit and, therefore, they do implicitly change in sign. The isoangles change in sign, as indicated earlier. However, they are now defined with respect to a negative–definite unit and, as such, they remain essentially invariant.

To illustrate this point, let us consider a conventional positive angle ϕ, which is defined with respect to the conventional unit +1. The lifting to the isoangle $\phi \rightarrow \hat\phi = T_\phi \phi$ implies no actual change of the value of the angle because the unit is lifted in the amount inverse of the deformation, $1 \rightarrow \hat{1}_\phi = T_\phi^{-1}$. For the case of isoduality we have essentially the same occurrence. In fact, we have the conjugation $\hat\phi \rightarrow \hat\phi = -\hat\phi$, while the unit is also jointly changed in sign, $\hat{1}_\phi \rightarrow \hat{1}^d_\phi = -\hat{1}_\phi$. In conclusion, both the isotopies and isodualities of the angle ϕ preserve their original value. On the contrary, the local coordinates a, y, and the magnitude r *do* change in sign under isodualities because they are preserved unchanged in structures (6.5.17) and (6.5.18).

From the above expressions we see that the isodual isotopic element and isodual isounit are given by

$$T^d_\phi = -b_1 b_2 = -(\det T)^{1/2}, \quad \hat{1}^d_\phi = (T^d_\phi)^{-1} = -(b_1 b_2)^{1/2}. \quad (6.5.19)$$

The *isodual irreducible isoreps* of $S\hat{O}^d(2)$ are then characterized by the isodual isoexponentiation

$$\hat{\Re}^d(\phi^d) = (\hat e^{iM\phi})^d = \hat e_{T^d}^{i^d M^d \phi^d} = \{e^{i^d M^d T^d_\phi \phi^d}\}\hat{1}^d_\phi =$$

$$= \{e^{iM T_\phi \phi}\}\hat{1}^d_\phi = -\hat{\Re}(\phi), \quad (6.5.20)$$

with isodual isoeigenvalue equation[62]

$$<\hat\phi|\ast^d \hat{L}^d_z = i<\hat\phi|\frac{\partial}{\partial\hat\phi}\hat{1}^d_\phi = i<\hat\phi|\frac{\partial^d}{\partial\hat\phi^d} = \quad (6.5.21a)$$

$$= <\hat\phi|(-M^d \hat{1}^d_\phi) = <\hat\phi|\frac{-\tilde M^d}{-b_1 b_2}|\hat\phi_M> = <\hat\phi|\frac{-\tilde M^d}{D^d}, \quad (6.5.22b)$$

and isodual isoeigenvalues

$$\tilde M^d = 0, -1, -2, ..., \quad D^d = -(\det T)^{1/2} = -b_1 b_2. \quad (6.5.14b)$$

The interested reader can then easily derive the remaining properties under

---

[62] One should recall from Sect. I.6.3 that the states $|\hat\phi>$ of an isohilbert or other isospace change into their conjugates $<\hat\phi| = (|\hat\phi>)^\dagger$ under isoduality. Also, the basis $\hat{L}_k$ of $S\hat{O}(3)$ changes sign under isoduality, $\hat{L}^d_k = -\hat{L}_z$, as the reader is encouraged to verify.

isoduality.

6.6: ISOLEGENDRE FUNCTIONS, ISOSPHERICAL HARMONICS, ISOREPRESENTATIONS OF SÔ(3) AND THEIR ISODUALS

We now study the isorepresentations (or isoreps for short) of the three-dimensional SÔ(3) isogroup. No contribution in the topic has appeared as of this writing (early 1994) to our best knowledge[63]. The isoreps of SÔ(3), including the first identification of the isolegendre polynomials and isospherical harmonics, are therefore introduced here for the first time.

The basic carrier space is the three-dimensional isoeuclidean space

$$\hat{E}(r,\hat{\delta},\hat{R}):\quad \hat{\delta} = T\delta,\quad T = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2),\quad \hat{1} = \text{diag.}\,(b_1{}^{-2}, b_2{}^{-2}, b_3{}^{-2}),\quad (6.6.1a)$$

$$r^2 = (\,x\,b_1{}^2\,x\, +\, y\,b_2{}^2\,y\, +\, z\,b_3{}^2\,z\,)\hat{1}\, \in\, \hat{R}(\hat{n},+,*),\qquad (6.6.1b)$$

when written in isospherical coordinates

$$x = r\,b_1{}^{-1}\sin\hat{\theta}\cos\hat{\phi},\quad y = r\,b_2{}^{-1}\sin\hat{\theta}\sin\hat{\phi},\quad z = r\,b_3{}^{-1}\cos\hat{\theta},\qquad (6.6.2a)$$

$$r = (\,x\,b_1{}^2\,x\, +\, y\,b_2{}^2\,y\, +\, z\,b_3{}^2\,z\,)^{1/2},\quad \hat{\theta} = b_3\,\theta,\qquad \hat{\phi} = b_1\,b_2\,\phi.\quad (6.6.2b)$$

The isogroup SÔ(3) can be first introduced as the isosymmetry of isoseparation (6.6.1b) (Sect. 11.6.2). Consider an isorotation of an angle α along a given axis $\hat{g} = \hat{g}(\alpha) \in$ SÔ(3). The image of the point $r = \{r^k\} = \{x, y, z\}$ under $\hat{g}$ is given by the isoorthogonal isotransform

$$r' = \hat{g} * r,\quad r'^k = \hat{g}^{ki}\,T_{ij}\,r^j,\qquad \hat{g} * \hat{g}^t = \hat{g}^t * \hat{g} = \hat{1},\qquad (6.6.3)$$

(where t denotes transposed), with identity isotransform $\hat{1} \in$ SÔ(3)

$$r' = \hat{1}*r = T^{-1}\,T\,r \equiv r.\qquad (6.6.4)$$

Consider now a conventional isofunction $\hat{f} = \hat{f}(r) = f(r)\hat{1}$ on $\hat{E}(r,\hat{\delta},\hat{R})$. Let $\hat{\mathfrak{R}}_{\hat{g}}$ be the isotransformation of the function $f(r)$ corresponding to $\hat{g}$. It is then easy to prove the following result as an isotopy of the conventional case (see, e.g., re. [12], Sect. 1.5)

$$\hat{\mathfrak{R}}_{\hat{g}} * \hat{f}(r) = \hat{}f(\hat{g}^{-1} * r).\qquad (6.6.5)$$

---

[63] We are here referring, specifically, to the isotopic case. On the contrary, there exists a considerable literature on the so-called q-special functions which are reviewed in App. II.6.D.

Similarly, it is easy to se that the succession of two isotransforms $\hat{\mathfrak{R}}_{\hat{g}}$ and $\hat{\mathfrak{R}}_{\hat{g}'}$ verifies the rule

$$\hat{\mathfrak{R}}_{\hat{g}} * \hat{\mathfrak{R}}_{\hat{g}'} * \hat{r}(r) = \hat{r}(\hat{g}^{-1} * \hat{g}'^{-1} * r). \tag{6.6.6}$$

We therefore have an *isolinear* isorep $\hat{g} \to \hat{\mathfrak{R}}_{\hat{g}}$ of $\hat{SO}(3)$ because at each element $\hat{g}$ of the isogroup there corresponds an isotransform $\hat{\mathfrak{R}}_{\hat{g}}$ such that

$$\hat{\mathfrak{R}}_{\hat{g}} * \hat{\mathfrak{R}}_{\hat{g}'} = \hat{\mathfrak{R}}_{\hat{g}*\hat{g}'}, \qquad \hat{\mathfrak{R}}_1 = 1. \tag{6.6.7}$$

The $\hat{SO}(3)$ isogroup can be constructed as the isosymmetry of the isosphere of unit radius $r = 1$. In this case, the isofunctions $\hat{f}$ have the dependence $\hat{f}(\theta, \phi)$ and the isotransforms (6.6.5) read

$$\hat{\mathfrak{R}}_{\hat{g}} * \hat{f}(\theta, \phi) = \hat{f}(\theta', \phi'). \tag{6.6.8}$$

Consider now an isorotation $\hat{g} = \hat{g}(\alpha) \in \hat{SO}(3)$. A simple extension of Lemma 6.5.5 yields the following

Lemma 6.6.1: *The isomeasure on the isosphere* $\partial\Omega(\theta, \phi)$ *which is isoinvariant under* $\hat{SO}(3)$, $\partial\Omega(\theta, \phi) \equiv \partial\Omega(\theta', \phi')$, *is given by*

$$\partial\Omega(\theta, \phi) = \sin\hat{\theta} \, \partial\theta \, \partial\phi = \sin(T_\theta \theta)(T_\theta d\theta)(T_\phi d\phi) =$$

$$= b_1 \, b_2 \, b_3 \sin\hat{\theta} \, d\theta \, d\phi = \sin\hat{\theta} \, d\hat{\theta} \, d\hat{\phi}, \tag{6.6.9}$$

We consider now isofunctions $\hat{f}(\theta, \phi)$ which are *isosquare integrable* (Ch. I.6) on the surface of the isosphere. Their isoproduct is given by

$$(\hat{f}, \hat{f}) = \int_\Omega \sin\theta \, d\hat{\theta} \, d\hat{\phi} * |\hat{f}(\theta, \phi)|^2 = \int_\Omega \sin(T_\theta \theta) \, d(T_\theta\theta) \, d(T_\phi\phi) \, \overline{\hat{f}(\theta, \phi)} * \hat{f}(\theta, \phi). \tag{6.6.10}$$

The invariant character of the isomeasure then implies the property

$$(\hat{\mathfrak{R}}_{\hat{g}} * \hat{f}_1, \hat{\mathfrak{R}}_{\hat{g}} * \hat{f}_2) = \int_\Omega \sin(T_{\theta'}\theta') \, d(T_\theta\theta') \, d(T_\phi\phi') \, \overline{\hat{f}_1(\theta', \phi')} * \hat{f}_2(\theta', \phi') =$$

$$= \int_\Omega \sin(T_\theta\theta) \, d(T_\theta\theta) \, d(T_\phi\phi) \, \overline{\hat{f}_1(\theta, \phi)} * \hat{f}_2(\theta, \phi) = (\hat{f}_1, \hat{f}_2). \tag{6.6.11}$$

We therefore have the following

Lemma 6.6.2: *The isoreps* $\hat{\mathfrak{R}}_{\hat{g}}$ *of* $\hat{SO}(3)$ *are isounitary with respect to the isoproduct (6.6.10).*

We now construct the *irreducible isoreps* $\hat{\mathfrak{R}}_{\hat{g}}$ of $\hat{SO}(3)$ acting on

isofunctions $\hat{T}(\theta, \phi)$ for given *isoweights* (Ch. I.4). The explicit form of $\hat{\Re}_{\hat{g}}$ has already been calculated in Ch. II.5 and it is given by the *hadronic angular momentum isooperator*

$$\hat{\Re}_{\hat{g}} * \hat{T} = (\hat{L}^2 * \hat{T} , \hat{L}_z * \hat{T}), \tag{6.6.12}$$

in realizations (II.5.6.12) or (II.5.6.19). Our task is therefore reduced to the identification of the isobasis $\hat{T}(\theta, \phi)$ which we here call *isospherical harmonics* and denote with the symbol $\hat{Y}(\theta, \phi)$ because they are the isotopic image of the conventional spherical harmonics $Y(\theta, \phi)$ [1,10–12] although now defined on the isosphere.

In the conventional case, the irreducible reps of SO(3) are characterized by the spherical harmonics $Y_L^M(\theta, \phi)$, where the weight is L = 0, 1, 2, ..., and the dimension is 2L+1 as characterized by the values M = L, L–1, ..., –L.

We know from the isorepresentation theory of Lie–isotopic algebras that the isotopies do not alter the dimension of a given irreducible rep. Therefore, before doing any calculation, we can anticipate that the isospherical harmonics preserve the above dimensions, that is, we must expect that they are characterized by an isoweight $\hat{L}$ = 0, 1, 2, ..., with dimension $2\hat{L}$+1 characterizable with the number $\hat{M}$ = $\hat{L}$, $\hat{L}$–1, ..., –$\hat{L}$, and we shall write $\hat{Y}_{\hat{L}}^{\hat{M}}(\theta, \phi)$, where the indices L and $\hat{L}$, M and $\hat{M}$ have been differentiated to avoid confusions.

Our task is now to find the explicit form of the isospherical harmonics and, above all, their *deviation* from the conventional form. For this purpose we recall the following properties

$$\frac{\partial}{\partial \phi} \hat{e}^{\,iM\phi} = i\,MT_\phi\, \hat{e}^{\,i\,M\phi} , \quad \frac{\partial}{\partial \hat{\phi}} \hat{e}^{\,iM\phi} = i\,M\,\hat{e}^{\,iM\phi} , \quad \frac{\partial}{\partial \hat{\phi}} \hat{e}^{\,iM\phi} = iM\hat{l}_\phi\, \hat{e}^{\,iM\phi} \tag{6.6.13a}$$

$$\frac{\partial}{\partial \theta} \cos\hat{\theta} = -\,T_\theta \sin\hat{\theta}, \quad \frac{\partial}{\partial \hat{\theta}} \cos\theta = -\sin\hat{\theta}, \quad \frac{\partial}{\partial \hat{\theta}} \cos\hat{\theta} = i\hat{l}_\theta \sin\hat{\theta} \tag{6.6.13b}$$
$$(6.6.4b$$

$$T_\phi = b_1\,b_2 , T_\theta = b_3, \quad T_\phi \neq T_\theta, \tag{6.6.13c}$$

and similarly for sin $\hat{\theta}$.

The transition from the conventional to the isotopic reps is essentially characterized by the lifting from the conventional sphere to the isosphere which, in turn, is realizable via the liftings $\theta \rightarrow T_\theta\theta$ and $\phi \rightarrow T_\phi\phi$, plus proper isonormalizations. These simple rules permit the introduction of the following

Definition 6.6.1: *The "regular isolegendre polynomials" are given by*

$$\hat{P}_{\hat{L}}^{\hat{M}}(\tilde{x}) = \hat{1}_\theta \frac{(-1)^{\hat{M}}}{2^{\hat{L}}\,\hat{L}!} (1 - \tilde{x}^2)^{\hat{M}/2} \frac{d^{\hat{L}+\hat{M}}}{dx^{\hat{L}+\hat{M}}} (\tilde{x}^2 - 1)^{\hat{L}}, \quad \tilde{x} = \cos\hat{\theta}, \tag{6.6.14}$$

and the "regular isospherical harmonics" are given by

$$\hat{Y}_{\hat{L}}^{\tilde{M}}(\theta, \phi) = 1_\phi \, 1_\phi \left(\frac{(\hat{L} - \tilde{M})(2\hat{L} + 1)}{4\pi (\hat{L} + \tilde{M})!} \, 1_\theta 1_\phi \right)^{1/2} \hat{P}_{(1)\hat{L}}^{\tilde{M}}[\cos(T_\theta \, \theta)] *_\theta \hat{e}^{i\tilde{M} \, 1_\phi \phi}$$

(6.6.15)

which are defined for the hadronic spectrum

$$\hat{L} = L b_3 = 0, 1, 2, \ldots \qquad \tilde{M} = M b_1 b_2 = \hat{L}, \hat{L} - 1, \ldots, -\hat{L}. \tag{6.6.16}$$

As one can see, the above expressions exhibit the *conventional functional dependence* only referred to the isoangles $\hat{\theta}$ and ϕ, plus different coefficients requested for the isorenormalization. Despite such simplicity, the isotopies of the angles $\theta \to \hat{\theta} = T_\theta \theta$, $\phi \to \hat{\phi} = T_\phi \phi$ are sufficient to require a redefinition of the quantum numbers, as desired.

It is easy to prove the following isoorthogonality property

$$\int_\Omega^\wedge d\Omega \, (T_\theta \, T_\phi) \, \hat{Y}_{\hat{L}'}^{\tilde{M}'} \, (T_\theta \, T_\phi) \, \hat{Y}_{\hat{L}}^{\tilde{M}} = \delta_{\hat{L}' \, \hat{L}} \, \delta_{\tilde{M}' \, \tilde{M}}, \tag{6.6.17}$$

which confirms the correct value of isomeasure (6.6.9).

We now recall realization (11.5.6.19) of the hadronic angular momentum in isospherical coordinates, i.e.,

$$\hat{L}^2 * \hat{Y} = -\frac{1}{D^2} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \hat{\phi}^2} \right\} \hat{Y} \tag{6.6.18a}$$

$$\hat{L}_z * \hat{Y} = -i \, b_1^{-1} b_2^{-1} \frac{\partial}{\partial \hat{\phi}} \hat{Y} = -i \frac{b_3}{D} \frac{\partial}{\partial \hat{\phi}} \hat{Y}, \tag{6.6.18b}$$

$$D = [\det T]^{1/2} = b_1 \, b_2 \, b_3 = T_\theta \, T_\phi. \tag{6.6.18c}$$

Lemma 6.6.3: *The "regular irreducible isoreps" of* $S\hat{O}(3)$ *are given, up to isotopic degrees of freedom, by*

$$\hat{L}^2 * \hat{Y}_{\hat{L}}^{\tilde{M}} = D^{-2} \hat{L}(\hat{L} + 1) \hat{Y}_{\hat{L}}^{\tilde{M}}, \quad \hat{L} = b_3 \, L, \tag{6.6.19a}$$

$$\hat{L}_z * \hat{Y}_{\hat{L}}^{\tilde{M}} = \frac{\tilde{M}}{T_\phi^2} \hat{Y}_{(1)\hat{L}}^{\tilde{M}} = \frac{b_3 \, M}{D} \hat{Y}_{\hat{L}}^{\tilde{M}}. \tag{6.6.19b}$$

As one can see, Eq.s (6.6.19) characterize *regular* isoreps because, according to our definition (Sect.s 1.4.7 and 11.6.1), the deformation of the quantum numbers is completely factorizable. These isoreps are sufficient for innovative physical applications, such as for the novel structure model of hadrons indicated in the preceding section and for the corresponding prediction of a new form of hadronic energy.

Even though the isotopies are axiom–preserving and the structures $S\hat{O}(3)$ and SO(3) coincide at the abstract level, Lemma 6.6.3 shows that hadronic mechanics implies a structural departure from quantum mechanics. In fact, even though the dimensionality of the representation remains unchanged, the conventional integer spectrum for the orbital angular momentum is mapped under isotopies into a continuously varying spectrum

$$\left(\begin{array}{l} L(L+1), L = 0, 1, ... \\ M = L, L-1, ..., -L \end{array} \right) \longrightarrow \left(\begin{array}{l} D^{-2}\hat{L}(\hat{L}+1), \quad \hat{L} = b_3 L = 0, 1, 2, ... \\ D^{-1} b_3 M, \quad \hat{M} = b_1 b_2 = \hat{L}, \hat{L}-1, ..., -\hat{L} \end{array} \right)$$

$$(6.6.20)$$

The simplest possible example illustrating the structural departure from quantum mechanics is given for the value

$$D = \exp\{\gamma t\}. \tag{6.6.21}$$

for which, as one can see, *the hadronic angular momentum decays continuously in time to the null value.* This confirms the avoidance of the "perpetual motion" for particles orbiting within physical media, as desired. We assume the reader is aware of the fact that "orbital angular momenta with continuously varying eigenvalues" are prohibited in the conventional, quantum mechanical SO(3) symmetry [10–12].

We should keep in mind that Lemma 6.6.3 is defined for *one* particle within a hadronic medium considered as *external,* thus resulting in an *open nonconservative system.*If we consider an *isolated collection* of hadronic particles, then the isotopic addition of hadronic angular momenta must be subjected to the constraint of reproducing conventional total eigenvalues, as studied later on in this chapter. This is due to the need that nonconservations can only be internal mutual exchanges, but always such to verify total conservation laws for systems isolated from the rest of the universe.

By no means the preceding regular isorep is unique. Consider the *general isospherical coordinates* (II.5.5.14), i.e.,

$$x = rB_{22}^{-1}\sin\hat{\theta}\, B_{11}^{-1}\cos\hat{\phi}, \quad y = rB_{22}^{-1}\sin\hat{\theta}\, B_{12}^{-1}\sin\phi, \quad z = rB_{21}^{-1}\cos\hat{\theta}, \tag{6.6.22a}$$

$$\hat{\theta} = B_{11} B_{12}\, \theta, \qquad \hat{\phi} = B_{21} B_{22}\, \phi, \tag{6.6.22b}$$

$$T_\phi = B_{11} B_{12} \equiv T_\theta = B_{21} B_{22}. \tag{6.6.22c}$$

We now recall realization (II.5.6.12) of the hadronic angular momentum

$$\hat{L}^2 * \hat{Y} = -\left\{ \frac{1}{\sin\hat{\theta}} \frac{\partial}{\partial\hat{\theta}}\sin\hat{\theta}\frac{\partial}{\partial\hat{\theta}} + \frac{1}{\sin^2\hat{\theta}} \frac{\partial^2}{\partial\hat{\phi}^2} \right\} \hat{Y} \tag{6.6.23a}$$

$$\hat{L}_z * \hat{Y} = -i\frac{\partial}{\partial\hat{\phi}}\hat{Y}_{(1)} = -i\hat{1}_\phi\frac{\partial}{\partial\phi}\hat{Y}. \tag{6.6.23b}$$

$$\frac{\partial}{\partial \phi} \qquad\qquad \frac{\partial}{\partial \phi}$$

Corollary 6.6.3A: *Another form of the regular irreducible isoreps of* SÔ(3) *are given by*

$$\hat{L}^{2}* \hat{Y}_{L}^{\tilde{M}} = -\left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^{2}\theta} \frac{\partial^{2}}{\partial\hat{\phi}^{2}} \right\} \hat{Y}_{L}^{\tilde{M}} =$$

$$= -\hat{1}_{\phi}^{2} \left\{ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^{2}\theta} \frac{\partial^{2}}{\partial\hat{\phi}^{2}} \right\} \hat{Y}_{L}^{\tilde{M}} =$$

$$= \hat{1}_{\phi}^{2} \hat{L}(\hat{L}+1) \hat{Y}_{L}^{\tilde{M}}, \qquad\qquad (6.6.24a)$$

$$\hat{L}_{z}* \hat{Y}_{L}^{\tilde{M}} = -i\frac{\partial}{\partial\hat{\phi}} \hat{Y}_{L}^{\tilde{M}} = -i\hat{1}_{\phi}\frac{\partial}{\partial\hat{\phi}} Y_{L}^{\tilde{M}} = \hat{1}_{\phi} \tilde{M} \hat{Y}_{L}^{\tilde{M}}. \qquad (6.6.24b)$$

Note in the above structure that the factorization depends on the identity T_{ϕ} = $B_{11}B_{12} \equiv T_{\theta} = B_{21}B_{22}$ which, in turn, depends on the selection of the general isospherical coordinates. In fact, for the ordinary isospherical coordinates we have in general $T_{\theta} = b_{3} \neq T_{\phi} = b_{1}b_{2}$.

Corollary 6.6.3B: *The "standard irreducible isoreps" of* SÔ(3) *are given by isorep (6.6.24) with* $T_{\theta} = T_{\phi} = 1$ *which is therefore expressible in term of the "hidden variable"*

$$B_{11} = B_{12}^{-1} = B_{21} = B_{22}^{-1} = \lambda. \qquad\qquad (6.6.25)$$

Note that the standard isoreps can also be obtained from isorep (6.6.19) for b_{1} = $b_{2}^{-1} = \lambda$ and $b_{3} = 1$.

We therefore confirm the extension to SÔ(3) of the isotopic realization of a "hidden variable" for SÔ(2) in the preceding section. This result has rather deep physical implications studied in Vol. III. At this point we note the following

Corollary 6.6.3C: *The conventional orbital angular momentum with spectrum* L = 0, 1, 2, ..., M = L, L−1, ..., −L *is in actuality characterized by three quantities, the conventional expressions* L *and* M *plus an arbitrary nowhere null "hidden variable" (or "hidden function")* λ

Ang. Mom. = { (L, M, λ), L = 0, 1, 2, ..., M = L, L−1, ..., −L, λ ∈ R(n,+,×), λ ≠ 0 } (6.6.26)

The irregular irreducible isoreps of SO(3) (those in which the hadronic spectrum is not completely factorizable into the conventional one) are unknown at this writing. This indicates the expected existence of other forms of isolegendre and isospherical harmonics whose study is here left to the interested

reader.

In closing this section, note that the irreducible isoreps of $S\hat{O}(3)$ are a *covering* of the corresponding irreducible reps of $SO(3)$ [10-12]. In fact, the latter are recovered identically for $\hat{1} = I = $ diag. $(1, 1, 1)$. The classification of all possible irreducible isoreps of $S\hat{O}(3)$ therefore includes the conventional ones as particular cases. When studying $S\hat{O}(3)$ the reader should therefore keep in mind that the conventional setting is preserved in its entirety and merely *add* new formulations which are not treatable with conventional theories.

We now outline the *isodual irreducible isoreps* of $S\hat{O}^d(3)$ as a direct generalization of those of $S\hat{O}^d(2)$ of the preceding section. The basic carrier space is the *isodual isoeuclidean space*

$$\hat{E}^d(r,\delta^d,\hat{R}^d): \quad \delta^d = T^d\delta, \quad T^d = -\text{diag.}\,(\,b_1^2\,,\,b_2^2,\,b_3^2\,)\,,\quad \hat{1}^d = (\,T^d\,)^{-1} = -\hat{1}, \quad (6.6.27a)$$

$$r^2 = (-x\,b_1^2\,x\, -\, y\,b_2^2\,y\, -\, z\,b_3^2\,z\,)\,\hat{1}^d \equiv r^2 \in \hat{R}^d(\hat{n}^d,+,*^d)\,, \quad (6.6.27b)$$

when written in *isodual isospherical coordinates*

$$x = r\,b_1^{-1}\sin\hat{\theta}^d\cos\hat{\phi}^d, \quad y = r\,b_2^{-1}\sin\hat{\theta}^d\sin\hat{\phi}^d, \quad z = r\,b_3^{-1}\cos\hat{\theta}^d, \quad (6.6.28a)$$

$$r = (x\,b_1^2\,x\, +\, y\,b_2^2\,y\, +\, z\,b_3^2\,z\,)^{1/2}, \quad \hat{\theta}^d = -b_3\,\theta, \quad \phi^d = -b_1\,b_2\,\phi\,. \quad (6.6.28b)$$

The *isodual isotopic elements* and related *isounits* are therefore given by

$$T^d_\theta = -b_3, \quad \hat{1}^d_\phi = -b_1\,b_2, \quad \hat{1}_\theta = -\hat{1}_\theta, \quad \hat{1}^d_\phi = -\hat{1}_\phi\,. \quad (6.6.29)$$

The isodual isogroup $S\hat{O}(3)$ is the isosymmetry of the isodual isosphere r^{2d}. Note that the latter coincides with the isosphere, $r^{2d} \equiv r^2$. But all numbers are now defined on isodual isofields. We therefore expect that the isodual irreducible isoreps are characterized by the same spectrum of $S\hat{O}(3)$ although changed in sign.

In fact, we have $\hat{L}^d_z = -\hat{L}_z$ and $\hat{L}^{2d} = -\hat{L}^2$ with equations for the *isodual hadronic angular moments*

$$\hat{\Upsilon}^\dagger *^d \hat{L}^{2d} = \hat{\Upsilon}^\dagger\left\{\frac{1}{\sin\hat{\theta}^d}\frac{\partial}{\partial\hat{\theta}^d}\sin\hat{\theta}^d\frac{\partial}{\partial\hat{\theta}^d} + \frac{1}{\sin^2\hat{\theta}^d}\frac{\partial^2}{\partial\hat{\phi}^{2d}}\right\}\frac{1}{T_\phi^{d2}} \quad (6.6.30a)$$

$$\hat{\Upsilon}^\dagger *^d \hat{L}_z = i\,\hat{\Upsilon}^\dagger\frac{\partial^d}{\partial\phi^d} = i\,\hat{\Upsilon}^\dagger\frac{\partial}{\partial\hat{\phi}^d}\hat{1}^d_\phi\,. \quad (6.6.30b)$$

The *isodual isolegendre functions* and *isodual isospherical harmonics* are then constructed with respect to the isodual isoangles and related isounits and they are defined for the isodual spectrum

$$\hat{L}^d = T^d_\theta\,L = 0, -1, -2, \ldots\ldots \quad \hat{M}^d = T_\phi\,M = -\hat{L}, -\hat{L}+1, \ldots, +\hat{L}\,. \quad (6.6.31)$$

The *isodual regular irreducible isoreps* of $\hat{SO}^d(3)$ in general isospherical coordinates are given by

$$\hat{\Upsilon}^\dagger \ast^d \hat{L}^{2d} \;=\; \hat{\Upsilon}^\dagger (- \hat{L}^d (\hat{L}^d + 1) T_\phi{}^{d-2}) \tag{6.6.32a}$$

$$\hat{\Upsilon}^\dagger \ast^d \hat{L}_z \;=\; \hat{\Upsilon}^\dagger (- \tilde{M} D^{d-1}) \tag{6.6.32b}$$

The interested reader can work out the remaining aspects such as the isodual regular isoreps, standard isoreps, isomeasure, etc.

6.7: ISOREPRESENTATION OF sû(2) AND THEIR ISODUALS

We now study of *the isotopic* $\hat{U}(2)$ *symmetries* first introduced in ref. [4] and then studied in detail in ref.s [5,6]. They can be defined as the largest possible nonlinear, nonlocal and noncanonical, simple, Lie–isotopic symmetries of the *complex two–dimensional isoeuclidean spaces*

$$\hat{E}(z,\hat{\delta},\hat{C}): \quad z = (z_i, z_2), \quad \hat{\delta} = T \delta \equiv T = \text{diag.} (g_{11}, g_{22}) = \hat{\delta}^\dagger > 0 , \tag{6.7.1a}$$

$$z^2 = z^\dagger_i \, \hat{\delta}_{ij}(t, z, z^\dagger, ...) z_j = \bar{z}_1 \, \hat{g}_{11} z_1 + \bar{z}_2 \, \hat{g}_{22} z_2 = \text{inv.} \tag{6.7.1b}$$

$$\hat{C} = \hat{C}(\hat{c},+,\ast), \quad \hat{1} = T^{-1} = \hat{\delta}^{-1} = \text{diag.} (g_{11}{}^{-1}, g_{22}{}^{-1}) > 0. \tag{6.7.1c}$$

Introduce the isohilbert space with isoinner product and isonormalization

$$\mathfrak{K}: \quad < \hat{u} \uparrow \hat{v} > \; = \; < \hat{u} | T | \hat{v} > \hat{1}, \quad < \hat{u} \uparrow \hat{u} > \; = \; \hat{1} . \tag{6.7.2}$$

Then, the isosymmetry $\hat{U}(2)$ can be defined as the Lie–isotopic group of isounitary operators on $\mathfrak{K}$

$$\hat{U} \ast \hat{U}^\dagger \;=\; \hat{U}^\dagger \ast \hat{U} \;=\; \hat{1} \;=\; T^{-1} \equiv \hat{\delta}^{-1}, \tag{6.7.3}$$

and can be decomposed into *connected, special isounitary isosymmetries* $S\hat{U}(2)$ for

$$\det (\hat{U} \, \hat{\delta}) \;=\; + 1, \tag{6.7.4}$$

plus a discrete part which is similar to that for $\hat{O}(3)$ and which is here ignored.

The connected $S\hat{U}(2)$ components admit the realization in terms of the generators $\mathfrak{J}_k$ and parameters θ_k

$$0 = e_{\xi}^{i \vee_k \sqrt{k}} = 1 \{ e^{i \vee_k \cdot \sqrt{k}} \} = \{ e^{\omega_k \cdot \vee_k} \} 1 , \tag{6.7.5}$$

under the basic conditions

$$\text{tr} (\hat{J}_k \hat{\delta}) \equiv 0, \qquad k = 1, 2, 3. \tag{6.7.6}$$

The *isoreps of the isoalgebras* $\hat{su}(2)$ were studied in ref.s [4] for the case in which, by specific requirement, the isocommutation rules have the same structure constants of SU(2), i.e., for the rules[64]

$$\hat{su}(2) \quad [\hat{J}_i \,\hat{,}\, \hat{J}_j] = \hat{J}_i \hat{\delta} \hat{J}_j - \hat{J}_j \hat{\delta} \hat{J}_i = i \,\epsilon_{ijk} \hat{J}_k. \tag{6.7.7}$$

which establish the local isomorphisms $\hat{SO}(2) \approx SU(2)$ *ab initio*.

Let $| b_k^d >$, $k = 1, 2, .. d$, be the d–dimensional basis of su(2) and let $| \hat{b}_k^d >$ be the corresponding isobasis of $\hat{SO}(2)$ with isoorthogonal conditions

$$< \hat{b}_i^d | * | \hat{b}_j^d > = \delta_{ij}, \quad i, j = 1, 2, ..., n. \tag{6.7.8}$$

As in the conventional case, $\hat{J}^2$ and $\hat{J}_3$ are a maximal set of isocommuting operators and they can therefore be simultaneously diagonalized. Also, recall that they remain Hermitean under isotopy because of the identity of the isotopic element of the isoenvelope and of the isohilbert space assumed earlier (see Sect. I.6.3 for details).

By noting that the isocasimir is

$$\hat{J}^2 = \sum_k \hat{J}_k * \hat{J}_k , \tag{6.7.9}$$

by putting as in the conventional case

$$\hat{J}_\pm = \hat{J}_1 \pm \hat{J}_2 , \tag{6.7.10}$$

and by repeating the same procedure as the familiar one (see, e.g., any of ref.s [10–12]) we have

$$\hat{J}_3 * | \hat{b}_k^d > = b_k^d | \hat{b}_k^d > , \tag{6.7.11a}$$

---

[64] We are now in a position to elaborate on the used symbols. In general, the symbol J_k denotes *the physical angular momentum components*, as in Eq.s (8.25), in which case the structure constants of $\hat{SO}(2)$ are $\epsilon_{ijk} b_k^{-2}$, as in Eq.s (8.30). On the contrary, the *redefined angular momentum components* are indicated with the symbol $\hat{J}_k$, as in Eq.s (8.31), in which case the structure constants are the conventional ones ϵ_{ijk}, as in Eq.s (8.32).

$$\mathfrak{J}^2 * | \mathfrak{b}_k{}^d > = b_1{}^d (b_1{}^d - 1) | \mathfrak{b}_1{}^d >, \tag{6.7.11b}$$

$$d = 1, 2, ..., \quad k = 1, 2, ..., d,$$

under the restrictions

$$b_{1d} \equiv -b_d{}^d, \quad K^2 = b_1{}^d (b_1{}^d - 1) \equiv b_d{}^d (b_d{}^d + 1). \tag{6.7.12}$$

The expected consequence is that *the dimensions of the isoreps of* SÛ(2) *remain the conventional ones*, i.e., they can be characterized by the familiar expression

$$n = 2j + 1 = 0, 1, 2, ... \tag{6.7.13}$$

as expected from the isomorphisms SÛ(2) ≈ SU(2).

However, *the explicit form of the matrix representations are different than the conventional ones*, as expressed by the rules [4]

$$(\mathfrak{J}_1)_{ij} = \tfrac{1}{2} i < \mathfrak{b}_i{}^d | * (\mathfrak{J}_- - \mathfrak{J}_+) * | \mathfrak{b}_j{}^d >, \tag{6.7.14a}$$

$$(\mathfrak{J}_2)_{ij} = \tfrac{1}{2} i < \mathfrak{b}_i{}^d | * (\mathfrak{J}_- + \mathfrak{J}_+) * | \mathfrak{b}_j{}^d >, \tag{6.7.14b}$$

$$(\mathfrak{J}_3)_{ij} = < \mathfrak{b}_i{}^d | * \mathfrak{J}_3 * | \mathfrak{b}_j{}^d >, \tag{6.7.14c}$$

under the condition identified earlier

$$tr (\mathfrak{J}_k \, \mathfrak{S}) \equiv 0, \quad k = 1, 2, 3 . \tag{6.7.15}$$

The explicit construction for the most important two- and three-dimensional isorepresentations are studied in the next sections.

Note that, unlike the realization of SÔ(3) in terms of isospherical harmonics (where we assumed the independence of the isotopic element from the local variables), the matrix isotopies of SÛ(2) studied in this section leave the the functional dependence of the isotopic element T = diag. $(g_{11}, g_{22}) > 0$ completely unrestricted. Thus the characteristic quantities g_{kk} have a general nonlinear and nonlocal-integral dependence on time t, the local t, z, ż, z̈ and their complex conjugated, the wavefunctions $\hat{\psi}$, $\partial\hat{\psi}$, $\partial\partial\hat{\psi}$ and their Hermitean conjugated, as well as a dependence on the local density μ of the medium in which the particle considered is immersed, its temperature τ, and any other needed quantity,

$$g_{kk} = g_{kk}(t, z, ż, z̈, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \mu, \tau, ...) > 0 . \tag{6.7.16}$$

under the condition of recovering the value $g_{kk} = 1$ when motion returns to be in vacuum, e.g., for null density of the interior medium, $\mu = 0$.

This functional dependence is important to understand the transition from the rigid and perennial notion of spin of quantum mechanics to the notion of *hadronic spin* studied later on in this and in the following chapters, which is a locally quantity dependent on the local characteristics of the physical medium in which it is immersed.

The *isodual irreducible isoreps* of $\hat{SU}^d(2)$ are particularly simple in matrix form because they merely change sign.

6.8: ISOTOPIES AND ISODUALITIES OF PAULI'S MATRICES

Some of the most celebrated matrices of quantum mechanics are given by the familiar Pauli matrices (see any of ref.s [1,10–12]) as the adjoint (or fundamental or regular) irreducible reps of the SU(2) algebra

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6.8.1)$$

which verify the properties under the conventional associative product of the envelope $\xi(su(2))$

$$\sigma_i \sigma_j = -i \epsilon_{ijk} \sigma_k, \quad (6.8.2)$$

with commutation rules

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = -2i \epsilon_{ijk} \sigma_k. \quad (6.8.3)$$

and familiar eigenvalues for the maximal commuting set

$$\sigma_3 |b_i^2> = \pm\tfrac{1}{2}|b_i^2>, \qquad \sigma^2|b_i^2> = \tfrac{1}{2}(\tfrac{1}{2}+1)|b_i^2>, \quad (6.8.4)$$

on the normalized basis $|b^2_k> =$ column $(1, 0)$.

The isotopies $SU(2) \to \hat{su}(2)$ permit a structural generalization of Pauli's matrices as two-dimensional irreducible isoreps of the $\hat{su}(2)$ algebra while preserving the local isomorphism $\hat{su}(2) \approx su(2)$. The isotopic images of Pauli matrices were first identified in ref. [4] under the name of *isopauli matrices*, then studied in detail in ref.s [5], while their connection to q–deformations was studied in ref. [6].

The isopauli matrices can be easily derived via the isorep theory of the preceding section. They confirm the three classes of irregular, regular and standard irreducible isoreps already established for $\hat{SO}(3)$ earlier in this chapter.

With reference to the notation of the preceding section, we consider the adjoint case characterized by the two-dimensional isometric $\hat{\delta} = $ diag. $(g_{11}, g_{22}) >$

0. The basis $| \hat{b}_k{}^2 >$, $k = 1, 2,$ is also two-dimensional and derivable from the conventional basis via the rule $| \hat{b}_k{}^2 > = T^{-1/2} | b_k{}^2 >$ under which we have the correct isonormalization $< \hat{b}_i{}^2 | * | \hat{b}_j{}^2 > = \delta_{ij}$. We then have the following:

Lemma 6.8.1: *The "irregular, adjoint, irreducible isoreps" of* SO(2), *called "irregular isopauli matrices", are given, up to isotopic degrees of freedom, by*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \quad \hat{\sigma}_3 = \begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix} = \Delta \hat{1} \sigma_3, \quad (6.8.5)$$

and verify the isoassociative and isocommutation rules [65]

$$\hat{\sigma}_1 * \hat{\sigma}_2 = i \hat{\sigma}_3, \quad \hat{\sigma}_2 * \hat{\sigma}_3 = i \Delta \hat{\sigma}_1, \quad \hat{\sigma}_3 * \hat{\sigma}_1 = i \Delta \hat{\sigma}_2, \quad (6.8.6a)$$

$$[\hat{\sigma}_1, \hat{\sigma}_2] = 2 i \hat{\sigma}_3, \quad [\hat{\sigma}_2, \hat{\sigma}_3] = 2 i \Delta \hat{\sigma}_1, [\hat{\sigma}_1, \hat{\sigma}_1] = 2 i \Delta \hat{\sigma}_2, \quad (6.8.6b)$$

$$\Delta = \det \hat{\delta} = g_{11} g_{22} > 0, \quad g_{kk} > 0, k = 1, 2, \quad (6.8.6c)$$

with isoeigenvalue equations

$$\hat{\sigma}_3 * | \hat{b}_i{}^2 > = \pm \Delta | \hat{b}_i{}^2 >, \quad \hat{\sigma}^2 * | \hat{b}_i{}^2 > = \Delta (\Delta + 2) | \hat{b}_i{}^2 > . \quad (6.8.7)$$

Again, the isorep is irregular in our terminology because the spectrum of isoeigenvalues is not completely factorizable into the conventional eigenvalues.

As it was the case for the irregular isoreps of sô(3), the irregular isopauli matrices are important for the most innovative applications of hadronic mechanics studied in Vol. III, those requiring the highest possible departures from quantum mechanics, such as the new structure model of hadrons with physical constituents, the prediction of a consequential new source of "hadronic energy", and others.

The most visible departure of the isopauli from the Pauli matrices is the alteration $\frac{1}{2} \rightarrow \frac{1}{2}\Delta$ of the basic eigenvalues of the spin $\frac{1}{2}$ of the original particle, called a *mutation* in these volumes in order to distinguish it from other procedures of contemporary physics, such as the *deformations*.[66] The understanding which will be illustrated shortly is that the hadronic mutation of

---

[65] The reader should keep in mind to sandwich the isotopic element T between the $\hat{\sigma}$'s for the correct reproduction of the results. Owing to extended use of the familiar associative product, the above necessary rule is forgotten more often than expected.

[66] The term *deformations* of Lie algebras is used in the mathematical literature when the Lie product is preserved while we deform the generators and/or the eigenvalues. Isorep (6.8.5)–(6.8.7) implies instead an alteration of the structure of the Lie product. The use of the term *deformation* could therefore be algebraically misleading. For this reason the term *mutation* was introduced in Vol. 1 and shall be used throughout this volume to denote alterations of quantum eigenvalues due to isotopies.

spin is not unique, but depends on the particular structure one assumes, that is, on the particular interior characteristics.

The second most visible difference is that, while the eigenvalues of Pauli's spin are constants, those characterized by the isopauli matrices have an arbitrary nonlinear, nonlocal and noncanonical dependence on all local quantities of type (6.7.16). In fact, we can introduce the following first form of

Hadronic spin $\frac{1}{2}\Delta$ $= \frac{1}{2}$ $\Delta(t, r, \dot{r}, \ddot{r}, \psi, \psi\dagger, \partial\psi, \partial\psi\dagger, \mu,. \tau, n, ...)$.   (6.8.8)

which will be made more precise when studying later on in this volume the isotopies of the Galilei and of the Poincaré symmetries and we identify the novel notion of *isoparticle* characterized by them.

The origin of the differences between the quantum and hadronic spin is that the particle is subjected to the point–like abstraction for the Pauli matrices, thus resulting in a rigid, perennial and immutable spin. On the contrary, the hadron represented by the isopauli matrices has an extended wavepacket and charge distribution thus resulting in a spin mutation, if nothing else, because that particle is no longer "free to spin" when immersed in a hyperdense medium. Equivalently, we can say that the factor Δ in the hadronic spin represents the effect of the interior medium in the intrinsic angular momentum.

If the extended particle in interior conditions is abstracted to a point, all internal nonlinear–nonlocal–noncanonical effects disappear, the isounit $\hat{1}$ recovers the conventional unit 1, and the isopauli matrices recover the Pauli matrices identically.

In this volume we shall consider a number of quantitative studies of the isopauli matrices and related hadronic spin for the experimental resolution of the issue whether or not the spin $\frac{1}{2}$ of a particle when in vacuum is preserved when the same particle is immersed in hyperdense hadronic media.

The inequivalence between the conventional and isopauli matrices is made clear by the following

Corollary 6.8.1A: *The irregular isopauli matrices are not unitarily equivalent to the conventional Pauli matrices.*

In fact, there is no unitary transforms such that

$$\hat{\sigma}_k = U \sigma_k U^\dagger, \qquad U U = U^\dagger U = 1, \qquad (6.8.9)$$

as the reader is encouraged to verify. In reality, the irregular isopauli matrices are derivable from Pauli matrices via *nonunitary transformations*, as anticipated in Sect. 11.6.1.

Lemma 6.8.2: *The "regular, adjoint, irreducible, isoreps" of* $SO(2)$, *called '*

"regular isopauli matrices", are given, at to isotopic degrees of freedom, by

$$\hat{\sigma}_1 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & g_{11} \\ g_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \Delta^{-\frac{1}{2}}\begin{pmatrix} 0 & -ig_{11} \\ +ig_{22} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \Delta^{-\frac{1}{2}}\begin{pmatrix} g_{22} & 0 \\ 0 & -g_{11} \end{pmatrix}, \quad (6.8.10)$$

and verify the isoassociative and isocommutation rules

$$\hat{\sigma}_i * \hat{\sigma}_j = i\Delta^{\frac{1}{2}}\epsilon_{ijk}\hat{\sigma}_k , \qquad (6.8.11a)$$

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i T \hat{\sigma}_j - \hat{\sigma}_j T \hat{\sigma}_i = 2i\Delta^{\frac{1}{2}}{}_{ijk}\hat{\sigma}_k , \qquad (6.8.11b)$$

and isoeigenvalue equations

$$\hat{\sigma}_3 * |\mathfrak{b}_i^2> = \pm \Delta^{\frac{1}{2}}|\mathfrak{b}_i^2>, \quad \hat{\sigma}^2 * |\mathfrak{b}_i^2> = 3\Delta|\mathfrak{b}_i^2>, i = 1, 2, \qquad (6.8.12)$$

Again, the isoeigenvalues are completely factorizable into conventional eigenvalues and the isorep is therefore regular.

Corollary 6.8.2A: *The regular isopauli matrices are derivable via Klimyk's rule with realization*

$$\hat{\sigma}_k = \sigma_k P, \quad P = \Delta^{\frac{1}{2}}1, \quad T = \text{diag.}(g_{11}, g_{22}), \quad k = 1, 2, 3. \qquad (6.8.13)$$

In fact, under the above rule we have the image of the original Pauli algebra

$$[\sigma_i, \sigma_j] = [\hat{\sigma}_i \hat{,} \hat{\sigma}_j]\Delta^{-1}1 = -2i\Delta^{-\frac{1}{2}}\epsilon_{ijk}\hat{\sigma}_k 1 , \qquad (6.8.14)$$

from which isoalgebra (6.8.11b) follows.

Despite the mathematical simplicity of their derivation, the physical implications of the regular isopauli matrices are far from being irrelevant. In fact, they also imply a *mutation of the quantum spin into the hadronic spin,* although of a type different than the preceding one.

Lemma 6.8.3: *The "standard, adjoint, irreducible reps" of sû(2), called ' "standard isopauli matrices", are given, up to isotopic degrees of freedom, by*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-1} \\ g_{11}^{-1} & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-1} \\ ig_{11}^{-1} & 0 \end{pmatrix}, \hat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \qquad (6.8.115)$$

and verify the isoassociative and isocommutation rules

$$\hat{\sigma}_i * \hat{\sigma}_j = i_{ijk}\hat{\sigma}_k' \qquad [\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = i\epsilon_{ijk}\hat{\sigma}_k , \qquad (6.8.16)$$

with isoeigenvalues

$$\hat{\sigma}_3 * | \hat{b} > = \pm | \hat{b} > , \quad \hat{\sigma}^2 * | \hat{b} > = 3 | \hat{b} > \tag{6.8.17}$$

As one can see, the structure of Pauli matrices is generalized, but the isoeigenvalues are fully conventional, thus yielding a standard irreducible rep of $\hat{su}(2)$.

Corollary 6.8.3A: *The standard isopauli matrices are derivable from Pauli matrices via the following realization of Klimyk's rule*

$$\hat{\sigma}_k = \sigma_k \hat{1} , \quad \det \hat{1} = 1 . \tag{6.8.18}$$

Despite the simplicity of their derivation and the preservation of the conventional eigenvalues, the physical and epistemological implications of the standard isopauli matrices are rather deep. We have shown in App. II.4.C that they imply the inapplicability of Bell's inequality and von Neumann theorem, thus permitting a completion of quantum mechanics into hadronic mechanics much along the original E–P–R argument. The connection with these results is made clear by the following

Corollary 6.8.3B: *The standard isopauli matrices provide an explicit realization of the "hidden variable"* λ

$$g_{11} = g_{22}^{-1} = \lambda \neq 0 , \tag{6.8.19}$$

holding for $\det T = \det \hat{\delta} = g_{11} g_{22} = 1$, *with explicit form of the matrices*

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda \\ i\lambda^{-1} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix} . \tag{6.8.20}$$

Pauli's matrices are essentially unique in the sense that their transformations under unitary equivalence

$$\tilde{\sigma}_k = U \sigma_k U^\dagger , \quad U U^\dagger = U^\dagger U = 1 , \tag{6.8.21}$$

do not yield significant changes in their structure, as well known [1,10–12].

The situation is different for the isopauli matrices, because isoreps possess "degrees of freedom" which are absent in the conventional su(2) theory, such as:

1) infinitely possible isotopic elements T;

2) formulation of the isoalgebra in terms of structure isofunctions;

3) use of an isotopic element $G > 0$ for the isohilbert space $\hat{\mathcal{H}}$ which is different than the isotopic element T of the isoalgebra (Sect. I..6.2);

and other aspects not studied in this section for brevity.

We here limit ourselves to the study of the "degrees of freedom" of the isopauli matrices characterized by the class of equivalence of hadronic

mechanics, that of *isounitary transforms*,

$$\hat{\sigma}'_k = \hat{U} * \hat{\sigma}_k * \hat{U}^\dagger, \quad \hat{U} * \hat{U}^\dagger = \hat{U}^\dagger * \hat{U} = \hat{1} \neq I, \tag{6.8.22}$$

In fact, we have the following second example of irregular isopauli matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & g_{22}^{-\frac{1}{2}} \\ g_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -ig_{22}^{-\frac{1}{2}} \\ ig_{11}^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} g_{11}^{-1} & 0 \\ 0 & -g_{22}^{-1} \end{pmatrix}, \tag{6.8.23}$$

with isocommutation rules and isoeigenvalues for $J_k = \frac{1}{2}\hat{\sigma}_k$

$$[J_1 \hat{,} J_2] = i \Delta J_3, \quad [J_2 \hat{,} J_3] = i J_1, \quad [J_3 \hat{,} J_1] = i J_2, \tag{6.8.24a}$$

$$J_3 * |b_i^2\rangle = \pm\frac{1}{2}|b_i^2\rangle, \quad J^2 * |b_i^2\rangle = \frac{1}{2}(\frac{1}{2} + \dot{\Delta})|b_i^2\rangle, \tag{6.8.24b}$$

where, as one can see, the eigenvalue of the third component is conventional, but that of the magnitude is generalized with a nonfactorizable isotopic contribution. This illustrates yet another type of mutation of the quantum spin.

Irregular isorepresentations also become standard under the condition detT = 1. We therefore have the second form of standard isopauli matrices derived from isorep (6.8.5)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \sigma_2, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{6.8.25}$$

as well as the following third form derived from structure (6.8.23)

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} \\ \lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i\lambda^{\frac{1}{2}} \\ i\lambda^{-\frac{1}{2}} & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & -\lambda \end{pmatrix}, \tag{2.26b}$$

The latter examples are important to illustrate the fact that *there exists standard isoreps of Lie–isotopic algebras which are not derivable from Klimyk's rule.*

The irregular and regular isopauli matrices are useful for interior dynamical problems where we expect a deviation from the conventional spin eigenvalues, e.g., the description of a neutron immersed in the hyperdense medium in the core of a neutron star or, along similar lines, for a hadronic constituent. The standard isopauli matrices are useful instead in applications where conventional spin eigenvalues hold, such as in nuclear physics.

In summary, we can say that the isotopic techniques have significant implications in the most fundamental notion of contemporary physics, that of spin $\frac{1}{2}$ Fermions, by identifying degrees of freedom and generalizations which are simply beyond the representational capabilities of quantum mechanics and related Lie theory, let alone quantitative treatment needed for experimental

verifications.

6.9: CONNECTION BETWEEN SÔ(3) AND SÛ(2), ISOSPINORS AND THEIR ISODUALS

We now study the connection between the isotopic SÔ(3) and SÛ(2) groups presented in this section apparently for the first time. The objectives are the verification that the angle isotopies

$$\theta \to \hat{\theta} = T_\theta\,\theta,\ T_\theta = b_3,\quad \phi \to \hat{\phi} = T_\phi\,\phi,\ T_\phi = b_1\,b_2, \qquad (6.9.1)$$

do carry over to SÛ(2) (because, as now familiar, the mutation of conventional eigenvalues originates precisely from the above mutation) and the identification of the SÛ(2) image of the isorotations (Sect.s II.6.3 and II.6.4)

$$r' = \hat{\Re}(\phi)*r = \hat{S}(\phi)\,r = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\hat{\phi} - y\,b_1^{-1}b_2\sin\hat{\phi} \\ x\,b_1\,b_2^{-1}\sin\hat{\phi} + y\cos\hat{\phi} \\ z \end{pmatrix}, \qquad (6.9.2a)$$

$$\hat{\Re}(\phi) = \begin{pmatrix} b_1^{-2}\cos\hat{\phi} & 1_\phi\sin\hat{\phi} & 0 \\ 1_\phi\sin\hat{\phi} & b_2^{-2}\cos\hat{\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (6.9.2b)$$

$$\hat{S}_\delta(\phi)\,r = \begin{pmatrix} \cos\hat{\phi} & b_1^{-1}b_2\sin\hat{\phi} & 0 \\ b_1\,b_2^{-1}\sin\hat{\phi} & \cos\hat{\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (6.9.2c)$$

with properties

$$x'\,b_1^2\,x' + y'\,b_2^2\,y' + z'\,b_3^2\,z' \equiv x\,b_1^2\,x + y\,b_2^2\,y + z\,b_3^2\,z, \qquad (6.9.3a)$$

$$\det\hat{\Re}(\phi) = \det 1, \qquad \det\hat{S}(\phi) = I. \qquad (6.9.3b)$$

The connection between SÔ(3) and SÛ(2) can be studied via suitable isotopies of the various conventional treatments [1,10–12]. Let us consider first the isotopies of Cartan's approach as presented, e.g., by Biedenharn and Louck [11], Sect. 2.4. For this purpose we introduce a vector $x = (x_1, x_2, x_3)$ of zero length in a three-dimensional isospace

$$x_1\,b_1^2\,x_1 + x_2\,b_2^2\,x_2 + x_3\,b_3^2\,x_3 = 0. \qquad (6.9.4)$$

The isotopies of Cartan's methods, here called *isocartan's method,* essentially occur by projecting the three-dimensional vector x into a two-dimensional vector $\xi = (\xi_1, \xi_2)$ via the rules

$$x_1 = b_1^{-1}(\xi_1^2 - \xi_2^2), \quad x_2 = i\, b_2^{-1}(\xi_1^2 - \xi_2^2), \quad x_3 = b_3^{-1}\xi_1\xi_2, \quad (6.9.5a)$$

$$\xi_1 = \pm[\tfrac{1}{2}(b_1 x_1 - i b_2 x_2)]^{1/2}, \quad \xi_2 = \pm[-\tfrac{1}{2}(b_1 x_1 + i b_2 x_2)]^{1/2}, \quad (6.9.5b)$$

$$\frac{\xi_1}{\xi_2} = -\frac{b_1 x_1 - i b_2 x_2}{b_3 x_3} = \frac{b_3 x_3}{b_1 x_1 + i b_2 x_2}, \qquad (6.9.5c)$$

The above property cannot any longer be represented via the conventional Pauli matrices, but they admit a full representation via the covering isopauli matrices. Let us denote with the symbol " * " the isotopic product in three-dimensional isospace with isotopic element $T_b = \text{diag.}\,(b_1^2, b_2^2, b_3^2)$ and with the symbol " $\circ$ " the isotopic product in two-dimensional isospace with diagonal elements $T_g = \text{diag.}\,(g_{11}, g_{22})$.

Then, properties (6.9.5) can be represented, say, with the regular isopauli matrices (6.78.10) in the form

$$(x * \hat{\sigma}) \circ \xi = \qquad (6.9.6)$$

$$= \Delta^{-1/2}\begin{pmatrix} g_{22}\, b_3 x_3 & g_{11}(b_1 x_1 - i b_2 x_2) \\ g_{22}(b_1 x_1 + i b_2 x_2) & -g_{11} b_3 x_3 \end{pmatrix}\begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} =$$

$$= \Delta^{1/2}\begin{pmatrix} b_3 x_3 \xi_1 + (b_1 x_1 - i b_2 x_2)\xi_2 \\ (b_1 x_1 + i b_2 x_2)\xi_1 - b_3 x_3 \xi_2 \end{pmatrix} = 0.$$

The equivalence of Eq.s (6.9.5c) and (6.9.6) establishes the one-to-one correspondence

$$\{x_1, x_2, x_3\} \longleftrightarrow x * \hat{\sigma} = \Delta^{-1/2}\begin{pmatrix} g_{22}\, b_3 x_3 & g_{11}(b_1 x_1 - i b_2 x_2) \\ g_{22}(b_1 x_1 + i b_2 x_2) & -g_{11} b_3 x_3 \end{pmatrix}, \quad (6.9.7)$$

which constitutes the desired correspondence between $S\hat{O}(3)$ and $S\hat{U}(2)$. Note the mixing of the characteristic quantities b_k of $S\hat{O}(3)$ and g_{ii}, of $S\hat{U}(2)$. Note also that, unlike the conventional case, the quantity $x*\hat{\sigma}$ *is not* traceless, although the quantity $(x*\hat{\sigma})T_g$ is traceless.

The correspondence between adjoint irreducible representations of $S\hat{O}(3)$ and $S\hat{U}(2)$ also follows from an isotopy of the conventional one [loc. cit.]. Let $\hat{U}$ be a two-dimensional isounitary matrix, i.e., such that

$$\hat{U} \circ \hat{U}^\dagger = \hat{U}^\dagger \circ \hat{U} = 1_g = \text{diag.}\,(g_{11}^{-1}, g_{22}^{-1}) \; := \hat{\sigma}_0. \qquad (6.9.8)$$

with isotransforms $\xi' = \hat{U}\circ\xi$. Introduce the map

$$x * \hat{\sigma} \;\rightarrow\; x' * \hat{\sigma} = \hat{U} \circ (x * \hat{\sigma}) \circ \hat{U}^\dagger. \qquad (6.9.9)$$

Then a simple isotopy of the conventional case [loc. cit.] yields the rules

$$x_k{'} = \tfrac{1}{2} \operatorname{tr} [\, \hat{\sigma}_k \odot \hat{U} \odot (\, x * \hat{\sigma}\,) \odot \hat{U}^I \,] = \hat{\Re}_{ki} \, T_{ij} \, x_j,$$ (6.9.10a)

$$\hat{\Re}_{ki} = \tfrac{1}{2} \operatorname{Tr} (\, \hat{\sigma}_k \odot \hat{U} \odot \hat{\sigma}_i \odot \hat{U}^\dagger\,).$$ (6.9.10b)

The above rules then imply that for each element $\hat{U} \in \hat{U}(2)$, $\hat{\Re}$ is real, proper and isoorthogonal and that the mapping $S\hat{O}(3) \to S\hat{U}(2)$ is onto (i,.e., there exists an element $\hat{U} \in \hat{U}(2)$ such that $\hat{U} \to \hat{\Re}$ for each $\hat{\Re} \in S\hat{O}(3)$). The study via other isopauli matrices is left to the interested reader.

The double-valuedness of $S\hat{U}(2)$ over $S\hat{O}(3)$ is already clear from Cartan's signs $\pm$ which persist under isotopies in Eq.s (6.9.5b). However, this aspect deserves an additional inspection because of an unexpected occurrence indicated below.

Let us re-examine the relationship between $S\hat{O}(3)$ and $S\hat{U}(2)$, this time, via an isotopy of the stereographic projection of the real sphere in Euclidean space into the complex two-dimensional space as done, e.g., by Vilenkin [11], Sect. 1.4.

By lifting the conventional treatment, we can study the projection of the three-dimensional isospace with real coordinates $\{x, y, z\}$ with isotopic element T = diag. $(b_1{}^2, b_2{}^2, b_3{}^2) > 0$ into the two-dimensional isospace with complex coordinates $\{\xi_1, \xi_2\}$ with isotopic element T = diag. $(g_{11}, g_{22}) > 0$ characterized by

$$\xi_1 = (\, b_1 \, x + i \, b_2 \, y\,) / (\tfrac{1}{2} - b_3 \, z\,), \quad \xi_2 = (\, b_2 \, y + i \, b_3 \, z\,) / (\tfrac{1}{2} - b_1 \, x\,).$$ (6.9.11)

The use of the isorotations (6.9.2) in the angle ϕ then implies the transformation properties

$$\xi_1{'} = (b_1 \, x{'} + i \, b_2 \, y{'}) / (\tfrac{1}{2} - b_3 \, z{'}) = e^{i \hat{\phi}} (b_1 \, x + i \, b_2 \, y\,) / (\tfrac{1}{2} - b_3 \, z\,) = e^{i \hat{\phi}} \xi$$ (6.9.12)

where one should note the necessary appearance of the isoangle $\hat{\phi}$. As in the conventional case, the preceding expression can be written $\xi_1{'} = e^{i\hat{\phi}/2} \, \xi_1 / e^{-i\hat{\phi}/2}$ and turned into the two-by-two matrix acting on $\xi = \{\xi_1, \xi_2\}$

$$\xi{'} = \{\xi_1{'}, \xi_2{'}\} = \hat{g}(\phi) * \xi = \tilde{g}(\phi) \, \xi,$$ (6.9.13a)

$$\hat{g}(\phi) = \begin{pmatrix} g_{11}{}^{-1} \, e^{i\phi/2} & 0 \\ 0 & g_{22}{}^{-1} \, \hat{e}^{-i\phi/2} \end{pmatrix}, \quad \tilde{g}(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & \hat{e}^{-i\phi/2} \end{pmatrix},$$ (6.9.13b)

which does leave invariant the $S\hat{U}(2)$ isoseparation

$$\xi{'}^\dagger * \xi{'} = \overline{\xi_1}{'} \, g_{11} \, \xi_1{'} + \overline{\xi_2}{'} \, g_{22} \, \overline{\xi_2}{'} = \overline{\xi_1} \, g_{11} \, \xi_1 + \overline{\xi_2} \, g_{22} \, \xi_2 = \xi^\dagger * \xi.$$ (6.9.14)

The repetition of the same procedure for the remaining Euler angles then yields the $S\hat{U}(2)$ image $\hat{g}(\phi_1, \theta, \phi_2)$ of a general isorotation $\hat{\Re}(\phi_1, \theta, \phi_2)$ (Sect. II.6.2)

$$\hat{g}(\phi_1, \theta, \phi_2) = \begin{pmatrix} g_{11}^{-1} e^{i(\psi_2 + \theta + \psi_1)/2} & 0 \\ 0 & g_{22}^{-1} e^{-i(\phi_2 + \theta + \phi_1)/2} \end{pmatrix}, \tag{6.9.15}$$

which also verifies invariance law (6.9.14). The latter property does established the desired result, namely, that SÔ(2) is double valued for *each* of the Euler angles of SÔ(3). Thus, the isoangles of SÔ(3) do carry over to the SÔ(2) structure, and the isorotations do admit an SÔ(2)-invariant image as expected.

Consider now the remaining part of the conventional connection between SO(3) and SU(2), i.e., the expression (see ref. [12], p. 10)

$$\frac{\xi' + 1}{\xi' - 1} = e^{i\theta} \frac{\xi + 1}{\xi - 1}, \tag{6.9.16}$$

and ensuing property

$$\xi' = \frac{\xi(e^{i\theta} + 1) + (e^{i\theta} - 1)}{\xi(e^{i\theta} - 1) + (e^{i\theta} + 1)} = \frac{\xi \cos \theta/2 + i \sin \theta/2}{i \xi \sin \theta/2 + \cos \theta/2}, \tag{6.9.17}$$

with the conventional transformation

$$\xi' = \{\xi_1', \xi_2'\} = g(\theta) \xi, \tag{6.9.18a}$$

$$g(\theta) = \begin{pmatrix} \cos \theta/2 & i \sin \theta/2 \\ i \sin \theta/2 & \cos \theta/2 \end{pmatrix}, \tag{6.9.18c}$$

which does leave invariant the SU(2) separation

$$\xi'^\dagger \xi' = \bar{\xi}_1' \xi_1' + \bar{\xi}_2' \xi_2' \equiv \bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2 = \xi^\dagger \xi. \tag{6.9.19}$$

In attempting to construct the isotopies of transforms (6.9.18), we have the unexpected occurrence that, while isotransforms (6.9.2a) do leave invariant isoseparation (6.9.3a) in three-dimensional *real* space, the corresponding isotransforms in the two-dimensional *complex* space

$$\xi' = \{\xi_1', \xi_2'\} = \hat{g}(\theta) * \xi = \tilde{g}(\theta) \xi, \tag{6.9.20a}$$

$$\tilde{g}(\theta) = \begin{pmatrix} \cos \theta/2 & i g_{11}^{-1/2} g_{22}^{1/2} \sin \theta/2 \\ i g_{11}^{1/2} g_{22}^{-1/2} \sin \theta/2 & \cos \theta/2 \end{pmatrix}, \tag{6.9.20c}$$

do not leave invariant the SÔ(2) isoseparation, i.e.,

$$\bar{\xi}_1' g_{11} \xi_1' + \bar{\xi}_2' g_{22} \xi_2' \neq \bar{\xi}_1 g_{11} \xi_1 + \bar{\xi}_2 g_{22} \xi_2, \tag{6.9.21}$$

as the reader is encouraged to verify. The only isotransforms leaving invariant the SU(2) isoseparation which are known at this writing (early 1994) are the isocartan form (6.9.7) and the diagonal one (6.9.15), while the isotopy of form

(6.9.18) is still unknown.

Definition 6.9.1: *An "isospinor" is a two component vector* $\xi = \{\xi_1, \xi_2\}$ *which transforms in accordance with the isocartan method. Isospinors are of Type I, II and III depending on whether the underlying isopauli matrices are irregular, regular or standard, respectively. An "isodual isospinor"* is a two-component vector $\xi = \{\xi_1 \; \xi_2\}$ which transforms according to the isocartan methods with the isodual isopauli matrices.

As we shall see, the notion of isospinor plays an important role in the construction of the isotopies of the Dirac equations, studied in Ch. 10, as well as in the novel applications of hadronic mechanics studied in Vol. III.

6.10: MATRIX ISOREPRESENTATIONS OF THE HADRONIC ANGULAR MOMENTUM

Consider the conventional 3-dimensional representation of SU(2) [1,10-12] on the normalized basis $| b_k{}^3 >$, k = 1, 2, 3,

$$
J_1 = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, J_2 = 2^{-\frac{1}{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

(6.10.1)

with familiar commutation rules

$$
[\, J_i \,, J_j \,] \;=\; J_i \, J_j \;-\; J_j \, J_i \;=\; i \, \epsilon_{ijk} \, J_k \,,
$$

(6.10.2)

and eigenvalues

$$
J_3 \,|\, b_k{}^3 > \;=\; m \,|\, b_k{}^3 > \,, \;\; M = 1, 0, -1 \,,
$$

(6.10.3a)

$$
J^2 \,|\, b_k{}^3 > \;=\; J_i \, J_i \,|\, b_k{}^3 > \;=\; 1 \, (\, 1 + 2) \,|\, b_k{}^3 > \,.
$$

(6.10.3b)

In this section wee study the three-dimensional irreducible isoreps of the $\hat{su}(2)$ isoalgebra, that is, the matrix equivalent of the representation in terms of isospherical harmonics of Sect. 11.6.6. This topic was briefly considered only in ref. [4] without any additional treatment known to this author at this writing (early 1994). Most of the results of this section are therefore presented here for the first time.

Let us consider the realization of $S\hat{O}(2)$ as a vector isospace with isometric

$$
\hat{\delta} = T\delta, \quad T = \text{diag.} \; (g_{11,} g_{22}, g_{33}) \,, \quad \delta = \text{diag.} \, (1, 1, 1) \,,
$$

(6.10.4)

and isonormalized isobasis $|\,\hat{b}_k{}^3> \,= T^{-\frac{1}{2}}|\,b_k{}^{3\prime}>$. The isorepresentation theory of Sect. 11.6.7 then implies the following

Lemma 6.10.1: *The three-dimensional irregular irreducible isoreps of* $\hat{su}(2)$ *are given, up to isotopic degrees of freedom, by*

$$\mathfrak{I}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & g_{11}g_{33} & 0 \\ g_{22}g_{33} & 0 & g_{11}\,g_{22}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}} \\ 0 & g_{11}\,g_{22}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}} & 0 \end{pmatrix}, \qquad (6.10.5a)$$

$$\mathfrak{I}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i\,g_{11}\,g_{33} & 0 \\ ig_{22}\,g_{33} & 0 & -i\,g_{11}\,g_{22}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}} \\ 0 & i\,g_{11}\,g_{22}{}^{\frac{1}{2}}\,g_{33}{}^{\frac{1}{2}} & 0 \end{pmatrix}, \qquad (6.10.5b)$$

$$\mathfrak{I}_3 = \begin{pmatrix} g_{11}g_{22}{}^{2}\,g_{33}{}^{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -g_{11}{}^{2}\,g_{22}{}^{2}\,g_{33} \end{pmatrix}, \qquad (6.10.5c)$$

with isocommutation rules [67]

$$[\mathfrak{I}_1\hat{,}\mathfrak{I}_2] = i\,\mathfrak{I}_3, \quad [\mathfrak{I}_2\hat{,}\mathfrak{I}_3] = i\,\Delta^2\,\mathfrak{I}_1/\sqrt{2}, \quad [\mathfrak{I}_3\hat{,}\mathfrak{I}_1] = i\,\Delta^2\,\mathfrak{I}_2/\sqrt{2}, \quad (6.10.6a)$$

$$\Delta = \det\,T = g_{11}\,g_{22}\,g_{33}, \qquad (6.10.6b)$$

and isoeigenvalue equations

$$\mathfrak{I}_3 * |\,\hat{b}_k{}^3> \;=$$

$$= \Delta^2 \begin{pmatrix} g_{11}{}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -g_{33}{}^{-1} \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} |\,\hat{b}_i{}^3> \;=\; \Delta^2\,M\,|\,\hat{b}_k{}^3>, \qquad (6.10.7a)$$

$$\mathfrak{I}^2 * |\,\hat{b}_i{}^3> \;=\; \mathfrak{I}_i * \mathfrak{I}_i * |\,\hat{b}_k{}^3> \;=\; (\mathfrak{I}_1\,T\,\mathfrak{I}_1 + \mathfrak{I}_2\,T\,\mathfrak{I}_2 + \mathfrak{I}_3\,T\,\mathfrak{I}3)\,T\,|\,\hat{b}_k{}^3> \;=$$

$$= \begin{pmatrix} \Delta^2(\Delta^2+1)\,g_{11}{}^{-1} & 0 & 0 \\ 0 & 2\,\Delta^2\,g_{22}{}^{-1} & 0 \\ 0 & 0 & \Delta^2(\Delta^2+1)\,g_{33}{}^{-1} \end{pmatrix} \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} |\,\hat{b}_k{}^3> \qquad (6.10.7b)$$

or, equivalently, in their explicit form

$$\mathfrak{I}_3 * |\,\hat{b}_i{}^3> \;=\; \mathfrak{I}_3\,T\,|\,\hat{b}_i{}^3> \;=\; \Delta^2\,\tilde{M}\,|\,\hat{b}_i{}^3>, \quad \tilde{M} = 1, 0, -1. \qquad (6.10.8a)$$

---

[67] The reader should keep in mind again that the computation of rules (6.9.6) requires the sandwiching of the isotopic element T between the $\mathfrak{I}$'s.

$$\hat{J}^2 * |\hat{b}_i{}^3 > = \hat{J}^2 T |\hat{b}_{+1}{}^3 > = \Delta^2(\Delta^2 + 1)|\hat{b}_{+1}{}^3 >, \quad \hat{M} = \hat{L} = 1, \qquad (6.10.8b)$$

$$\hat{J}^2 * |\hat{b}_i{}^3 > = \hat{J}^2 T |\hat{b}_0{}^3 > = 2\Delta^2 |\hat{b}_0{}^3 >, \quad \hat{M} = \hat{L} - 1 = 0, \qquad (6.10.8c)$$

$$\hat{J}^2 * |\hat{b}_i{}^3 > = \hat{J}^2 T |\hat{b}_{-1}{}^3 > = \Delta^2(\Delta^2 + 1)|\hat{b}_{-1}{}^3 >, \quad \hat{M} = -L = -1, \qquad (6.10.8d)$$

An inspection of the above isorep illustrates another novelty of the isotopic $S\hat{U}(2)$ symmetry, the fact that *different isoeigenvalues of* $\hat{J}^2$ *are possible for corresponding different isoeigenvalues of* $\hat{J}_3$. Note that this novelty is not transparent in the isofunctional treatment of the problem of Sect. II.6.6, by therefore illustrating the richness of the isorepresentation theory of Lie–isotopic algebras. The occurrence also confirms the expectation of isospherical harmonics different than those identified in Sect. II.6.6.

The primary difficulty in understanding and appraising the above occurrence is, again, of *conceptual* nature. Because of protracted use over generations, we are accustomed at the quantum mechanical notion of equal probability for all states of the same multiplet. This notion is evidently correct in the arena of applicability of quantum mechanics, point–like particles freely moving in vacuum under action–at–a–distance interactions.

In the transition to hadronic mechanics these conditions are altered. In fact, particles are no longer point–like and are no longer free, but are extended and constrained by the surrounding medium. As an illustration, suppose that this medium is highly anisotropic, e.g., because of the presence of a high value of the intrinsic angular momentum of the medium itself (we are therefore referring to a spinning particle totally immersed *inside* a heavier particle which a higher spin). A point particle in such a medium will continue to have equal probabilities for different quantum states of the same spin. But the situation for an extended particle is structurally different. In fact, as we shall soon see, hadronic isorep (6.10.5), under the above conditions, only *one* out of the three possible states of Lemma 6.10.1 will be admitted under the anisotropic conditions here considered.

In different terms, not only the probabilities for different spin states are different in hadronic mechanics because of the anisotropy of the underlying medium, but under certain conditions to be studied in the next sections, only *one* of the $2\hat{L}+1$ states can be allowed.

Expressions (6.10.8) illustrate that, more rigorously, the isoeigenvalue $\Delta^2(\hat{\Delta}+1)$ should be referred to an *isostate of maximal (or minimal) weight*.

Lemma 6.10.2: *The three-dimensional, regular, irreducible isoreps of* $S\hat{U}(2)$ *are given, up to the isotopic degrees of freedom, by*

$$\hat{J}_1 = \frac{K}{\sqrt{2}} \begin{pmatrix} 0 & g_{22}^{-1} & 0 \\ g_{11}^{-1} & 0 & g_{33}^{-1} \\ 0 & g_{22}^{-1} & 0 \end{pmatrix}, \qquad \hat{J}_2 = \frac{K}{\sqrt{2}} \begin{pmatrix} 0 & -i\, g_{22}^{-1} & 0 \\ i\, g_{11}^{-1} & 0 & -i\, g_{33}^{-1} \\ 0 & i\, g_{33}^{-1} & 0 \end{pmatrix},$$

$$(6.10.9a)$$

$$\mathfrak{J}_3 = K \begin{pmatrix} g_{11}^{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -g_{33}^{-1} \end{pmatrix}, \qquad (6.10.9b)$$

where K is a non–null constant, with isocommutation rules

$$[\mathfrak{J}_i \overset{\frown}{,} \mathfrak{J}_j] = i K^{-1} \epsilon_{ijk} \mathfrak{J}_k, \qquad (6.10.10)$$

and isoeigenvalue equations

$$\mathfrak{J}_3 * | \mathfrak{b}_k^3 > = K \hat{M} | \mathfrak{b}_k^3 >, \quad \mathfrak{J}^2 * | \mathfrak{b}_k^3 > = K^2 \hat{L}(\hat{L}+1) | \mathfrak{b}_k^3 > . \qquad (6.10.11)$$

As one can see, the above isorep confirms the expectation of Sect. 11.6.1 in regard to the structure of the isoeigenvalues under Klimyk's rule.

Lemma 6.10.3: *The three-dimensional standard, irreducible isorep of* sû(2) *are given, up to isotopic degrees of freedom, by*

$$\mathfrak{J}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & g_{11}^{-1} g_{22}^{-1} & 0 \\ 1 & 0 & g_{33}^{-1} \\ 0 & g_{22}^{-1} & 0 \end{pmatrix}, \mathfrak{J}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i g_{11}^{-1} g_{22}^{-1} & 0 \\ i & 0 & -i g_{33}^{-1} \\ 0 & i g_{22}^{-1} & 0 \end{pmatrix},$$

$$\mathfrak{J}_3 = \begin{pmatrix} g_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g_{33}^{-1} \end{pmatrix}, \qquad (6.10.12)$$

with isocommutation rules with conventional structure constants

$$[\mathfrak{J}_i \overset{\frown}{,} \mathfrak{J}_j] = \mathfrak{J}_i T \mathfrak{J}_j - \mathfrak{J}_j T \mathfrak{J}_i = i \epsilon_{ijk} \mathfrak{J}_k, \qquad (6.10.13)$$

and isoeigenvalue equations with conventional eigenvalues

$$\mathfrak{J}_3 * | \mathfrak{b}_k^3 > = \hat{M}, \quad \hat{M} = 1, 0, -1, \mathfrak{J}^2 * | \mathfrak{b}_k^3 > = 2 | \mathfrak{b}_k^3 > . \qquad (6.10.14)$$

Since the elements g_{kk} are unrestricted in isorep (6.10.12), we have the following

Corollary 6.10.3A: *The standard isorep (6.10.12) establishes that the conventional eigenvalues of the angular momentum* L = 1, M = 1, 0, -1, *possess three unrestricted "hidden variables" in the explicit realization*

$$\lambda_1 = g_{11} \neq \lambda_2 = g_2 \neq \lambda_3 = g_{33}, \ \det T = \lambda_1 \lambda_2 \lambda_3 \neq 0, 1, \qquad (6.10.15)$$

with isorep

$$
\mathfrak{J}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \lambda_1^{-1}\lambda_2^{-1} & 0 \\ 1 & 0 & \lambda_3^{-1} \\ 0 & \lambda_2^{-1} & 0 \end{pmatrix}, \mathfrak{J}_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i\lambda_1^{-1}\lambda_2^{-1} & 0 \\ i & 0 & -i\lambda_3^{-1} \\ 0 & i\,g_{22}^{-1} & 0 \end{pmatrix},
$$

$$
\mathfrak{J}_3 = \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix}. \tag{6.10.16}
$$

The above lemma illustrates another property of the isorepresentation theory which can be expressed via the following

Proposition 6.10.1: *Lie–isotopic algebras admit standard isorepresentations with value of the determinant of the isotopic element different than one,* det T $\neq$ 1.

In Sect. 6.8 we have verified the existence of standard isoreps which are not derivable from the Klimyk rule from the regular isoreps for det T = 1. Proposition 6.10.1 establishes the further property that det. T = $g_{11}g_{22}g_{33}$ can be different than one and the isorep can still be standard.

The study of additional aspects is left to the interested reader, such as the construction of additional irregular isoreps, regular isoreps not derivable from Klimyk's rule, standard isoreps derivable from the Klimyk rule, etc.

The properties identified in this section illustrate again the richness of the isorepresentation theory of the Lie–isotopic algebras, particularly when compared to the conventional representation theory, because of the emergence of properties in the former which simply cannot be formulated in the latter.

The implications of these results are significant. Mathematically, *the isorepresentations* of $\hat{su}(2)$ *confirm that the Lie–isotopic theory is a genuine covering of the conventional Lie's theory.* In fact, the isoreps are nonlinear, nonlocal and noncanonical, they possess isoeigenvalues nonexistent in the Lie theory while admitting the conventional linear, local and canonical reps and related eigenvalues as particular cases. Even when the isoeigenvalues coincide with the conventional eigenvalues, we have a richness of structure with hidden variables which simply cannot be identified with the conventional Lie theory.

Physically, the isorepresentations of $\hat{su}(2)$ imply that *the hadronic angular momentum and spin of one individual particle in interior conditions become <local> quantities,* that is, they can only be defined at a specific space–time point, and *<vary>* from point to point, as expected in the physical conditions here considered, such as a proton moving in the core of a star.

It is evident that we are referring to local, internal variations, while the

total angular momentum will be formulated shortly under the subsidiary constraint of recovering *conventionally quantized values* as measured in the exterior particle problem. Despite that, nontrivial, isotopic, internal degrees of freedom will persist for two or more hadronic bound states.

6.11: ADDITION OF HADRONIC SPIN

In the preceding sections we have studied the differences between the hadronic angular momentum and spin and the conventional quantum notions any time $\det\hat{I} \neq 1$. Other fundamental differences emerge in the study of the sum of hadronic angular momenta and spins.

The most insidious difference, particularly for the noninitiated reader, is of *conceptual* rather than technical nature. In fact, because of protracted use, one is accustomed to the several couplings of particles at large mutual distances admitted by quantum mechanics. By comparison, hadronic mechanics generally admits only *one* stable bound state at mutual distances smaller than the size of the wavepacket, if any at all. These comparatively severe restrictions, rather than being a limitation of the theory, represent instead its most interesting profile for novel applications as we shall see in Vol. III.

As an anticipation of these applications, consider a possible bound state of an electron and a proton at mutual distances *smaller* than the charge radius of the proton. The use of quantum mechanics would immediately lead to the infinite states of the hydrogen atom which, for the case considered, are against experimental evidence. The function of hadronic mechanics is therefore that of *suppressing* such an atomic-type spectrum and reduce the possible coupling to *only one state*. The understanding that any excitation of such hadronic state would imply the electron exiting the proton, thus regaining the infinite atomic spectrum of the hydrogen atom.

The best way to resolve a possible conceptual impasse is by noting that *the conventional theory of angular momentum and spin is a particular case of the hadronic theory*. Thus, the limited number of particle couplings admitted by the isotopic $S\hat{U}(2)$ theory at very small distances are *in addition to*, and not in substitution of the conventional variety of couplings of $SU(2)$ at large mutual distances.

For notational convenience as well as for comparative needs, let us consider first the conventional quantum mechanical case of the sum of two spin $\tfrac{1}{2}$ particles (see, e.g., Bohm, in ref. [13], p. 398). Denote the two quantum particles with the index $\alpha = 1, 2$ and assume that each one possesses spin $S_\alpha = \tfrac{1}{2}$. The total spin is then given by the operators

$$S = S_1 + S_2 \qquad\qquad (6.11.1a)$$

$$(S_1 + S_2)^2 = 3/2 + 2 S_1 \times S_2 \tag{6.11.1b}$$

To compute the basis, we recall that each particle can exist in only one of the two states $\pm \frac{1}{2}$ with eigenstates

$$\psi_{+ a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \psi_{- a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.11.2}$$

Now, in quantum mechanics the coupling can occur either in a state of *singlet* (antiparallel spin) or of *triplet* (parallel spin). Introduce then the eigenstates

$$\psi_{++} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_{+-} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \psi_{-+} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \psi_{--} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.11.3}$$

where we assume the tensor product of different bases.

The possible, normalized, stable states permitted by the conventional SU(2) theory are then the following ones

$$\text{Singlet:} \quad \psi_{\text{sin}} = \frac{1}{\sqrt{2}}(\psi_{+-} - \psi_{-+}), \quad S = 0 \tag{6.11.4a}$$

$$\text{Triplet:} \quad \psi_{\text{trpl}} = \frac{1}{\sqrt{2}}(\psi_{+-} + \psi_{-+}), \quad S = 1, \tag{6.11.4b}$$

which can be derived from the use of two independent Pauli's representations and result in the following respective expressions

$$\psi^{\dagger}_{\text{sin}} S_1 \times S_2 \, \psi_{\text{sin}} = -3, \quad \psi^{\dagger}_{\text{trpl}} S_i \times S_2 \, \psi_{\text{trpl}} = +1 \quad . \tag{6.11.5}$$

In the transition to hadronic mechanics, the first important difference is given by the fact that *the only admissible hadronic state is the singlet because the triplet is highly unstable.*

We shall first formally prove this important property via the lemma below. The same property is then conceptually elaborated in Figure 6.9.1, it is finally illustrated with practical applications and subjected to experimental verifications in Vol. III.

Lemma 6.11.1 [4]: *The singlet isostate is the only admissible, stable coupling of two hadronic particles with spin characterized by an irregular irreducible isorep of SO(2) when coupled one inside the other at mutual distances equal or smaller than their size .*

COUPLING OF SPINNING PARTICLES IN
QUANTUM MECHANICS
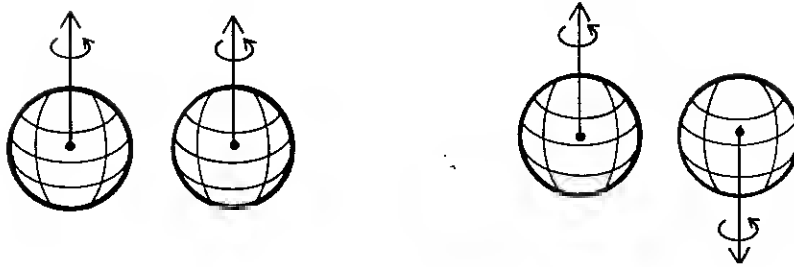
COUPLING OF SPINNING PARTICLES
IN HADRONIC MECHANICS
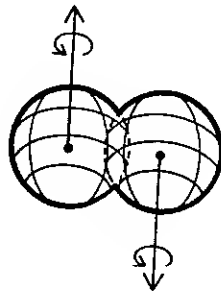
FIGURE 6.11.1: The coupling of spinning particles provides a most visible difference between quantum and hadronic mechanics. In quantum mechanics one considers large mutual distances (or, equivalently, point particles which, as such, cannot overlap). In this case spinning particles can be coupled either in a singlet or in a triplet, as well known. In the transition to hadronic mechanics the situation is different. In fact, we now have *extended* particles (that is objects with an extended charge distribution and/or extended wavepacket) which have to be coupled while being in condition of mutual penetration/overlapping. It is evident that, in the latter case, the triplet state is highly unstable because it would imply one particle spinning *against* the spinning of the other, thus implying destabilizing drag effects. The occurrence was identified since the original proposal to build hadronic mechanics, ref. [14], p. 852, and illustrated via the so-called <*gear model*>. In fact, gears can only be coupled in a singlet state. As we shall see in Vol. III, the lack of existence of a stable triplet state at mutual distances smaller than 1 fm as predicted by hadronic mechanics has far reaching implications, particularly for the new hadronic technology now emerging in the scientific horizon.

Proof : The hadronic addition theory of spin can be constructed via a

step-by-step isotopy of the conventional theory. Consider then the isotopic sum of two particles with spin as in Eq.s (11.6.8.7),

$$\hat{S}^2_\alpha = \tfrac{1}{2}\Delta^{\frac{1}{2}}(\tfrac{1}{2}\Delta^{\frac{1}{2}} + 1), \qquad \hat{S}_{3\alpha} = \pm\tfrac{1}{2}\Delta^{\frac{1}{2}}, \qquad \alpha = (1), (2), \qquad (6.11.6)$$

under the simplified condition that the two particles are identical (say, two electrons), in which case the isometrics $\hat{\delta} = T\delta = \text{diag.}\ (g_{11}, g_{22})$, and therefore $\det T =_a = g_{11}g_{22}$, are the same for both particles because each particle is the medium of the other. If the two particles are different (say, one electron and one proton), then the two isometrics are different.

Let us again denote the two particles with the symbol $\alpha = (1), (2)$. The properly isonormalized isobases are given by

$$\hat{\psi}_{++} = \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}_{(1)} \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}_{(2)}, \qquad \hat{\psi}_{--} = \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}_{(1)} \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}_{(2)}, \qquad (6.11.6a)$$

$$\hat{\psi}_{+-} = \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}_{(1)} \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}_{(2)}, \qquad \hat{\psi}_{-+} = \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}_{(1)} \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}_{(2)}, \qquad (6.11.6b)$$

which preserve the conventional orthogonality conditions.

The isotopy of the conventional treatment [13] then yields the existence of the following two basis

$$\text{Isosinglet:}\quad \hat{\psi}_{\text{sin}} = \frac{1}{\sqrt{2}}(\hat{\psi}_{+-} - \hat{\psi}_{-+}), \qquad\qquad (6.11.7a)$$

$$\text{Isotriplet:}\quad \hat{\psi}_{\text{trpl}} = \frac{1}{\sqrt{2}}(\hat{\psi}_{+-} + \hat{\psi}_{-+}), \qquad\qquad (6.11.7b)$$

The *total hadronic spin* is then characterized by the isoeigenvalues

$$\hat{S}_{\text{Tot}\,3\,\beta} = \hat{\psi}^\dagger_\beta * ((\hat{S}_1 + \hat{S}_2)_3) * \hat{\psi}_\beta, \qquad \beta = \text{singl, tripl} \qquad (6.11.8a)$$

$$\hat{S}^2_{\text{Tot}\,\beta} = \hat{\psi}^\dagger_\beta * (\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 * \hat{S}_2) * \hat{\psi}_\beta. \qquad\qquad (6.11.8b)$$

It is now a tedious but straightforward repetition of the calculations in ref. [13], pp. 398–400, under isotopy to prove that, for the case of the isopauli matrices (11.6.8.5), the *total hadronic spin for a singlet coupling* is given by

$$\hat{S}_{\text{Tot}\,3\,\text{singl}} = \hat{\psi}^\dagger_{\text{singl}} * ((\hat{S}_1 + \hat{S}_2)_3) * \hat{\psi}_{\text{singl}} \equiv 0, \qquad\qquad (6.11.9a)$$

$$\hat{S}^2_{\text{Tot singl}} = \hat{\psi}^\dagger_{\text{singl}} * (\hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 * \hat{S}_2) * \hat{\psi}_{\text{singl}} \equiv 0, \qquad (6.11.9b)$$

which is manifestly consistent with the isosymmetry $S\hat{U}(2)$, while that for the

total hadronic spin in a triplet state is given by the values

$$\hat{S}_{\text{Tot 3 trpl}} = \hat{\psi}\dagger_{\text{trpl}} * \{ (\hat{S}_1 + \hat{S}_2)_3 \} * \hat{\psi}_{\text{trpl}} \equiv \Delta^{\frac{1}{2}} , \qquad (6.11.10a)$$

$$\hat{S}^2_{\text{Tot trpl}} = \hat{\psi}\dagger_{\text{trpl}} * \{ \hat{S}_1^2 + \hat{S}_2^2 + 2\hat{S}_1 * \hat{S}_2 \} * \hat{\psi}_{\text{trpl}} \equiv 2\Delta^{\frac{1}{2}} \qquad (6.11.10a)$$

which are impossible for the isotopic $S\hat{U}(2)$ symmetry because they should read either

$$\hat{S}_{\text{Tot 3 trpl}} = 0, \qquad \hat{S}^2_{\text{tot trpl}} = 2\Delta^{\frac{1}{2}} \qquad (6.11.11)$$

as requested by the corresponding isoeigenvalues $L_3 = 0$ and $L^2 = 2\Delta^{\frac{1}{2}}$, or, in an eigenstates of maximal weight, they imply the value

$$\hat{S}_{\text{Tot 3 trpl}} = \Delta^{\frac{1}{2}}, \qquad \hat{S}^2_{\text{tot trpl}} = \Delta^{\frac{1}{2}}(\Delta^{\frac{1}{2}} + 1). \qquad (6.11.12)$$

which are incompatible with the original ones (6.11.6).

The assumption of different isometrics for the two particles implies additional restrictions on the total value confirming the above result. **q.e.d.**

It is instructive for the interested reader to prove the following

Corollary 6.11.1A: *The singlet isostate is the only admissible stable coupling of two hadronic particles with spin characterized by regular irreducible isorep of* s$\hat{u}$(2) *with different isometrics (for different particles). However, both singlets and triplets states are admitted for hadronic spins characterized by regular isoreps with the same isometrics (same particles) or for the standard isoreps.*

In summary, the case of the irregular isoreps for two particles (whether equal or not) and the case of the regular isoreps for different particles imply the largest departures from quantum mechanics. The simpler cases of two regular isoreps for the same particles or the standard isoreps imply the same predictions of quantum mechanics.

The above results should be kept in mind for Vol. III. In fact, if one wants to *preserve* quantum mechanical descriptions, and only add hidden isotopic degrees of freedom, e.g., for the reconstruction of exact symmetries when believed to be conventionally broken, it is evident that the regular isoreps with the same isometrics or the standard isoreps must be used.

If, on the contrary, one wants to seek fundamentally *novel* predictions not possible for quantum mechanics, and submit them to experimental tests, the irregular isoreps or the standard isoreps with different isometrics must be used.

6.12: ADDITION OF HADRONIC ANGULAR MOMENTUM AND SPIN

The addition of angular momentum and spin is another topic illustrating the departures of hadronic from quantum settings. It is also one of the topics of hadronic mechanics most important for novel applications, such as the chemical synthesis of electrons and protons into neutrons studied in Vol. III.

Conventionally, one studies the addition of angular momentum and spin for a particle moving in vacuum under external electromagnetic interactions, in which case the familiar quantum mechanical results are exactly valid. If interior physical conditions are considered, one usually approximates the particle as being point-like, thus regaining motion in vacuum.

In this section we shall study extended particles moving inside a hadronic medium. This implies motion of extended wavepackets within a medium composed of other wavepackets which results in in the nonlinear–nonlocal–noncanonical interactions studied in these volumes. Still in turn, this implies the *inapplicability* (and not the violation) of the conventional quantum theory for numerous independent reasons studied during the course of our analysis, e.g., because of its strictly local–canonical character.

We can therefore safely state that the inapplicability of the conventional theory for the addition of angular momentum and spin for a particle inside hadronic matter (say, for an electron in the core of a star) is out of scientific doubts whenever that particle is not approximated as a point. The only scientifically open issue is the identification of the applicable generalization-covering of the quantum theory.

In this section we shall review the current theoretical knowledge as of early 1994 on the sum of an orbital hadronic angular momentum $\Delta_L (\hat{L} = 1)$ with the hadronic spin $\frac{1}{2} \Delta (\hat{J} = \frac{1}{2})$. The problem of its experimental verification will be considered in Vol. III. The studies herein reviewed are those of ref. [5].

As now familiar, the methodological tool we use is the isotopy of the conventional treatment, resulting in the lifting of conventional metrics (units) into isometrics (isounits). The novelty of this section is that we necessarily have two *different* isotopies, one for the *spin*, characterized by the isotopy of the 2-dimensional complex carrier space $E(z,\delta,C) \rightarrow \hat{E}(z,\hat{\delta},\hat{C})$, with

$$\text{Spin isotopic element } T_S = \text{diag.} (g_{11}, g_{22}), \quad \Delta_S = g_{11} g_{22}, \qquad (6.12.1a)$$

and one for the *orbital* angular momentum, characterized by the lifting of the real three-dimensional Euclidean space $E(r,\delta,R) \rightarrow \hat{E}(r,\hat{\delta}<\hat{R})$ with

$$\text{Orbital isot. elem. } T_L = \text{diag.} (b_1^2, b_2^2, b_3^2), \quad \Delta_L = b_1^2 b_2^2 b_3^2. \qquad (6.12.1b)$$

Each isoproduct must then necessarily occur in the related two- or three-dimensional space. Since there is no possible ambiguity, both isoproducts will be

indicated with the same symbol "*".

We shall use the isopauli matrices (II.6.8.5) for the hadronic spin $\frac{1}{2}\Delta_S$ ($J = \frac{1}{2}$), and representation (II.6.10.5) for the hadronic angular momentum Δ_L ($\hat{L} = 1$) where we shall ignore for simplicity powers of the determinants Δ_S and Δ_L owing to the degrees of freedom indicated earlier. The composite system (hadronic angular momentum $\hat{L}$ and spin $\hat{J}$) has the total isounit

$$1_{tot} = 1 = 1_L \times 1_S = T_L^{-1} \times T_S^{-1},\qquad (6.12.2)$$

and it is characterized by the tensorial product

$$\hat{L} + \hat{J} = \hat{L} \times 1_S + 1_L \times \hat{J},\qquad (6.12.3)$$

which shall be tacitly assumed hereon.

The basis of the composite isostate is given by the tensorial product of the corresponding bases. Since we shall restrict the isostates to those of maximal weight, the total basis is a combination of one of the following isonormalized terms

$$\hat{\psi}_{++} = \begin{pmatrix} b_1^{-1} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}, \ \hat{\psi}_{--} = \begin{pmatrix} 0 \\ 0 \\ b_3^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}, (6.12.4a)$$

$$\hat{\psi}_{+-} = \begin{pmatrix} b_1^{-1} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ g_{22}^{-\frac{1}{2}} \end{pmatrix}, \ \hat{\psi}_{-+} = \begin{pmatrix} 0 \\ 0 \\ b_3^{-1} \end{pmatrix} \begin{pmatrix} g_{11}^{-\frac{1}{2}} \\ 0 \end{pmatrix}. \quad (6.12.4b)$$

At this point the isotropy or anisotropy of the medium in which motion occurs implies considerable differences between the isotopies of the conventional cases $J = L + \frac{1}{2}$ and $J = L - \frac{1}{2}$.

Recall that empty space is isotropic and, as such, it has no influence in the two cases here considered. On the contrary, physical media are generally anisotropic and, as such, they do imply an influence on the dynamics within their interior.

In particular, if the medium in which motion occurs has no intrinsic angular momentum (at the particle level), it can indeed be assumed as being isotropic. On the contrary, the presence of an intrinsic angular momentum implies the existence of a privileged direction in the medium itself, with the understanding that the underlying space remains isotropic. This results in an anisotropy which directly affects the addition of orbital angular momentum and spin.

ADDITION OF HADRONIC ANGULAR MOMENTUM AND SPIN
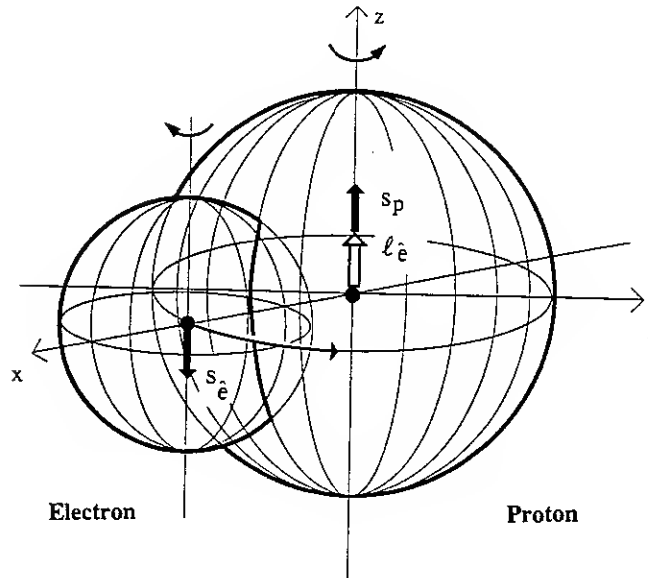
FIGURE 6.12.1: A pictorial view of the addition of hadronic angular momentum and spin for a particle here assumed to have a small mass when penetrating within a spinning hadron here assumed to be much heavier to allow its approximation as being at rest. First, we note that the medium considered is no longer isotropic as empty space because it has the preferred direction of spinning here assumed to be along the z-axis. Within such a medium, the particle is not free to orbit in any direction in space *inside* the other particle, but it is evidently forced to orbit along the direction z of the intrinsic angular momentum of the heavier particle in which it moves. Second, the direction of the spin of the particle is also not free in space, but restricted to form a singlet coupling with that of the basic medium as per Lemma II.6.11.1. This leads to the configuration of this figure for the hadronic angular momentum and spin which is restricted to the states of maximal weight, as studied in the text. The main result is that, starting with a quantum mechanical particle with spin $\frac{1}{2}$ and arbitrary angular momentum, the isotopic $S\hat{U}(2)$ theory restricts the allowable values of the total angular momentum to be identically null. Equivalently, one can obtain the same results by considering the limit of the particle considered when at rest in the center of the medium. In this case the *orbital angular momentum must coincide in magnitude with the intrinsic spin, thus resulting in a null total angular momentum.* As studied in detail in Sect. 11.5, the value $\frac{1}{2}$ of the *orbital* momentum is prohibited in quantum mechanics but fully allowed in the covering hadronic mechanics because essentially dependent on the local

physical characteristics. The above result is at the foundation of one of the most important applications of hadronic mechanics, the chemical synthesis of hadrons from ligther hadrons studied in Vol. III.

In fact, under the assumption that the medium is much heavier than the particle considered, the orientation of the orbital motion in an isotropic hadronic medium can clearly be arbitrary, as in the conventional case, while that for an anisotropic medium cannot (see Fig. 6.12.1 for more details).

Let us consider first the case of the *addition of hadronic angular momentum and spin within an isotropic hadronic medium* . It is tedious but easy to verify that the conventional case $J = 1 + \frac{1}{2}$ carries over under isotopies in its entirety. In fact, by using as the isobasis $\hat{\psi}_{++}$ (or, equivalently, $\hat{\psi}_{--}$), representations (11.6.8.5) and (11.6.10.5) yield

$$\hat{\psi}\dagger_{++} * (\hat{L}_3 + \hat{J}_3) * \hat{\psi}_{++} = \Delta_L + \frac{1}{2} \Delta_S, \tag{6.12.6a}$$

$$\hat{\psi}\dagger_{++} * (\hat{J}^2 + \hat{J}^2 + 2\hat{L} * \hat{J}) * \hat{\psi}_{++} = (\Delta_L + \frac{1}{2}\Delta_S)(\Delta_L + \frac{1}{2}\Delta_S + 1), \tag{6.12.6b}$$

where one should recall our notation in which

$$2\hat{L} * \hat{J} * \hat{\psi}_{++} = 2(\hat{L}\,T_L)(\hat{J}\,T_S)\psi_{++}, \tag{6.12.7}$$

and each of the two terms $\hat{L}\,T_L$ (and $\hat{J}\,T_S$) solely operates on the corresponding three-dimensional (two-dimensional) basis. The rest of the calculations are carried out with the same methods as those of the preceding section.

Results (6.12.7) essentially imply that, as expected, the *total hadronic angular momentum* is given by

$$J_{tot} = \Delta_L + \frac{1}{2} \Delta_S, \tag{6.12.8}$$

and it is a local quantity.

Note, in particular, that one can have $\Delta L = \Delta_S = 1$, but still admit nontrivial isotopic degrees of freedom as in the standard isoreps.

We now study the addition of hadronic angular momentum and spin in an anisotropic hadronic medium with a preferred direction in space $\hat{n}$. The most probable case is then when the orbital angular momentum is parallel to $\hat{n}$, while the hadronic spin is antiparallel (see Fig. 6.12.1 for details). These conditions imply the restriction to eigenstates of maximal weight parallel or antiparallel to $\hat{n}$. The following hadronic property is of fundamental significance for the applications of hadronic mechanics in Vol. III.

Lemma 6.12.1 [4]: *Under the irregular isoreps of the isotopic $S\hat{U}(2)$ symmetry of Class 1 an extended particle immersed within a hadronic medium with its own intrinsic angular momentum in the z-direction, the only admissible coupling is that in a singlet (Lemma 6.11.1) and the only admissible difference total angular momentum is that with*

null value,

$$J_{tot} = \Delta_L - \tfrac{1}{2}\Delta_S \equiv 0, \qquad \Delta_L \equiv \tfrac{1}{2}\Delta_S \qquad (6.12.9)$$

which can only hold for one of the values

$$\Delta_L = 1, \qquad \Delta_S = 2, \qquad (6.12.10a)$$

$$\Delta_L = \tfrac{1}{2}, \qquad \Delta_S = 1. \qquad (6.12.10b)$$

Proof: Under the assumed conditions, the eight possible bases are given by

$$\hat{\Psi}_1 = \hat{\Psi}_{++}, \quad \hat{\Psi}_2 = \hat{\Psi}_{--}, \quad \hat{\Psi}_3 = (1/\sqrt{2})(\hat{\Psi}_{++} + \hat{\Psi}_{--}), \quad \hat{\Psi}_4 = (1/\sqrt{2})\{\hat{\Psi}_{++} - \hat{\Psi}_{--}\}, \quad (6.12.11a)$$

$$\hat{\Psi}_5 = \hat{\Psi}_{+-}, \quad \hat{\Psi}_6 = \hat{\Psi}_{-+}, \quad \hat{\Psi}_7 = (1/\sqrt{2})(\hat{\Psi}_{+-} + \hat{\Psi}_{-+}), \quad \hat{\Psi}_4 = (1/\sqrt{2})(\hat{\Psi}_{++} - \hat{\Psi}_{--}). \quad (6.12.11b)$$

The only cases resulting in the difference of hadronic angular momentum and spin for the third components $\hat{L}_3$ and $\hat{J}_3$ are given by

$$(\hat{L}_3 - \hat{J}_3) * \hat{\Psi}_1 = (\Delta_L - \tfrac{1}{2}\Delta_S)\hat{\Psi}_1, \quad (\hat{L}_3 - \hat{J}_3) * \hat{\Psi}_2 = (-\Delta_L + \tfrac{1}{2}\Delta_S)\hat{\Psi}_2, \quad (6.12.12a)$$

$$(\hat{L}_3 - \hat{J}_3) * \hat{\Psi}_3 \equiv 0, \quad (\hat{L}_3 - \hat{J}_3) * \hat{\Psi}_4 = (\Delta_L - \tfrac{1}{2}\Delta_S)\hat{\Psi}_4, \quad (6.12.12b)$$

and

$$(\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_5 = (\Delta_L - \tfrac{1}{2}\Delta_S)\hat{\Psi}_5, \quad (\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_6 = (-\Delta_L + \tfrac{1}{2}\Delta_S)\hat{\Psi}_6, \quad (6.10.13a)$$

$$(\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_7 \equiv 0, \quad (\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_8 = (\Delta_L - \tfrac{1}{2}\Delta_S)\hat{\Psi}_8, \quad (6.12.13b)$$

Cases 3 and 7 are those of the Lemma. In fact, they imply the isoeigenvalues for the case 7

$$\hat{\Psi}^\dagger_7 * (\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_7 = 0, \qquad (6.12.14a)$$

$$\hat{\Psi}^\dagger_7 * (\hat{L}^2 + \hat{J}^2 + 2\hat{L} * \hat{J}) * \hat{\Psi}_7 = \Delta_L(\Delta_L + 1) + \tfrac{1}{2}\Delta_S(\tfrac{1}{2}\Delta_S + 1) - 2\Delta_L\Delta_S, \quad (6.12.14b)$$

which can hold in a way compatible with $S\hat{U}(2)$ only when the second isoeigenvalue is also identically null.

It is easy to see that the compatibility of all other cases with $S\hat{U}(2)$ requires degenerate or negative-definite isometrics which is contrary to the assumptions of the Lemma. As an example, case 5 is characterized by the isoexpectation values

$$\hat{\Psi}^\dagger_5 * (\hat{L}_3 + \hat{J}_3) * \hat{\Psi}_5 = \Delta_L - \tfrac{1}{2}\Delta_S, \qquad (6.12.15a)$$

$$\hat{\Psi}^\dagger_5 * (\hat{L}^2 + \hat{J}^2 + 2\hat{L} * \hat{J}) * \hat{\Psi}_5 = \Delta_L(\Delta_L + 1) + \tfrac{1}{2}\Delta_S(\tfrac{1}{2}\Delta_S + 1) - \Delta_L\Delta_S, \quad (6.12.15b)$$

which, to be consistent under irregular isoreps of $S\hat{U}(2)$, should imply the possibility of rewriting the second isoexpectation value as

$$(\Delta_L - \tfrac{1}{2} \Delta_S)(\Delta_L - \tfrac{1}{2} \Delta_S + 1).$$

$$(6.12.16)$$

This requirement is possible only for $\Delta_S \equiv 0$, i.e., for a degenerate isometric $\hat{\delta}$, or for negative values of the isodeterminant,

$$\Delta L = \Delta_S \equiv 0, \quad \text{or} \quad \Delta_L = \tfrac{1}{2} \Delta_S = -\tfrac{1}{2}.$$

$$(6.12.17)$$

For case 8 we have

$$\hat{\psi}^\dagger_8 * (\hat{L}_3 + \hat{J}_3) * \hat{\psi}_8 = \Delta_L - \tfrac{1}{2} \Delta_S,$$

$$(6.12.18a)$$

$$\hat{\psi}^\dagger_8 * (\hat{L}^2 + \hat{J}^2 + 2\hat{L} * \hat{J}) * \hat{\psi}_8 = \Delta_L(\Delta_L + 1) + \tfrac{1}{2}\Delta_S(\tfrac{1}{2}\Delta_S + 1),$$

$$(6.12.18b)$$

which is evidently incompatible with the conditions herein considered. All other cases are either equivalent to case 7 or to cases 4, 5, 8. **q.e.d.**

By recalling that the isometrics for the angular momentum and spin are now necessarily different even for regular isoreps, we have the following

Corollary 6.12.1A: *The results of Lemma 6.12.1 hold also for regular isoreps, while conventional total values are admitted for the standard isoreps.*

Additional studies, such as the isodualities of the above composition of hadronic angular momentum and spin, are left to the interested reader.

APPENDIX 6.A: LIE–ADMISSIBLE GENOTOPIES OF ANGULAR MOMENTUM AND SPIN AND THEIR ISODUALITIES

We now briefly outline the problem of the (Lie–admissible) genotopies of the quantum mechanical angular momentum and spin which has not been investigated at this writing (early 1994), to our best knowledge.

Let us recall that the conventional Lie group SO(3) transforms a quantum mechanical operator Q according to the rule

$$Q \quad \rightarrow \quad Q' = \Re(\theta_1, \theta_2\,\theta_3)\, Q\, \Re^\dagger(\theta_1, \theta_2, \theta_3)\,, \tag{6.A.1}$$

where $\Re(\theta_1, \theta_2\,\theta_3)$ is the general form of a rotation in terms of Euler's angles. Note the action to the right $\Re \Rightarrow Q = \Re Q$ and to the left $Q \Leftarrow \Re^\dagger = Q\Re^\dagger$. At the level of the underlying algebra, the emerging structure is a *Lie bimodule* (Sect. I.7.6) in which, from the anticommutativity of the Lie product $[A, B] = A \Rightarrow B - B \Leftarrow A = AB - BA$, the action to the right $A \Rightarrow v$ is related to that to the left $v \Leftarrow A$ by the simple rule $A \Rightarrow v = -v \Leftarrow A$, where A is a generators and v an element of the vector space in which the representation is constructed. Owing to the simple relationship between the two actions, the study of the rotational Lie group SO(3) can be effectively reduced to that of only one action, say, that to the right, as routinely done in presentations in the field [1,10–12].

In the transition to the Lie–isotopic group $S\hat{O}(3)$ we have the isotransform

$$Q \quad \rightarrow \quad Q' = \hat{\Re}(\theta_1, \theta_2\,\theta_3) * Q * \hat{\Re}^\dagger(\theta_1, \theta_2, \theta_3)\,, \tag{6.A.2}$$

where $\hat{\Re}(\theta_1, \theta_2\,\theta_3)$ is given by Eq. (II.6.2.12). In this case we have the isotopic actions to the right $\hat{\Re} \Rightarrow Q : \hat{\Re} * Q = \hat{\Re}TQ$ and to the left $Q \Leftarrow \hat{\Re}^\dagger := Q * \hat{\Re}^\dagger = QT\hat{\Re}^\dagger$. But the operator T is Hermitean. At the level of underlying algebra we therefore have a *Lie isobimodule* (Sect. I.7.6) in which, from the anticommutativity of the Lie–isotopic product $[\hat{A,}B] = A \Rightarrow B - B \Leftarrow A = ATB - BTA$, the action to the right $A * v$ is connected to that to the left $v * A$ by the rule $A \Rightarrow v = -v \Leftarrow A$. As a result of this simple conjugation, the study of the Lie–isotopic group $S\hat{O}(3)$ can be effectively reduced to that of only one action, say, that to the right, as done in this chapter.

As studied in Ch. I.7, and outlined in Sect. II.3.3, the *genotopies* $^<S\hat{O}^>(3)$ of *the rotational Lie group* SO(3) are based, first, on an isotopic lifting and, second, on the differentiation of the isotopic action to the right and to the left,

$$Q \quad \rightarrow \quad Q' = \hat{\Re}^>(\theta_1, \theta_2\theta_3) > Q < {}^<\hat{\Re}^\dagger(\theta_1, \theta_2, \theta_3)\,, \tag{6.A.3}$$

where the action to the right $>$ is characterized by an isotopic element S, $A>B := ASB$ and that to the left $<$ by a different isotopic element R, $A<B := ARB$, with the two isotopic elements R and S interconnected by a conjugation, e.g., $R = S^\dagger$.

The emerging structure is then essentially characterized by two different Lie–isotopic groups, one for the isoaction to the right

$$S\hat{O}^>(3) : \quad \hat{\Re}^> = \Big(\prod_{k=1,2,3} e_{|\xi}^{\,i\,J_k S\,\theta_k}\Big)1^>\,, \quad 1^> = S^{-1}\,, \tag{6.A.4}$$

and one for the action to the left

$$^<S\hat{O}(3) : \quad {}^<\hat{\Re}(\theta) = {}^<1\,\Big(\prod_{k=1,2,3} e_{|\xi}^{\,-i\,\theta_k \hat{T} J_k}\Big)\,, \quad {}^<1 = R^{-1}\,. \tag{6.A.5}$$

At the level of underlying algebra this implies the loss of Lie and Lie–

isotopic algebras in favor of the covering Lie-admissible algebras with general product $(A, B) = A \Rightarrow B - B \Leftarrow A = ARB - BSA$ which is no longer totally anticommutative. In particular, we have a *Lie-admissible isobimodule* (Sect. 1.7.6) in which the action to the right $A{>}a = ASv$ is no longer connected to that to the left $v{<}A = vRA$ when working in the original vector space of SO(3) or isovector space of SÔ(3), and this illustrates the Lie-admissible character of the genotopy.

However, as recalled in Sect. 11.3.3, if the isobimodular structure is represented in its own two-sided genospace, the algebraic-group theoretical profiles are changed. In fact, jointly with the genotopies $A \Rightarrow B = ATB \rightarrow A{>}B = ASB$ and $A \Leftarrow B = ATB \rightarrow A{<}B = ARB$ we have the joint change of the underlying unit in the amount *inverse* of the deformations $1 \rightarrow 1^> = S^{-1}$ and $1 \rightarrow {}^<1 = R^{-1}$. In this representation the Lie-admissible isobimodule coincides at the abstract level with the Lie-isotopic and conventionally Lie bimodules. Specifically, the genoaction to the right $A{>}v = ASv$, when referred to the isounit $1^> = S^{-1}$ is equivalent to the opposite of the genoaction to the left, $-v{<}A = -ARA$ when the latter is referred to its own genounit ${}^<1 = R^{-1}$.

In conclusion the *Lie-admissible genorotational group* ${}^<sO^>(3)$ can be constructed via the use of two different Lie-isotopic groups, one for the genoaction to the right $sÔ^>(3)$ with genotopic element S and related genounit $1^> = S^{-1}$ and one for the genoaction to the left ${}^<sÔ(3)$ with genotopic element R and genounit ${}^<1 = R^{-1}$ interconnected by a given conjugation, with general transformation law (6.A.3). When passing to the underlying algebra, the Lie-isotopic algebra is lost in favor of the covering Lie-admissible algebras when working in the original SÔ(3) space, that with trivial unit $1 = $ diag. $(1, 1, 1)$ or the Hermitean isounit $1 = T = T^\dagger$. However, when working in the appropriate two-sided genospace with units $1^>$ and ${}^<1$, we reach the abstract identity

$$ {}^<sÔ^>(3) \approx sÔ(3) \approx SO(3). \qquad (6.A.6) $$

Despite that, the *genorepresentation theory* of the ${}^<sô^>(3)$ algebra is considerably richer than that of the isotopic sô(3) or conventional so(3) algebras because, e.g., it implies different genoeigenvalues for the action to the right and that to the left, for instance, of the type

$$ J^{>2>} {>} |b^>{>} = 1^>_\phi{}^2 J^> (J^> + 1)|b^>{>}, \quad J^>_3 {>} |b^>{>} = 1^>_\phi \hat{M}^> |b^>{>}, \quad (6.A.7a) $$

$$ {<}{}^<b| {<}{}^<J^{2<} = {<}{}^<b|(1 + {}^<J){}^<J \, {}^<1_\phi{}^2, \quad {<}{}^<b| {<}{}^<J_3 = {<}{}^<b|{}^<\hat{M} \, {}^<1_\phi, \quad (6.A.7b) $$

$$ J^> = 0, \tfrac{1}{2}, 1, ..., \quad \hat{M}^> = J^>, J^> - 1, ..., -J^> \qquad (6.A.7c) $$

$$^{<}J = 0, \tfrac{1}{2}, 1, ..., \qquad ^{<}\tilde{M} = {}^{<}J, {}^{<}J - 1, ..., - {}^{<}J . \qquad (6.A.7d)$$

The above mathematical rules yield the desired physical result: *the axiomatization of angular momentum and spin under irreversible conditions.* In fact, we have one set of eigenvalues for the direction > of time, say that forward to future time, and a *different* set of eigenvalues for the conjugate different direction < of time, that forward from past time.

The *isodual Lie-admissible rotational group* $^{<}S\hat{O}^{>d}(3)$ are then characterized by the isodualities $1^{>d} = -1^{>}$ and $^{<}1^{d} = -{}^{<}1$, yielding the remaining two possible directions of time (Sect. II.3.3), that backward from future time $>^d$ and that backward in past time $<^d$.

In turn, this illustrates the reduction of macroscopic irreversibility to a primitive origin at the particle level which is prohibited by quantum mechanics, as well known, but permitted by the covering hadronic mechanics,

APPENDIX 6.B: ISOPLANE WAVE EXPANSIONS

We now study an aspect of the isoscrödinger's equation which is important for the applications of hadronic mechanics.

Recall that the conventional Schrödinger equation in spherical polar coordinates

$$\{ - \frac{1}{2 m \, r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{2m \, r^2} L^2 \} \ \psi(r, \theta, \phi) = E \, \psi(r, \theta, \phi), \qquad (6.B.1)$$

admits the familiar solutions

$$L^2 (\theta, \phi) \, Y_{Lm}(\theta, \phi) = L \, (\, L + 1) \, Y(\theta, \phi), \, L = 0, 1, 2, ... \qquad (6.B.2a)$$

$$L_z \, Y_{Lm}(\theta, \phi) = M \, Y_{Lm}(\theta, \phi), \ M = L, L-1, ..., -L , \qquad (6.B.2b)$$

$$\psi(r, \theta, \phi) = R(r) \, Y_{Lm}(\theta, \phi) \qquad (6.B.2c)$$

$$R(r) = N \, J_L(kr), \ k = (2mE)^{\tfrac{1}{2}}, \qquad (6.B.2d)$$

where $Y_L{}^M(\theta, \phi)$ are the familiar *spherical harmonics* and $J_L(kr)$ are the equally familiar *Bessel functions.*

The above results permit the known partial wave expansion (see, e.g., ref. [13])

$$e^{i \mathbf{k} \times \mathbf{r}} = \sum_{L=0,...,\infty} \sum_{M=L,..,-L} N_{LM} J_L(kr) Y_L^M(\theta, \phi), \qquad (6.B.3a)$$

$$e^{i k z} = e^{i k r \cos \theta} = \sum_L N_L J_L(kr) P_L(\cos\theta), \qquad (6.B.3b)$$

where $P_L(\cos \theta)$ represents the *Legendre polynomials*, and the N's are suitable renormalization constants.

The isotopies of the above properties are important for a number of applications of hadronic mechanics. As a first step, they can be expressed with the same conventional properties, although now referred to the isospherical coordinates. We can then write

$$\{ - \frac{1}{2\,m\,r^2} \frac{\partial}{\partial r} \hat{r}^2 \frac{\partial}{\partial r} + \frac{D^{-2} \hat{L}(\hat{L}+1)}{2m\,r^2} \} \hat{\psi}(r, \theta, \phi) = E' \hat{\psi}(r, \theta, \phi), \quad (6.B.4)$$

where

$$\hat{L}^2 * \hat{Y}_L^{\tilde{M}}(\theta, \phi) = D^{-2} \hat{L}(\hat{L}+1) Y(\theta, \phi), \hat{L} = 0, 1, 2, ... \qquad (6.B.5a)$$

$$\hat{L}_z * \hat{Y}_L^{\tilde{M}}(\theta, \phi) = D^{-1} b_3 \tilde{M} \hat{Y}_L^{\tilde{M}}(\theta, \phi), \tilde{M} = \hat{L}, \hat{L}-1, ...,-\hat{L}, \qquad (6.B.5.b)$$

$$\hat{\psi}(r, \theta, \phi) = \hat{R}(r) \hat{Y}_L^{\tilde{M}}(\theta, \phi) \qquad (6.B.5c)$$

$$\hat{R}(r) = N_L J_L(kr), \quad k' = (2mE')^{\frac{1}{2}}, \qquad (6.B.5d)$$

where $\hat{Y}_L^{\tilde{M}}(\theta, \phi)$ are the *isospherical harmonics* (11.6.15) and $J_L(kr)$ are the conventional Bessel functions, although defined with respect to radial variables on isospace.[68]

As a result, we can introduce the following *isoplane-wave expansion* first studied in ref. [5]

$$e^{i \mathbf{k}\hat{T}\mathbf{r}} = \sum_{\hat{L}=0,...,\infty} \sum_{\tilde{M}=L,..,-L} N_{\hat{L}\tilde{M}} J_L(kr) \hat{Y}_L^{\tilde{M}}(\theta, \phi), \qquad (6.B.6)$$

with the simplified form for the expansion along the polar axis

$$e^{i k b_3^2 z} = e^{i k b_3 r \cos b_3 \theta} = \sum_{\hat{L}} N_{\hat{L}} J_{\hat{L}}(kr) \hat{P}_{\hat{L}}[\cos (b_3 \theta)], \qquad (6.B.7)$$

which are important for the nonpotential scattering theory studied later on ion this volume.

As one can see already from these preliminary aspects, the isotopy of the planewave expansion is not trivial, because it alters the arguments of the conventional expansions. In particular, the isotopic theory is expected to provide quantitative predictions of deviations from conventional treatments which are

---

[68] Recall from Sect. 11.5.6 that the isounit of the radial part is the unit 1, thus implying no change in the Bessel function other than a different meaning of the variables.

testable with experiments.

APPENDIX 6.C: ISO-CLEBSCH-GORDON COEFFICIENTS

The isotopy of the conventional Clebsch–Gordon coefficients was preliminarily studied in ref. [4] because of possible value in practical applications.

Consider two particles $\alpha = 1, 2$ with hadronic spins $\Delta_\alpha j_\alpha$ and respective isounits $\hat{1}_k$ in condition of total mutual penetration. The total spin is given by

$$\mathfrak{J} = \mathfrak{J}_1 \times \hat{1}_2 + \hat{1}_1 \times \mathfrak{J}_2, \tag{6.C.1}$$

and assume, from section II.6.11, that such total operator admits the total isoeigenvalue in a way compatible with $\hat{SO}(2)$

$$\Delta j = \Delta_1 j_1 - \Delta_2 j_2. \tag{6.C.2}$$

The total isostate is given by the tensorial product

$$|\Delta_1 j_1, \Delta_2 j_2; \Delta_1 m_1, \Delta_2 m_2 > = |\Delta_1 j_1; \Delta_1 m_1 > |\Delta_2 j_2; \Delta_2 m_2 >, \tag{6.C.3}$$

which must be properly isonormalized.

By using the isocommutation rules of $\hat{SO}(2)$ it is easy to prove that $\mathfrak{J}_z$ and $\mathfrak{J}^2$ isocommute with all other operators and therefore constitute a maximal isocommuting set suitable for the reformulation of the isobasis. We can therefore write

$$|\Delta_1 j_1, \Delta_2 j_2; \Delta_1 m_1, \Delta_2 m_2 > = \tag{6.C.4}$$

$$\Sigma_{\Delta_1 j_1 \Delta_2 m_2} |\Delta_1 j_1, \Delta_1 m_1; \Delta J \Delta M > < \Delta_2 j_2, \Delta_2 m_2; \Delta J \Delta M| T |\Delta_1 j_1, \Delta_2 j_2; \Delta_1 m_1, \Delta_2 m_2 >$$

where T is the total isotopic element and the summation is restricted to states of maximal weight.

The *iso-Clebsch-Gordon coefficients* are then given by

$$< \Delta_1 m_1, \Delta_2 m_2 | \Delta j, \Delta m > = < \Delta_1 j_1, \Delta_2 j_2; \Delta_1 m_1, \Delta_2 m_2 | T | \Delta_1 j_1, \Delta_2 j_2; \Delta j, \Delta m > \tag{6.C.5}$$

The study of the properties of the above coefficients (as well as of the isotopies of the various other coefficients of the conventional theory) is intriguing and instructive (e.g., for degenerate isotopic elements), but we cannot study these aspects here in detail for brevity.

APPENDIX 6.D: q-SPHERICAL HARMONICS

As indicated in this chapter, by no means, the isospherical harmonics are the only possible deformations of the conventional functions. In fact, they merely constitute one type of possible deformations, that in which the unit is the inverse of the deformation itself so as to achieve an abstract unity of the original and the deformed functions (the abstract unity of the sphere and isosphere).

A variety of other deformations of spherical harmonics have been constructed in the literature, see, e.g., ref.s [15–17] and literature quoted therein. Their primary difference with the deformations studied in these volumes is that the deformed spherical harmonics are still referred to the original field of numbers with the conventional unit +1. As a result, the deformed functions do not coincide any longer with the original one at the abstract level (there is no longer the notion of isosphere to reach abstract unity).

Even though mathematically impeccable, the latter deformations are afflicted by a number of problems of physical consistency studied by Lopez [18] and outlined in Appendices I.7.A1 and II.3.C. One of them is due to the fact that q-deformations are noncanonical thus implying nonunitary time evolutions. In turn, under such time evolution, the q-number is turned into a T-operator according to the rules

$$U\, U^\dagger \neq 1, \quad U\, (\, A\, q\, B\,)\, U^\dagger \;=\; A'\, T\, B', \quad A' = U\, A\, U^\dagger, \quad T = q\, (\, U\, U^\dagger\,)^{-1}, \quad (6.D.1)$$

by reaching in this way an isotopic structure even when not desired.

We therefore have the following

Proposition 6.D.1: *q-deformed spherical harmonics referred to the conventional field of numbers with unit +1 do not preserve their structure under the time evolution.*

This illustrates the reason for the preference in these volumes of the isotopic deformation. In fact, the latter does preserve their structure at all future or past times.

Note also that the reformulation of the q-deformed spherical harmonics with respect to the isofield with isounit $\hat{1} = q^{-1}$ would be trivial because q is a constant. To have a nontrivial isotopy of the spherical harmonics the deformation must be characterized by a $(2j+1) \times (2j+1)$ matrix T, thus returning in this way to the problem studied in Sect. II.6.6.

Despite the above aspects, a study of the q-deformed spherical harmonics is indeed recommended because instructive. As an illustration, we here outline the *Granovskii–Zhedanov q-spherical functions* [15] with deformed $SU_2(2)$ algebra

$$[J_0 , J_\pm] = \pm J_\pm , \quad [J_+ , J_-] = (\sinh 2\omega J_0) / \sinh\omega , \qquad (6.D.2a)$$

$$J_q{}^2 = J_+ J_- + [\cosh\eta\, \omega\, (2 J_0 + 1)] / 2 \sinh^2 \omega , \qquad (6.D.2b)$$

where $\omega = \omega(q) > 0$ and $J_q{}^2$ is the q–deformed Casimir invariant. Note the preservation of the conventional Lie product and the alteration instead of the structure constants.[69] Note also that the deformed algebra is referred to conventional fields $R(n,+,\times)$ of numbers n.[70]

Next, the studies of ref. [16] are conducted with the conventional SO(3) invariant measure and related inner product

$$(\phi , \psi) = \int_\Omega d\phi \, \sin\theta \, d\theta \, \phi^\dagger(\theta, \phi) \, \psi(\theta, \phi) \quad \in \quad C(c,+,\times) . \qquad (6.D.3)$$

This implies that the structure of the Hilbert space of SO(3) is preserved under the q–deformation.[71] Since the invariant measure is not changed, the conventional transformation to spherical coordinates apply.[72]

The studies then lead to the so–called *q–spherical harmonics with vertical recurrences* [15]

$$Y_{jm}(\theta, \phi) = e^{im\phi} G_m(\theta + i\omega) \prod_{k = 0,...,m} \sin [\theta +/ i \omega (m - 1 - 2 k)] , \quad (6.D.4)$$

where $G_m(\theta + i\omega)$ is an arbitrary function with period $2i\omega$.

The q–deformation of the spectrum of eigenvalues is evident. In fact, we have relations of the type [loc. cit.]

$$J_- J_+ Y_{jm} = [j - m][j + m + 1] Y_{jm} , \qquad (6.D.5)$$

where

---

[69] In the isotopic approach we generalized instead the structure of the Lie product and preserve the structure constants to ensure the isomorphism of the deformed and original algebras.

[70] In the isotopic approach we jointly lift the enveloping associative algebra $AB \rightarrow ATB$ and the underlying field $R(n,+,\times) \rightarrow \hat{R}(\hat{n},\hat{+},\hat{*})$ into forms with basic isounit $\hat{1} = T^{-1}$. This is done so that the isoassociative product ATB referred to the isounit $\hat{1} = T^{-1}$ is evidently equivalent to the original product AB with unit I.

[71] This occurrence is at the origin of the lack of preservation of the Hermiticity of the generators J_0 and $J_\pm$ under time evolution, as identified by Lopez [18]. In the isotopic approach we jointly lift the Lie algebra and the underlying Hilbert space in such a way to preserve the Hermiticity of the generators at all times.

[72] Under isotopy, the spherical coordinates are instead subjected to a lifting which is crucial for the main result of this chapter, *the deformation of the spectrum of eigenvalues of* SU(2) *under the local isomorphism* SO(2) $\sim$ SU(2).

$$[\,x\,]\,:=\,\sinh\omega x\,/\,\sinh\omega\,. \qquad (6.D.6)$$

The notion of verticality is referred to the deformation in the (j, m) space of SU(2) which is vertical with respect to the j–axis. Equivalently, the notion of verticality expresses the property that the J_ operator of $SU_q(2)$ preserves j and changes m.[73]

There exists no operator in $SU_q(2)$ which changes j and preserves m. As a result, we have no *horizontal recurrences* strictly speaking. Nevertheless, ref. [15] studies a weaker relation, by identifying additional deformations of the spherical harmonics. They must be computed with respect to a certain *weighting* of the inner product (6.D.3) of the type

$$(\,\phi\,,\,\psi\,)_W\,=\,\int_\Omega d\phi\,\sin\theta\,d\theta\,\phi^\dagger(\theta,\,\phi)\,W\,\psi(\theta,\,\phi)\,. \qquad (6.D.7)$$

which, as such, is much similar to the isoinner product.[74]

For brevity, we refer the interested reader to the locally quoted literature on the latter and numerous additional topics.

REFERENCES

1. A. R. EDMONDS, *Angular Momentum in Quantum Mechanics*, Princeton Univ. Press (1957); D. M. BRINK and G. R. SATCHLER, *Angular Momentum*, Clarendon Press, Oxford, England (1968)
2. R. M. SANTILLI, Hadronic J. 1, 228 (1978)
3. R. M. SANTILLI, Hadronic J. 8, 25 and 36 (1985)
4. R. M. SANTILLI, Hadronic J. Suppl. 4B, issue no. 2 (1989)
5. R. M. SANTILLI, in *Proceedings of Third International Wigner Symposium*, Int. J. Mod. Phys. B, Vol. 8 (1994); JINR Rapid Comm., 6, 24 (1993); Acta App. Math., in press.
6. A. U. KLIMYK and R. M. SANTILLI, Algebras, Groups and geometry 10, 323 (1993)
7. R. M. SANTILLI, *Isotopic Generalization of Galilei's and Einstein's Relativities*, first edition (1991), second edition, Academy of Sciences of the Ukraine, Kiev (1994)

---

[73] Note that the concept of verticality does not exist under isotopies because the Ĵ_ operator of SÛ(2) changes both j and m. Similarly, there exists no notion of horizontality because there exists no operator in SÛ(2) which preserves m and changes j.

[74] The primary difference is that the weighed inner product is referred to conventional field for the q-deformations, while in the isotopies it is referred to an isofield with isounit $\hat{1} = W^{-1}$ which render's it equivalent to the original composition.

8. A. K. ARINGAZIN, A. JANNUSSIS, M. NISHIOKA, D. F. LOPEZ AND B. VELJANOSKI, *Santilli's Isotopic Generalizations of Galilei's and Einstein's Relativities,* Kostarakis Publisher, Athens, Greece (1991)

9. J. V. KADEISVILI, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities,* first edition (1992), second edition, Academy of Sciences of the Ukraine, Kiev (1994)

10. N. Ja VILENKIN, *Special Functions and the Theory of Group Representations,* Amer. Math. Soc., Providence, R. I. (1968)

11. L. C. BIEDENHARN and J. D. LOUCK, *Angular Momentum in Quantum Physics* , Addison Wesley, Reading MA p(1981)

12. I. M. GEL'FAND and Z. Ya. SHAPIRO, *Representations of the Rotation and Lorentz Groups and their Applications,* Macmillan Company, New York (1963)

13. D. BOHM, *Quantum Theory,* Dover, New York (1979)

14. R. M. SANTILLI, Hadronic J. $\underline{1}$, 574 (1978)

15. Ya. I. GRANOVSKII and A. S. ZHEDANOV, J. Phys. A: Math Gen $\underline{26}$, 4331 (1993)

16. N. JA. VILENKIN and A. U. KLIMYK, *Representations of Lie Groups and Special Functions,* Vol.s I, II and III, Kluwer, Dordrecht (1992)

17. J. FUCHS, *Affine Lie Algebras and Quantum Groups,* Cambridge Univ. Press, Cambridge, England (1992)

18. D. F. LOPEZ, Hadronic J. $\underline{16}$, 429 (1993)

7: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF GALILEI'S RELATIVITY

7.1: STATEMENT OF THE PROBLEM

As now familiar, the main objective of these volumes is to study strong interactions with a nonlinear–nonlocal–nonhamiltonian component due to mutual penetrations of the charge distributions and wavepackets of hadrons. In particular, the study is expected to result in novel structural models of nuclei, hadrons and stars treated in Vol. III.

After having identified the foundations of hadronic mechanics, in this chapter we identify the nonrelativistic dynamical symmetry and relativity which are applicable under nonlinear–nonlocal–nonhamiltonian interactions. The relativistic and gravitational extensions will be studied in subsequent chapters.

Consider the body of knowledge originated by Galileo Galilei [1] and today known as *classical or quantum Galilei relativity* (see, e.g., ref.s [2,3] and literature quoted therein). The basic *Galilean invariant* for a system of N particles in Euclidean space is given by

$$t_a - t_b = \text{inv.}, \quad (r^i_a - r^i_b)\delta_{ij}(r^j_a - r^j_b) = \text{inv. at } t_a = t_b, \qquad (7.1.1)$$

$$i, j = 1, 2, 3 (= x, y, z), \quad a, b = 1, 2, ..., N .$$

Its maximal possible linear symmetry is characterized by the *Galilei transformations*

$$t' = t + t°, \qquad \text{translations in time},$$

$$r^k_a{}' = r^k_a + r^{°k}, \qquad \text{translations in space},$$

$$r^k_a{}' = r^k_a + t° v^k, \qquad \text{Galilean boosts}, \qquad (7.1.2)$$

$$r' = \Re(\theta_1, \theta_2, \theta_3) r, \qquad \text{r o t a t i o n s} \qquad ,$$

$$\tau t = -t, \quad \pi r = -r, \qquad \text{time and space inversions},$$

which constitute the connected *Galilei group* (prior to scalar extensions hereon ignored for simplicity)

$$G_o(3.1) \;=\; [\; SO(3) \odot T_t \circ(1)\;] \times [\; T_r \circ(3) \odot T_r \circ(3)\;]\,, \tag{7.1.3}$$

where $\odot$ represents the direct product and $\times$ the semidirect product (see, e.g., ref. [2] for the classical version and ref. [3] for the quantum one).

The Galilei symmetry and relativity are *exactly* valid for the *nonrelativistic exterior dynamical problems*, and they are assumed as the foundation of our nonrelativistic studies.

The relativity characterizes *closed Hamiltonian systems*, that is, systems of "massive points" (in Galilei's own words [1]) isolated from the rest of the universe which move in vacuum under action–at–a–distance interactions without collisions, with classical equations of motion and related ten Galilean total conservation laws in self–evident notation

$$m_{ka}\ddot{r}_{ka} \;=\; - \partial V(r) / \partial r^k{}_a\,, \quad k = 1, 2, 3, \quad a = 1, 2, ..., N\,, \tag{7.1.4a}$$

$$dE_{tot} / dt = 0, \quad dJ_{tot} / dt = 0, \quad dP_{tot} / dt = 0, \quad dG_{tot} / dt = 0\,, \tag{7.1.4b}$$

The isotopies $\hat{G}(3.1)$ of the Galilei symmetry $G(3.1)$, called *Galilei–isotopic (or isogalilean) symmetry*, and of the underlying relativity, called *Galilei–isotopic (or isogalilean) relativity*, emerged in systematic studies by this author [4,5,6] of the *classical, nonrelativistic, interior dynamical problem*.

The isogalilean relativity was first proposed in memoir [4]. The generalized relativity was then subjected to systematic studies in monographs [5] for internal forces which are nonhamiltonian but still local–differential (see the presentation of the isogalilean relativity in Ch. 6, Vol. II, of ref.s [5]). The generalized relativity was then subjected to systematic classical studies for nonlinear–nonlocal–nonhamiltonian interactions in monographs [6]. The classical *isodual isogalilean symmetry and relativity* were also introduced in ref.s [6]. The operator counterparts were studied in memoirs [7,8].

While the Galilean relativity characterizes closed Hamiltonian systems, the covering isogalilean relativity characterizes a structurally more general class of closed systems of extended particles with action–at–a–distance/potential as well as contact nonpotential interactions.

The latter systems are called *closed nonhamiltonian* [5,6] to indicate that they are closed–isolated as the conventional conservative systems, and thus verify all conventional Galilean conservation laws, yet they admit nonhamiltonian internal forces with equations of motion

$$m_{ka}\ddot{r}_{ka} \;=\; F_{ka}{}^{SA}(r) + F_{ka}{}^{NSA}(r, \dot{r}, \ddot{r})\,, \tag{7.1.5a}$$

$$\sum_a F_a^{NSA} = 0, \qquad \sum_a \dot{r}_a \cdot F_a^{NSA} = 0, \qquad \sum_a r_a \wedge F_a^{NSA} = 0 \qquad (7.1.5b)$$

$$dE_{tot} / dt = 0, \quad dJ_{tot} / dt = 0, \quad dP_{tot} / dt = 0, \quad dG_{tot} / dt = 0, \qquad (7.1.5c)$$

where SA stands for the verification of the conditions of variational selfadjointness, which are the integrability conditions for the existence of a potential [5] while NSA stands for their violation, and conditions (7.1.5b) are, in general, subsidiary constraints to the equations of motion (7.1.5a) (see Ch. 6, Vol. II, ref. [5] for details).

Note that conditions (7.1.5b) are seven in total for a system of N particles and therefore with 3N components F_{ka}^{NSA}. An algebraic solution in the nonpotential forces F_{ka}^{NSA} therefore always exist for $N \geq 3$, the case $N = 2$ admitting a special solution with acceleration–dependent forces (see for brevity App. III.A, Vol. II, ref. [6]).

Systems (7.1.5) are called closed nonhamiltonian because the knowledge of the Hamiltonian alone, as in systems (7.1.4), is insufficient to characterize the equations of motion, owing to the additional nonpotential forces F_{ka}^{NSA}. Ref.s [4,5,6] introduced the isotopies of Classical Hamiltonian mechanics precisely for the purpose of representing the additional nonpotential forces, first, via the generalized brackets of the theory [5] and then via a lifting of the basic unit of the theory.

The evident consistency of systems (7.1.5) establishes that, by no means, the validity of the ten total Galilean conservation laws (7.1.4b or (7.1.5c) implies that all internal forces are solely derivable from a potential V(r). In fact, systems (7.1.5) establish that total conventional conservation laws are verified also under nonpotential internal forces, provided that they verify conditions (7.1.5b).

In essence, global stability is reached in closed Hamiltonian systems via the stability of the orbits of each individual constituent. On the contrary, the global stability is generally[75] reached in the closed nonhamiltonian systems via a collection of orbits each of which is individually unstable. We merely have internal exchanges of energy and other physical quantities which however compensate each other in such a way to verify total conservation laws.

The best classical example of closed Hamiltonian systems is the planetary system (see Fig. 7.1.1). A majestic classical example of closed nonhamiltonian systems is the structure of Jupiter considered as isolated from the rest of the universe (see also Fig. 7.1.1).

The first most visible difference between the planetary system and the structure of Jupiter is that the former admits the heaviest constituent in the Keplerian center, while the latter admits no Keplerian center at all. This physical evidence is sufficient forceful, alone, to require a structural generalization of

[75] There are exceptions to this rule of internal instability given by the two–body and three–body in restricted or Lagrangian configurations whose orbits must be stable for certain dynamical occurrences (see App. III.A, Vol. II, ref. [6]).

Galilei's symmetry and relativity for interior problems [6]. In fact, the characterization of Jupiter via the Galilean symmetry would necessarily imply a *planetary-type* structure which is contrary to clear experimental evidence. On the contrary, one needs a generalization of Galilei's symmetry and relativity capable of admitting an arbitrary particle at the center, whether heavier or ligther than all other constituents.

The second clear difference between closed Hamiltonian and nonhamiltonian systems is that the interactions in the planetary structure are invariant under the Galilei symmetry, while the internal forces in Jupiter's structure, being of contact-nonpotential type, are manifestly *noninvariant* under the Galilean symmetry [4]. When these internal forces are represented in their actual nonlocal-integral character, they are beyond the analytic, algebraic and topological structure of the Galilei relativity without any hope of quantitative treatment. The need for a generalization of Galilei's symmetry and relativity to restore the form-invariant description of interior systems is then beyond scientific doubts.

But perhaps the limitations of Galilei's transformations (7.1.2) for interior conditions most important for these volumes is their *linearity* which is at variance with the *nonlinearity in the velocities* of the interior problem. We can safely state that, while certainly providing a meaningful approximation, the nonrelativistic characterization of interior systems via the Galilean symmetry will not resist the test of time because of their linearity.

For additional, equally forceful needs to generalize Galilei's symmetry and relativity for interior conditions, we refer the interested reader to monographs [5,6] whose results are implied hereon. In particular, the understanding of this chapter requires a technical knowledge of the integrability conditions for the existence of a potential or a Hamiltonian [5], and in particular, their *violation* by the interior systems of our everyday experience.

It is remarkable that the above main classical lines persist in their entirety in the transition to operator interior systems. The best example is the transition from the atomic to the nuclear structure. In fact, the first most visible difference between these systems is the presence of the nucleus in the former, while "there is no nucleus in the atomic nucleus". In fact, any nucleon can be at the center of the atomic nucleus.

As we shall see in Vol. III, this physical evidence alone is sufficient to establish that the Galilean relativity and underlying nonrelativistic quantum mechanics, even though providing an excellent approximation, cannot be *exactly* valid for the nuclear structure. Relativistic corrections have been unable to provide an exact, numerical, representation of several nuclear data, such as the total magnetic moment of few-body nuclei, thus favoring a *structural generalization* of the nuclear description beginning at *the nonrelativistic level,* and prior to any relativistic extension.

But there are deeper reasons to suggest the latter lines of research. We here

mention the experimental evidence derived from data on the volumes of nucleons and of nuclei, that nucleons in the nuclear structure are in average conditions of mutual penetration of about 10^{-3} units of their volume. This establishes the existence in nuclear physics of the conditions of mutual penetration which are at the foundations of hadronic mechanics. This experimental information also clarifies the excellent approximation provided by the conventional Galilean relativity and nonrelativistic quantum mechanics for the nuclear structure, and the fact that hadronic mechanics will essentially provide *small corrections* in nuclear physics.

When considering the structure of hadrons the above lines become more compelling. In fact, the constituents are now in condition of total mutual penetration because, as studied in undergraduate quantum mechanics, all massive particles have a wavepackets of the order of 1 fm, that is, of the order of the size of all hadrons. This results in conditions of total mutual penetration of the hadronic constituents with expected much bigger contributions of nonlocal–nonhamiltonian type as compared to nuclear physics.

Also, "there is no nucleus in the hadronic structure". Alternatively, the assumption of a Keplerian nucleus in the hadronic structure would imply an atomic type behaviour contrary to reality. This experimental evidence is sufficient to indicate the existence of profound dynamical differences between the atomic and hadronic structures, with consequential needs to represent them with correspondingly different symmetries and relativities.

Finally, in interior gravitational problems, such as gravitational collapse, we have the total mutual penetration of hadrons as well as their compression in large number into a small region of space. The consequential emergence of the most general possible nonlinear–nonlocal–nonhamiltonian conditions is then beyond any scientific doubt.

As we shall see in this chapter, the operator isogalilean relativity does indeed provide a quantitative nonrelativistic characterization of closed nonhamiltonian systems in a form suitable for experimental verifications. In the subsequent two chapters we shall study the compatible operator formulations of closed nonhamiltonian relativistic and gravitational systems.

The most important prediction of the emerging novel structure models of nuclei, hadrons and stars is a new form of subnuclear energy called *hadronic energy* [9] which will be studied in details in Vol. III. In this volume we shall outline the *antigravity* (Sect. 11.8.7) and the *space–time machine* (Sect. 11.9.7) for additional studies also in Vol. III.

But the closed–isolated character of systems is, per se, an approximation of more general open–nonconservative conditions because, strictly speaking, no systems can be considered as being truly isolated from the rest of the universe (Mach principle). This is the reason why the original proposal [4] submitted the classical *Lie–admissible generalization of Galilei's symmetry and relativity* of which the Lie–isotopic versions are particular cases. The latter generalizations were subsequently studied in ref.s [10] under the name of *Galilei–admissible (or*

genogalinean) *symmetry* and *relativity* for the characterization of particles in open–nonconservative–irreversible conditions. Additional classical studies were conducted in ref.s [6] with the introduction, in particular, of the *isodual Galilei-admissible symmetry* and *relativity* for the characterization of antiparticles, also in open–nonconservative–irreversible conditions. The operator genogalinean symmetry and relativity of this have been introduced in this chapter for the first time, to our best knowledge.

EXTERIOR MANY–BODY SYSTEMS

Keplerian center

INTERIOR MANY–BODY SYSTEMS
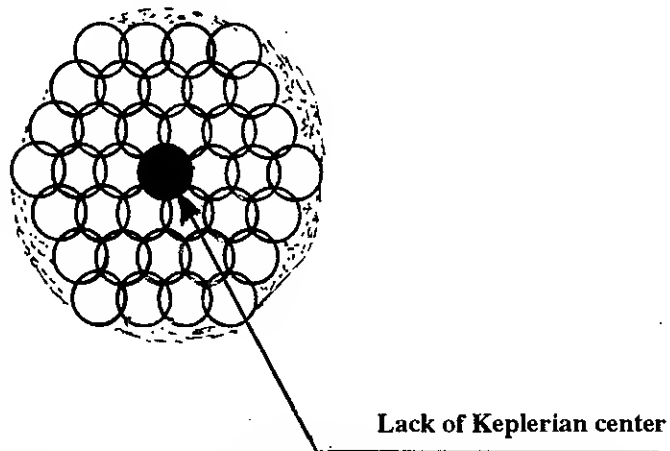

Lack of Keplerian center
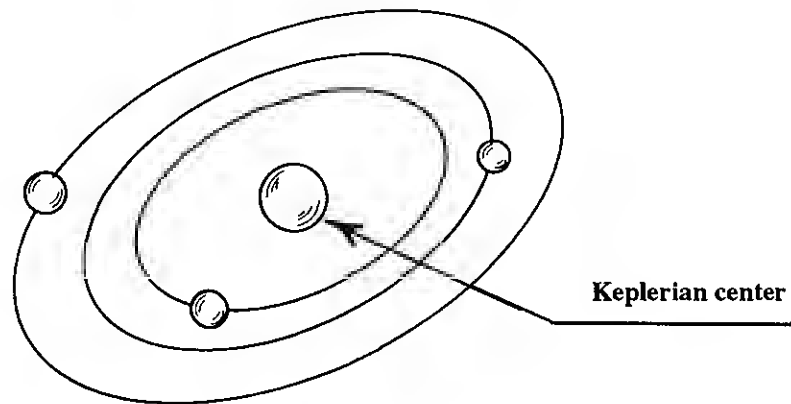
FIGURE 7.1.1: A schematic view of exterior dynamical systems, such as the planetary or atomic systems, and the corresponding interior dynamics systems,

such as the structure of Jupiter or the structure of an atomic nucleus, a hadron, or a star. The first visible difference is that the former admit the *Keplerian nucleus* in the center, while the latter do not. Similarly, "there is no nucleus in the atomic nucleus" or in a star. This structural difference is sufficient, alone, to require a generalization of Galilei's symmetry and relativity as studied in monographs [5,6] at the classical level and in this chapter at the operator level. From these first lines, one can already see the emergence of fundamentally novel structure models of nuclei, hadrons and stars.

In summary, quantum, mechanics admits only one nonrelativistic dynamical symmetry and relativity, the Galilean ones. Hadronic mechanics admits instead the following hierarchy of relativities:

1) The conventional **Galilean relativity** for the characterization of closed–isolated Hamiltonian systems of particles in reversible conditions;

2) The **isogalilean relativity** for the characterization of systems of particles which are still closed–isolated and with reversible center–of–mass trajectory, but admit nonhamiltonian internal forces;

3) The **genogalinean relativity** for the characterization of systems of particles which are in open–nonconservative–irreversible conditions;

4) The **isodual Galilean relativity** for the characterization of closed–isolated Hamiltonian systems of antiparticles in reversible conditions;

5) The **isodual isogalilean relativity** for the characterization of systems of antiparticles which are still closed–isolated with reversible center–of–mass trajectory but with nonhamiltonian internal forces; and

6) The **isodual genogalinean relativity** for the characterization of systems of antiparticles which are in open–nonconservative–irreversible conditions.

All the above relativities are unified by the **abstract Galilei-admissible relativity of Class III.** Nevertheless, to avoid an excessively abstract treatment particularly for a first exposure to hadronic mechanics, we shall study the generalized relativities in an individual basis.

A comparison between hadronic mechanics and the rather vast conventional literature on dissipative systems (see, e.g., ref.s [11-13] and literature quoted therein) is instructive. Both approaches deal with contact–nonpotential interactions. However, hadronic mechanics is the only one which "closes" these systems into a form verifying total conservation laws while preserving the nonpotential character of the internal forces. Also, conventional treatments are based on conventional Hilbert spaces and fields thus suffering of various problematic aspects under nonunitary time evolutions studied earlier, while hadronic mechanics is based on suitably generalized Hilbert spaces and fields. Finally, and perhaps most significantly, conventional approaches do not replace quantum mechanical symmetries with covering symmetries, while the central objective of hadronic mechanics is to replace them with structurally generalized

symmetries which, as such, can characterize structurally more general systems.

Needless to say, numerous types of generalizations of Galilei's symmetry and relativity can be attempted, e.g., via the *q-deformations, nonlinear theories* or *theories with nonassociative envelopes* (Sect. 1.7.9), but they are afflicted by various problematic aspects which prevent unambiguous physical applications. Also, we adopt the *isotopic* generalization because it is the only one achieving a *direct representation*[76] of closed-isolated interior systems under the local isomorphisms $\hat{G}(3.1) \approx G(3.1)$.

The latter property permits a remarkable nonrelativistic unification of the exterior and interior systems at the abstract geometric, algebraic and analytic levels.

Above all, the use of q-deformations or other approaches implies the necessary abandonment of the Galilean laws, thus creating the problem of identifying plausible new laws. By comparison, the isotopies permit *the preservation of the physical laws of the Galilean relativity at the abstract level* and merely provide more general *realizations* of the same laws.

7.2: CLASSICAL GALILEI-ISOTOPIC SYMMETRY AND ITS ISODUAL

Let us begin by recalling that the classical isogalilean symmetry $\hat{G}(3.1)$ characterizes closed nonhamiltonian systems *without* constraints (7.1.5b) [6]. In fact, the imposition of Galilei's symmetry to the equations of motion (7.1.4a) ensures the conservation of the generators of the symmetry, resulting in Eq.s (7.1.4b). The situation is fully similar at the isotopic level. In fact, by recalling that the basis of a Lie algebra is not changed under isotopies (Ch. 1.4), the imposition of the isogalilean symmetry to Eq.s (7.1.5a) ensures the validity of *conventional* conservation laws (7.1.5c), without need of subsidiary constraints.[77]

The presence of the nonpotential interactions ensures that the constituents

---

[76] A *direct analytic representation* is that occurring in the frame r of the experimenter without any use of the transformation theory [5]. The *indirect analytic representations*, those in a transformed frame r',are prohibited in these volumes because physical reality must be first represented in the frame of the observer, before the transformation theory may acquire a physical significance. At any rate the transformations of nonhamiltonian to Hamiltonian systems are highly nonlinear (*Lie-Koening theorem* [5]) and, as such, are *not realizable in laboratory* and do not preserve the *inertial character of reference frames*, by therefore leaving open the problem of the applicable generalized relativity.

[77] One can see these occurrences by nothing that, in the absence of the Galilean symmetry, the conventional equations of motion (7.1.4) would need exactly the same subsidiary constraints (7.1.5b) to verify the total conservation laws, although the constraints are now referred to potential forces F_{ka}^{SA}.

are in "contact" with each other, thus permitting an arbitrary particle to be at the center, called *isonucleus* [6]. In this way, for the unit 1 = diag. $(1, 1, 1)$, the nucleus is Keplerian, while for the isounits $\hat{1}(t, r, \dot{r}, \ddot{r}, ...)$ we have the isonucleus, i.e., the particle at the center can be ligther or heavier than the remaining constituents.

The above main lines have been realized as follows. The basic carrier spaces are the isospaces $\hat{E}(t, \hat{R}_t) \times T^* \hat{E}(r, \hat{\delta}, \hat{R})$ of Kadeisvili Class 1, whose characteristic functions have an arbitrary dependence on time t, coordinates r and their derivatives $\dot{r}$, $\ddot{r}$, local density of the interior medium μ, local temperature τ, local index of refraction n, and any other needed interior quantity

$$\hat{E}(t, \hat{R}_t): \quad \hat{1}_t = T_t^{-1} = b_4^{-2}, \qquad b_4(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...) > 0, \qquad (7.2.1a)$$

$$\hat{E}(r, \hat{\delta}, \hat{R}): \quad \hat{\delta} = T\delta, \ T = \text{diag.} (b_1^2, b_2^2, b_3^2), \ \hat{b}_k = b_k(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...) > 0, \quad (7.2.1b)$$

$$r^2 = (x\, b_1^2\, x + y\, b_2^2\, y + z\, b_3^2\, z)\hat{1} \in \hat{R}(\hat{n}, +, *), \quad \hat{1} = T^{-1}, \qquad (7.2.1c)$$

where $\hat{1}_t$ (T_t) is the time isounit (time isotopic element); $\hat{1}$ (T) is the space isounit (space isotopic element); $\hat{R}_t$ $(\hat{R})$ is the time (space) isofield; r^2 is the isoseparation; and $T^* \hat{E}(r, \hat{\delta}, \hat{R})$ represents the isophase space with local coordinates r^k and $p_k = m\dot{r}_k$.

The isosymmetry $\hat{G}(3.1)$ can be defined [6] as the Lie–isotopic group of nonlinear, nonlocal and noncanonical isotransforms on $\hat{E}(t, \hat{R}_t) \times T^* \hat{E}(r, \hat{\delta}, \hat{R})$ leaving invariant the following isoseparation for a system of N particles [78]

$$t_a - t_b = \text{inv.}, \qquad (7.2.2a)$$

$$(r_a^i - r_b^i)\,\delta_{ij}(t, r, p, ...)\,(r_a^j - r_b^j) = \text{inv.} \quad \text{at} \quad t_a = t_b, \qquad (7.2.2b)$$

$$i, j = 1, 2, 3, \quad a, b = 1, 2, ..., N .$$

The isoalgebra is expressed in terms of the Lie–isotopic brackets

$$[A\hat{,} B] = \frac{\partial A}{\partial r_a^k}\, b_k^{-2}(t, r, p, ...)\, \frac{\partial B}{\partial p_{ka}} - \frac{\partial B}{\partial p_{ka}}\, b_k^{-2}(t, r, p, ...)\, \frac{\partial A}{\partial r_a^k} \qquad (7.2.3)$$

where the b's are derived via the isoexterior isodifferential of a one–isoform (Sect. 1.5.4), thus verifying the Lie algebra axioms by construction because of the isopoincaré lemma 1.5.4.1).

By recalling the expressions in isoeuclidean space

---

[78] Note the uniqueness of the interior isometric for all particles. This is requested by evident geometrical needs for one single space, and it is not in conflict with the results of the preceding section owing to the arbitrariness of the isometric itself which can be reducible, i.e., given by the tensorial product of different isometrics.

$$r_k = \delta_{ki} r^i = b_k^2 r^k, \qquad\qquad r^i \delta_{ij} r^j = r^k r_k = r_k r^k, \qquad (7.2.4)$$

brackets (7.2.3) can be equivalently written

$$[A\hat{,} B] = \frac{\partial A}{\partial rka} \frac{\partial B}{\partial pka} - \frac{\partial B}{\partial rka} \frac{\partial A}{\partial pka}, \qquad (7.2..5)$$

which coincides with conventional brackets thus illustrating again the axiom-preserving nature of the isotopies. The understanding is however that the generalization of the original geometry is now embedded in the covariant coordinates $r_k = b_k^2(t, r, \dot{r}, \ddot{r}, \mu, \tau, n, ...) r^k$. A more formal expression of the brackets can be introduced via the isoderivatives but it is not necessary for this review.

The *Isogalilean algebra* $\hat{g}(3.1)$ is characterized by the conventional generators

$$X = \{ X_k \} = \{ J^i, P_i, G_i, H \}, \quad J_i = \sum_a \epsilon^{ijk} r_{ja} P_{ka}; \quad P_i = \sum_a P_{ia}, \quad (7.2.6a)$$

$$G_i = \sum_a (m_a r_{ia} - t P_{ia}), \quad H = P_{ka} b_k^2 P_{ka}/2m_a + V(r_{ab}) \}, \qquad (7.2.6b)$$

$$\hat{r_{ab}} = \{ (r^k_a - r^k_b) b_k^2 (r^k_a - r^k_b) \}^{\frac{1}{2}}, \qquad (7.2.6c)$$

although now defined in $\hat{E}(t,\hat{R}_t) \times T^* \hat{E}(r,\delta,\hat{R})$, with *isocommutation rules* (see also those for $S\hat{O}(3)$ of Sect. II.6.3) [6]

$$[J^i \hat{,} J^j] = \epsilon^{ijk} J^k, \qquad [J^i \hat{,} P_j] = \epsilon^{ijk} P_k, \qquad (7.2.7a)$$

$$[J^i \hat{,} G_j] = \epsilon^{ijk} G_k, \qquad [J^i \hat{,} H] = 0, \qquad (7.2.7b)$$

$$[G_i \hat{,} P_j] = \delta_{ij} M, \qquad [G_i \hat{,} H] = 0, \qquad (7.2.7c)$$

$$[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} H] = 0, \qquad (7.2.7d)$$

and (local) *isocasimir invariants*

$$\hat{C}^{(1)} = (P_i \delta^{ij} P_j - M H), \qquad (7.2.8a)$$

$$\hat{C}^{(2)} = (MJ - G \wedge P)^2 = (MJ - G\wedge P) * (MJ - G\wedge P), \qquad (7.2.8b)$$

The *connected isogalilean group* $\hat{G}_o(3.1)$ is characterized by the same generators (7.2.6) and the conventional (ordered set of) parameters

$$w = (w_k) = (\theta_i, r^o_i, v^o_i, t^o), \quad k = 1, 2,..., 10, \qquad (7.2.9)$$

with the isogroup structure (here written for simplicity with the conventional exponentiation) [6]

$$r' = \{ [\prod_k e^{w_k \omega^{\mu\sigma} \times I_\sigma{}^\nu (\partial_\nu X_k)(\partial_\mu)}] \hat{1} \} * r, \qquad (7.2.10)$$

The above structure characterizes a *finite*, ten–dimensional isogroup owing to the *isotopic Baker–Campbell–Hausdorff composition theorem* 1.4.5.1 and yields an isotopic structure fully equivalent to the conventional one (7.1.3), i.e.,

$$\hat{G}_0(3.1) \;=\; [\; S\hat{O}(3) \; \hat{\circ} \; \hat{T}_t(1)\;]\;\hat{\times}\;[\;\hat{T}_r(3) \;\hat{\circ}\; \hat{T}_r(3)\;]\;, \tag{7.2.11}$$

where hat denotes isotopy. The local isomorphism $\hat{G}(3.1) \approx G(3.1)$ is then evident, also in view of the preservation of the conventional structure constants in isoalgebra (7.2.7).

Note also the identity $\hat{G}(3.1) \equiv G(3.1)$ at the abstract level, that is, the complete loss of any differentiation between the Galilean and isogalilean structures at the realization–free level, as expected from the corresponding identity of the underlying carrier spaces and isospaces.

This latter property is important to illustrate that, by no means, the Galilean symmetry is abandoned in our studies of interior dynamical systems. On the contrary, the efforts are devoted to the preservation of the symmetry although realized in a way structurally more general than the conventional one [5,6]. The possibility to preserve the Galilean symmetry at the abstract level for interior problems also illustrates the preference of the isotopies over other approaches, such as q–deformations.

For detailed studies on the construction of isogalilean invariant systems, the necessary and sufficient conditions for the $\hat{G}(3.1)$ invariance, the construction of the symmetry from the equations of motion and other topics, we refer the reader to monographs [5,6].

Note that in the conventional case, one usually assigns the Hamiltonian (this is at times called *direct problem of Newtonian mechanics*) and verifies its $G(3.1)$–invariance, by (rarely) constructing the equations of motion. The situation for isotopies is reversed. In fact, nonhamiltonian equations of motions are assigned first. One must then construct their isoanalytic representation in terms of a Hamiltonian H and the isounit $\hat{1}$ (this is called the *inverse problem of Newtonian mechanics* [5,6]). The verification of the $\hat{G}(3.1)$–isoinvariance is the the third step.

Note also that *the Galilean transforms are unique, while there exist an infinite number of different isogalilean transforms,* although they all belong to the same class. In fact, different nonhamiltonian forces are represented via different isounits which enter in the isogroup structure, thus resulting in different explicit transforms (see next section).

The *isodual isogalilean symmetry* $\hat{G}^d(3.1)$ is characterized by the isodual generators $X_k{}^d = -X_k$, isodual parameters $w_k{}^d = -w_k$, isodual isotopic elements $T^d = -T$ and can be easily derived from the preceding studies on isoduality.

7.3: CLASSICAL GALILEI–ISOTOPIC RELATIVITY
AND ITS ISODUAL

The *classical isogalilean relativity* [6] can be defined as the form-invariance description of systems under the isogalilean symmetry $\hat{G}(3.1)$ in a way fully parallel to the conventional relativity [2].

The *general isogalilean transforms* of Class 1 can be readily computed from isoexponentials (7.2.10) plus the isodiscrete components from Sect. 11.6.2, and can be written [6]

$$t' = t + t° B_4^{-2}, \qquad \text{isotime translations} \qquad (7.3.1a)$$

$$r^{k \,\prime} = r^k + r^{°k} B_k(r°)^{-2}, \qquad \text{isospace translations} \qquad (7.2.3b)$$

$$r^{k \,\prime} = r^k + t° v^{°i} B_k(v°)^{-2}, \qquad \text{isogalilean boosts} \qquad (7.3.1c)$$

$$r' = \Re(\theta_1, \theta_2, \theta_3) * r, \qquad \text{isorotations} \qquad (7.3.1d)$$

$$r' = \hat{\pi} * r = -r, \quad t' = \hat{\tau}*t = -t, \qquad \text{isoinversions} \qquad (7.3.1e)$$

where:

1) isotransforms (7.3.1d) are the isorotations of Ch. 11.6;

2) the isoinversions generator can be assumed as given by $\hat{\pi} = \pi\,\hat{1}$, $\hat{\tau} = \tau\,\hat{1}_t$, where π and τ are conventional inversion for space and time, respectively; and

3) the B's are nonlinear, nonlocal and noncanonical functions in all variables characterized by [6]

$$B_4(t_0)^{-2} = b_4^{-2} + \; + t_0 [\, b_4^{-2} \hat{,} H\,]\,/\,2! + t_0^2\,[\,[\,b_4^{-2} \hat{,} H\,]\hat{,} H\,]\,/\,3! + \ldots \qquad (7.3.2a)$$

$$B_k(r°)^{-2} = b_k^{-2} + r^°_j [\, b_k^{-2} \hat{,} P_j\,]\,/\,2! + r^°_m r^°_n [\,[\, b_k^{-2} \hat{,} P_m\,]\hat{,} P_n\,]\,/\,3! + \ldots \qquad (7.3.2b)$$

$$B_k(v°)^{-2} = b_k^{-2} + v^°_k [\, b_k^{-2} \hat{,} G_j\,]\,/\,2! + v^°_m v^°_n [\,[\, b_k^{-2} \hat{,} G_m\,]\hat{,} G_n\,] + \ldots \qquad (7.3.2c)$$

where the brackets are given by Eq.s (7.2.3).

It is readily seen that isotransforms (7.3.1) leave invariant the isoseparation (7.2.2) in isospace. Note, from the functional degrees of freedom of the B's, the nonlinearity and nonlocality of the isotransforms, *as desired*.

The *restricted isogalilean transforms* of Class I [6] occur when the characteristic b-functions are averaged into constants,

$$b°_4 = \text{Aver}\,\{\,b_4(t, r, \ldots)\,\} > 0, \qquad b°_k = \text{Aver}\,\{\,b_k(t, r, \ldots)\,\} > 0. \qquad (7.3.3)$$

In this latter case, isotransforms (7.3.1) evidently become *linear and local*, but still more general than the conventional ones.

Note finally that, being nonlinear, general isotransforms (7.3.1) are

noninertial, and characterize a class of frames equivalent to an actual frame in our Earthly environment which, as well known, is noninertial[79]. By comparison, the restricted transforms are indeed inertial as the conventional ones.

Moreover, we should recall that the full nonlinear, nonlocal and nonhamiltonian dependence of the characteristic b-functions is needed only for the local internal description, say, of one constituent at a given internal space-time point. By recalling that in closed nonhamiltonian systems each constituent is in an unstable, and thus noninertial conditions, the noninertial character of the general isogalilean symmetry then confirms the achievement of a broader composite structure.

On the contrary, the exterior global treatment of a closed nonhamiltonian system as a whole requires the restricted isotransforms, owing to the need of the global behaviour of the interior medium, in which case the characteristic b-functions are averaged into b°-constants.[80]

A simple illustration is given by the characterization of the speed of light in interior conditions which, as studied better in the next chapter, is given by $c = c_0 b_4 = c_0/n$, where c_0 is the speed in vacuum and $n = b_4^{-1}$ is the usual index of refraction. When the value of the speed of light is needed at each point, say, of a planetary atmosphere, we have a rather complex functional dependence of $n = b_4^{-1}$ requiring the general isogalilean transforms. On the contrary, in most cases it is sufficient to study the *average* speed of light throughout the entire atmosphere considered. In this latter case the speed is represented by $c_0 b_4^{\circ} = c_0/n^{\circ} = const < c_0$ and the restricted, linear-inertial isogalilean transforms are sufficient.

The reader should therefore keep in mind that *only restricted and therefore inertial isotransforms are generally applicable for the global treatment of a stable, hadronic, composite systems, that is, for exterior experimental measures, while the general noninertial isotransforms are needed for the study of noninertial conditions of the individual constituents of the system.*

In summary, the fundamental transforms assumed in these volumes for the *exterior* nonrelativistic dynamical problem are the Galilean transforms

[79] We should recall that, in the final analysis, *inertial reference frames are a philosophical abstraction because they do not exist in our Earthly physical reality, nor are they attainable in our Solar or Galactic systems* .

[80] As elaborated in ref. [6], the assumption of an *inertial* framework for the *individual* constituents would evidently restrict the applicable symmetries and relativities to be *linear* , thus leading to a conventional Keplerian system with *stable* individual orbits without advances. On the contrary, a necessary condition for the achievement of generalized composite systems is the assumption *ab initio* of a *noninertial* setting for the *individual* constituents which then permits the identification of *nonlinear* symmetries for *unstable* individual orbits, all this in a way fully compatible with a stable total system, conventional total conservation laws, and inertial exterior-global treatment.

$$t' = t + t°, \qquad r' = \Re(\theta_1, \theta_2, \theta_3) \, r + t°v° + r°. \tag{7.3.4}$$

while those assumed for the *interior* nonrelativistic dynamical problem are the covering isogalilean transforms

$$t' = t + t° B_4^{-2}, \quad r' = \Re(\theta_1, \theta_2, \theta_3) * r + t°v'B(v°)^{-2} + r°B(r°)^{-2}. \tag{7.3.5}$$

From the viewpoint of the Lie–isotopic group $\hat{G}(3.1)$, the latter can be represented via the element $\hat{g}(\Re, t°, v°, r°)$. The isounit is then given by $\hat{g}(\hat{1}, 0, 0, 0)$ and its isotopic action is given by $r' = \hat{g} * r := \hat{g} \, T \, r, \hat{1} = T^{-1}$. The inverse element is given by $\hat{g}^{-1}(\Re^{-1}, - t°, - \Re* \, v°, - \Re*r° + t°\Re*v°)$ (see ref. [2], pp. 256–257 for the conventional case). The isocomposition of two isogalilean transforms follows the rule

$$\hat{g}_1(\Re_1, t°_1, v°_1, r°_1) * \hat{g}_2(\Re_2, t°_2, v°_2, r°_2) =$$

$$= \hat{g}_3(\Re_2 * \Re_1, t°_1 + t°_2, v°_2 + \Re_2* v°_1, r°_2 + \Re_2 * r°_1 + v°_2 \, t°\hat{}_1). \tag{7.3.6}$$

The *isodual isogalilean relativity* is the antiautomorphic image of the isorelativity under the time and space conjugations

$$\hat{1}_t \rightarrow \hat{1}_t^d = - \hat{1}_t, \quad \hat{1} \rightarrow \hat{1}^d = - \hat{1}. \tag{7.3.7}$$

The basic numbers are therefore characterized by the *isodual field for time* $\hat{R}_t^d(\hat{1}_t^d,+,*_t)$, where the *isodual isotime* is $\hat{1}^d = t\hat{1}_t^d = -\hat{t}$ and the *isodual field for space* $\hat{R}^d(\hat{n}^d,+,*)$, with *isodual space numbers* $\hat{n}^d = n\hat{1}^d$. Their most salient features are the isodual isoproducts

$$\hat{1}_1^d *_t \hat{1}_2^d = \hat{1}_t^d \, T_t^d \hat{1}_2^d = (t_1 \, t_2) \hat{1}_t^d, \tag{7.3.8a}$$

$$\hat{n}_1^d *^d \hat{n}^d_2 = \hat{n}_1^d T^d \hat{n}^d = (n_1 \, n_2) \hat{1}^d, \tag{7.3.8b}$$

and the isodual isonorms (Ch. 1.2)

$$|\hat{1}^d| \hat{1}^d = -|\hat{1}| < 0, \qquad |\hat{n}^d| \hat{1}^d = -|\hat{n}| < 0, \tag{7.3.9}$$

which are *negative-definite*.

The only possible directions of time are therefore those *backward from future time*, and *backward in past time*. Also, the magnitude of the energy, linear momentum, angular momentum and other physical quantities is *negative*.

The carrier spaces are the *isodual isoeuclidean spaces*

$$\hat{E}^d(t,\hat{R}_t^d): \ \hat{1}_t^d = (T_t^d)^{-1} = - b_4^{-2}, \qquad b_4(t, r, \dot{r}, \ddot{r}, \dots) < 0, \tag{7.3.10a}$$

$$\hat{E}^d(r,\delta^d,\hat{R}^d): \ \delta^d = T^d \delta, \ T^d = - \text{diag.} \, (b_1^2, b_2^2, b_3^2), \ b_k = b_k(t, r, \dot{r}, \ddot{r} \dots) < 0, \tag{7.3.10b}$$

$$r^{2d} = (- x \, b_1^2 x - y \, b_2^2 y - z \, b_3^2 z) \hat{1}^d \in \hat{R}^d(\hat{n}^d,+,*^d), \ \hat{1}^d = (T^d)^{-1}. \tag{7.3.10c}$$

The *isodual isogalilean symmetry* $\hat{G}^d(3.1)$ is the antiautomorphic image of $\hat{G}(3.1)$ under the conjugations here considered; 17 results to be largest possible invariance group of the separations

$$t_a - t_b = \text{inv.}, \tag{7.3.11a}$$

$$[(r^i_a - r^i_b)\, \delta_{ij}{}^d(t, r, p,...)\, (r^j_a - r^j_b)] \uparrow^d = \text{inv.} \quad \text{at} \quad t_a = t_b; \tag{7.3.11b}$$

and it is characterized by negative-definite generators, parameters and isotopic elements.

The *isodual isogalilean transformations* can then be written

$$t' = t + t° B_4^{-2}, \qquad\qquad \text{isodual isotime translations} \quad (7.3.12a)$$

$$r^{k\,\prime} = r^k + r^{ok} B_k(r°)^{-2}, \qquad \text{isodual isospace translations} \quad (7.3.12b)$$

$$r^{k\,\prime} = r^k - t° v^{oi} B_k(v°)^{-2}, \qquad \text{isodual isogalilean boosts} \quad (7.3.12c)$$

$$r' = \hat{\Re}^d(\theta_1, \theta_2, \theta_3) *^d r, \qquad\qquad \text{isodual isorotations} \quad (7.3.1d)$$

$$r' = \hat{\pi}^d *^d\, r = -r, \quad t' = \hat{\tau}^d *_t{}^d t = -t, \text{ isodual isoinversions} \quad (7.3.12e)$$

and, as such, they formally coincide with isotransforms (7.3.1).

Note the *independence of isoduality from the space and time inversions and isoinversions*, which is made clearer by the following invariance properties

$$r' = \hat{\pi} * r \equiv \hat{\pi}^d *^d r = -r, \quad t' = \hat{\tau} *_t t = \hat{\tau}^d *_t{}^d t = -t, \tag{7.3.14}$$

In fact, the inversions (isoinversions) reverse the sign of space and time coordinates while preserving the conventional unit +1 (isounits $\hat{1}$). On the contrary, the isodualities preserve the sign of space and time coordinates, while changing the sign of the basic units.

Such a distinction appears even stronger at the operator level because, while the inversions and isoinversions preserve their Hilbert space, isodualities imply the passage to the isodual Hilbert spaces (Sect. 1.6.3).

In Vol. II, ref. [6], p. 60, we proved the following

Proposition 7.3.1: *All systems which are invariant under the isogalilean symmetry are isodual isoinvariant.*

The term "isodual isoinvariant" refer to the construction of the isodual systems which result to be invariant under the isodual isogalilean symmetry. The above property permit the following new:

Universal invariance law under isoduality: *All physical laws which are Galilean or isogalilean invariant are also invariant under the isodual*

symmetry.

The above new invariance is the result of the *invariance of the basic separation under isoduality,* $r^{2d} \equiv r^2$. As a result, any given separation can equivalently be interpreted as either belonging to an isoeuclidean space or to its isodual.

For brevity we refer the interested reader to Vol. II, ref. [6] for the necessary and sufficient conditions for the invariance of a systems under the isogalilean symmetry and its isodual and other related topics.

7.4: OPERATOR GALILEI-ISOTOPIC RELATIVITY AND ITS ISODUAL

The operator formulation of the isogalilean symmetry was submitted, apparently for the first time, in memoirs [7] and then studied in more details in ref. [8]. Consider the (ordered set of) conventional generators of the quantum mechanical Galilei algebra $g(3.1)$,

$$X = \{ X_k \} = \{ J = \sum_a r_a \wedge p_a, P, \ G = \sum_a (m\, r_a - t\, p_a), H \} \qquad (7.4.1)$$

(where the subscript "tot" has been dropped for simplicity) although now considered in the isoenveloping operator algebra ξ and referred to the isohilbert space $\mathcal{H}$ over an isofield $\hat{C}(\hat{c},+,*)$ of isocomplex numbers. Assume all isostructures to be of Class 1 and have a common isotopic element $T > 0$ independent from the local coordinates r to avoid gravitational aspects (see the next chapters for the inclusion of local coordinates).

Then, the now familiar isotopic properties[81]

$$p_k * | \hat{\Phi} > = -i\, \hat{1}_k{}^i \nabla_i | \hat{\Phi} > = -i\, b_k^{-2} \nabla_k | \hat{\Phi} >, \quad k=1,2,3, \quad \hbar = 1. \qquad (7.4.2a)$$

$$[r^i \,\hat{,}\, r^j] * | \hat{\Phi} > = [p_i \,\hat{,}\, p_j] * | \hat{\Phi} > = 0, \ [r^i, p_j] * | \hat{\Phi} > = i\, b_i^{-2} \delta^i{}_j * | \hat{\Phi} >. \qquad (7.4.2b)$$

$$[r_i \,\hat{,}\, r_j] * | \hat{\Phi} > = [p_i \,\hat{,}\, p_j] * | \hat{\Phi} > = 0, \ [r_i, p_j] * | \hat{\Phi} > = i\, \delta^i{}_j * | \hat{\Phi} >. \qquad (7.4.2c)$$

imply the *operator isogalilean algebras* [7,8][82]

---

[81] One should keep in mind the *isotopic differential rule* tacitly assumed in this section

$$[A * B \,\hat{,}\, C] = A * [B \,\hat{,}\, C] + [A \,\hat{,}\, C] * B \qquad (a)$$

as studied in Sect. I.4.4.

[82] If one uses the original local coordinates r^k without their contraction to the covariant form $r_k = b_k k$, then the isogalilean algebra assumes the form

$$[J_i \,\hat{,}\, J_j] = -i\, \epsilon_{ijk}\, b_i^{-2} J^k, \ [J_i \,\hat{,}\, P_j] = -i\, \epsilon_{ijk}\, b_i^{-2}\, P_k,$$

$$[J^i \hat{,} J^j] = - i \, \epsilon^{ijk} J^k, \quad [J^i \hat{,} P_j] = - i \, \epsilon^{ijk} P_k, \tag{7.4.3a}$$

$$[J_i \hat{,} G_j] = - i \, \epsilon_{ijk} \, b_j^{-2} G_k, \quad [J_i \hat{,} H] = 0, \tag{7.4.3b}$$

$$[G_i \hat{,} P_j] = \delta_{ij} \, b_j^{-2} M, \quad [G_i \hat{,} H] = 0, \tag{7.4.3c}$$

$$[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} H] = 0, \quad i, j = 1, 2, 3, \tag{7.4.3d}$$

whose algebraic equivalence to rules (7.2.7) is evident. This establishes that the operator isogalilean algebra $\hat{g}(3.1)$ here considered is the unique and unambiguous operator version of the classical version of Sect. II.7.2.

The *operator isocasimirs* of $\hat{g}(3.1)$ are given by[83]

$$\hat{C}^{(o)} = \hat{1}, \qquad \hat{C}^{(1)} = P^2 - MH, \qquad \hat{C}^{(2)} = (MJ - G \wedge P)^2. \tag{7.4.4}$$

The construction of the isoexponentiation of rules (7.4.3) into the operator *isogalilean group* $\hat{G}(3.1)$ is consequential, resulting in the isounitary isotransforms[84]

$$\hat{U}(\hat{w}) = \hat{e}^{\, i X_k * \hat{w}_k} = \{ e^{\, i X_k \, T \, w_k} \} \hat{1}. \tag{7.4.5}$$

The explicit form of the isogalilean transforms remains the classical ones.

An inspection of the structure of isocasimirs (7.4.4) illustrates the generalized notion of systems characterized by the isogalilean symmetries as studied in more details in Sect. II.7.8.

The *operator isodual isogalilean symmetry* $\hat{G}^d(3.1)$ is also characterized by the isodualities $\hat{1}_t^d = - \hat{1}_t$, $\hat{1} = - \hat{1}$, but now referred to the isodual Hilbert space with elements $< \hat{\psi} |$ and isodual isoinner product $< \hat{\psi} | T^d | \hat{\phi} > \hat{1}^d \in \hat{C}^d(\hat{c}^d, +, *^d)$.

$$[J_i \hat{,} G_j] = - i \, \epsilon_{ijk} \, b_j^{-2} G_k, \quad [J_i \hat{,} H] = 0,$$

$$[G_i \hat{,} P_j] = \delta_{ij} \, b_j^{-2} M, \quad [G_i \hat{,} H] = 0,$$

$$[P_i \hat{,} P_j] = [G_i \hat{,} G_j] = [P_i \hat{,} H] = 0, \quad i, j = 1, 2, 3.$$

The local isomorphisms with $g(3.1)$ still holds because of the positive–definiteness of the b–terms. Equivalently, in the latter realization the original structure constants can be regaining via the redefinitions $J_1' = b_1 \, b_2 \, J_1$, $J_2' = b_1 \, b_3 \, J_2$, $J_3 = b_1 \, b_2 \, J_3$.

[83] Note that in the classical realization, only the functions multiplying the isounits are the isocasimir invariants, while in the operator case the entire structures are isocasimirs, including the isounits. This is evidently due to the fact that the isocommutation rules are computed among *functions* in the former case, and among *matrices/operators* in the latter case.

[84] The isounitary character is established by the isotopies of Wigner's theorem on quantum symmetries studied later on in Sect. II.7.6.

The fundamental isodual rules are given by

$$< \hat{\psi} | *^d p_k = i < \hat{\psi} | \nabla_i \hat{1}_k{}^i = i < \hat{\psi} | \nabla_k b_k^{-2}, \quad k=1, 2, 3, \quad \hbar = 1. \quad (7.4.6)$$

$$< \hat{\psi} | *^d [r^i \hat{,} r^j]^d = < \hat{\psi} | *^d [p_i \hat{,} p_j] = 0, < \hat{\psi} | *^d [r^i, p_j]^d = - i < \hat{\psi} | \delta^i{}_j b_i^{-2}, \quad (7.4.7b)$$

$$< \hat{\psi} | *^d [r_i \hat{,} r_j]^d = < \hat{\psi} | *^d [p_i \hat{,} p_j]^d = 0, \quad (7.4.7c)$$

$$< \hat{\psi} | *^d [r_i, p_j]^d = -i < \hat{\psi} | \delta^i{}_j . \quad (7.4.7d)$$

The *isodual isogalilean algebra* is then given by isoalgebra (7.3.7) with all signs reversed, while the *isodual isogalilean group* can be written in terms of the isodual isounitary isotransforms

$$\hat{U}^d(\hat{w}) = \hat{e}^{i^d X_k{}^d *^d \hat{w}_k{}^d} = \{ e^{i X_k{}^T w_k} \}\hat{1}^d = - \hat{U}(\hat{w}) . \quad (7.4.8)$$

The explicit form of the isodual transform s is then the same as the classical ones.

Recall that hadronic mechanics permits the identification of the hitherto unknown *isodual quantum mechanics*, which is the image of the conventional mechanics under the map if the units $+1 \rightarrow -1$ over the isodual Hilbert space $\mathcal{3C}^d$ with conventionally isodual elements $< \psi |^d = - (< |)^\dagger$ with isodual composition

$$\mathcal{3C}^d: \quad < \psi | \phi >^d = < \psi |^d (-1) | \phi >^d (-1)^{-1} \in C^d(c^d, +, *). \quad (7.4.9)$$

As one can see, the above isodual composition coincides with the conventional one, and this may be a reason for the lack of identification of the isodual quantum mechanics until now. Despite the above identity, the physical implications are nontrivial, as we shall see.

The fundamental symmetry for nonrelativistic isodual quantum mechanics is the *isodual Galilean symmetry* $G^d(3.1)$ which is a simple particular case of $\hat{G}^d(3.1)$.

We now study the isotopies and isodualities of the Galilean relativity. As well known, a most rigorous way of establishing the Galilei relativity in quantum mechanics is via *Mackey's imprimitivity theorem* [3] formulated within the context of a conventional enveloping associative operator algebra ξ and Hilbert space $\mathcal{3C}$ over a field of complex numbers $C(c,+,\times)$. In particular, this theorem also provides the largest possible linear, local and unitary invariance of Schrödinger's equation.

An important result of ref. [7] was the proof that Mackey's theorem admits consistent isotopic generalizations in enveloping isoassociative algebras $\hat{\xi}$ and isohilbert space $\hat{\mathcal{3C}}$ over the isocomplex field $\hat{C}(\hat{c},+,*)$ for the case of Class I

structures with common isotopic element T of Class I independent from the space and time coordinates[85]. This study established the consistency of the isogalilean relativity. In particular, the *isotopic imprimitivity theorem*, or *isoimprimitivity theorem* for short, characterizes the largest known invariance of the isoschrödinger's equation which is nonlinear, nonlocal and noncanonical when projected in the original space and isolinear, isolocal and isocanonical when formulated in isospace (App. II.4.C).

We shall present below the isoimprimitivity theorem for the simplest and most known class of irreducible isoreps of Lie–isotopic algebras, those of regular type. The extension to the more general class of irregular isoreps is left to the interested reader, while the reduction of the proof to the standard irreducible isoreps is trivial.

The basic properties which permit a virtual *verbatim*, line-by-line, isotopic lifting of Class I of Mackey's theorem are the following:

1) The abstract identity of ordinary fields $C(c,+,\times)$ and isofields $\hat{C}(\hat{c},+,*)$ as well as of algebras ξ over C, and isoalgebras $\hat{\xi}$ over $\hat{C}$,

$$C \approx \hat{C}, \qquad \xi \approx \hat{\xi}, \qquad\qquad (7.4.10)$$

As stressed in Vol. I, all distinctions are lost at the abstract level between the conventional unit I of quantum mechanics and the isounit $\hat{1}$ of hadronic mechanics, between the conventional product QP in ξ and the isotopic product $Q*P = QTP, \hat{1} = T^{-1}$ in $\hat{\xi}$, etc.

2) The abstract identity of the conventional and isohilbert spaces from the positive-definiteness of the isotopic element T,

$$\mathfrak{K}: \ <\phi|\psi> \ \in C(c,+,\times) \ \rightarrow \ \hat{\mathfrak{K}}: \ <\phi|T|\psi>\hat{1} \ \in \hat{C}(\hat{c},+,*). \qquad (7.4.11)$$

3) The abstract identity of the quantum and hadronic rules including the identity of: conventional and isotopic derivatives,[86] conventional Hermiticity and isohermiticity, etc.. In particular, the fundamental commutation rules and their isotopic coverings can be written in the unified notation for an abstract

---

[85] The results holds also for an explicit dependence on r and t following the construction of the isodifferential calculus for such a dependence done in the next chapter. The formulation of the imprimitivity theorem for the more general case in which the isohilbert space $\mathfrak{K}$ has an isotopic element G different than the element T of $\hat{\xi}$ and $\hat{C}(\hat{c},+,*)$ has not been investigate to date. Such an extension appears to be important for the possible reconstruction of the exact parity symmetry in isospace for weak interactions.

[86] Note that a considerable increase in complexity occurs for isounits with a dependence in the local coordinates when the isoderivatives are expressed in terms of the conventional form. However, if one remains at the abstract level and considers instead isoderivatives per se, the dependence on the local coordinates is considerably less complicated.

associative envelope ξ ($\hbar = 1$)

$$Q_i \times Q_j |_\xi - Q_j \times Q_i |_\xi = P_i \times P_j |_\xi - P_j \times P_i |_\xi = 0, \qquad (7.4.12a)$$

$$Q_i \times P_j |_\xi - P_j \times Q_i |_\xi = i\,I \quad . \qquad (7.4.12b)$$

Then, the conventional formalism of the imprimitivity theorem (see ref. [3], p. 180) holds for the simplest possible realization of the envelope ξ with product QP, as well as for the lesser trivial isotopic realization of $\hat\xi$ with isoproduct Q*P = QTP, $\hat 1 = T^{-1}$.

Once the above abstract identities are understood, the isotopies of the imprimitivity theorem is consequential.

In fact, one can prove the uniqueness of the following structure of the *isokinetic energy* up (to isoscalar terms) in $T^*\hat E(r,\delta,\hat R)$

$$\hat K = P^2 / 2{*}\hat m = \delta^{ij} P_i\, T\, P_j / 2\,m, \qquad (7.4.13)$$

the uniqueness of the structure of any admissible *isopotential* on $T^*\hat E(r,\delta,\hat R)$ (also up to isoscalar multiples)

$$\hat V = V(\,r_{ab}\,)\hat 1 \;, \quad r_{ab} = [\,(r^i_a - r^i_b)^t\, \delta_{ij}\,(r^j_a - r^j_b)\,]^{\frac{1}{2}}\;, \qquad (7.4.14)$$

and the uniqueness of the energy isoequation in $\hat E(t,\hat R_t)$

$$\hat H * \hat\psi = (\hat K + \hat V) * \hat\psi = [\,(\,1/2m\,)\,\delta^{ij} P_i\, T\, P_j + V\,\hat 1\,)\, T\,\hat\psi =$$

$$= \hat E\, T\,\hat\psi = E\,\psi, \qquad 7.4.15)$$

The isogalilean symmetry $\hat G(3.1)$ realized via isounitary operators (7.4.5)

$$\hat U * \hat H * \hat U^\dagger \equiv \hat H, \qquad (7.4.16)$$

then follows as the largest possible isosymmetry (see the next section on the isotopies of Wigner's theorem on unitary symmetries).

The nonlinearity, nonlocality and noncanonicity of the symmetries is established by the the integro–differential functional dependence of the isotopic element T appearing in the exponent of the isotopic structure (7.4.5).

The *operator isodual isogalilean relativity* and the operator *isodual Galilean relativity* can now be readily identified via the preceding techniques.

7.5: OPERATOR GALILEI-ADMISSIBLE RELATIVITY AND ITS ISODUAL

The reader familiar with the Lie–admissible generalization of Lie and Lie–isotopic methods can now construct the *Galilei–admissible symmetry* as a genotopy of the conventional symmetry G(3.1) or of the isotopic symmetry Ĝ(3.1).

The important steps are:

1) the differentiation between multiplication to the right Q>P, from multiplication to the left Q<P, their association with motion forward to future time and forward from past time, respectively, under the needed continuity, boundedness and regularity conditions,

$$Q > P = Q R P \neq Q < P = Q S P, \quad R, S, R \pm S \text{ nonsingular}, \quad (7.5.1)$$

2) the identification of corresponding right and left genounits and related conjugation here assumed to be ordinary Hermiticity,

$$1^> = R^{-1}, \quad {}^<1 = S^{-1}, \quad 1^> = ({}^<1)^\dagger, \quad (7.5.2b)$$

3) the assumption of genoreals and genocomplex numbers of Ch. 1.7 at the foundation of the theory, here written in the unified notation

$$<\hat{F}>(<\hat{a}>,+,<>), \quad (7.5.3)$$

and related *genoeuclidean spaces*

$$<\hat{E}>(t,<\hat{R}_t>): \quad {}^<1_t{}^> = (<T_t>)^{-1} = {}^<b_4{}^{>-2}, \quad (7.5.4a)$$

$$<\hat{E}>(r,<\hat{\delta}>,<\hat{R}>): \quad {}^<\delta^> = {}^<T^>\delta, \quad {}^<T^> = \text{diag.} ({}^<b^>_1{}^2, {}^<b^>_2{}^2, {}^<b^>_3{}^2), \quad (7.5.4b)$$

$$r^{<2>} = (x {}^<b^>_1{}^2 x - y {}^<b^>_2{}^2 y - z {}^<b^>_3{}^2 z) {}^<1^> \in <\hat{R}>(<\hat{n}>,+,<>), \quad (7.5.4c)$$

where only one direction of time can be selected and the characteristic b-quantities can now be complex.

The *Galilei–admissible genosymmetry* <Ĝ>(3.1) is the symmetry of genoinvariant (7.5.4c), it can then be uniquely constructed as a consequence of the above basic assumptions, and it is given by the following Lie–admissible group of genounitary transforms of a physical quantity Q

$$Q' = \hat{U}^>(w) > Q < {}^<\hat{W}^\dagger(w), \quad (7.5.5a)$$

$$\hat{U}^>(w) = \hat{e}^{\,i\,X_k > w^>_k} = \{e^{\,i\,X_k\,R\,w_k}\}1, \quad (7.5.5b)$$

$$<\hat{W}^<(w) = \hat{e}^{\,i\,{}^<w_k\,<{}^<w_k} = {}^<1\{e^{\,i\,w_k\,S\,X_k}\}, \quad (7.5.5c)$$

$$\hat{U}^> > \hat{U}^\dagger = \hat{U}^{>\dagger} > \hat{U} = 1^>, \quad {}^<\hat{W} < {}^<\hat{W}^\dagger = {}^<\hat{W}^\dagger < {}^<\hat{W} = {}^<1. \quad (7.5.5d)$$

We therefore have *two Galilei-admissible transforms,* one for motion forward to future time

$t' = t + t° B^>_4{}^{-2}$,	genotime translations	(7.5.6a)
$r^{k\,\prime} = r^k + r^{°k} B^>_k(r°)^{-2}$,	genospace translations	(7.5.6b)
$r^{k\,\prime} = r^k + t° v^{°i} B^>_k(v°)^{-2}$,	genogalinean boosts	(7.5.6c)
$r' = \Re^>(\theta_1, \theta_2, \theta_3) > r$,	genorotations	(7.5.6d)
$r' = \hat{\pi}^> > r = -r,\ t' = \hat{\tau}^> >_t t = -t$,	genoinversions	(7.5.6e)

and the other for motion forward from past time

$t' = t + t° {}^<B_4{}^{-2}$,	genotime translations	(7.5.7a)
$r^{k\,\prime} = r^k + r^{°k} {}^<B_k(r°)^{-2}$,	genospace translations	(7.5.7b)
$r^{k\,\prime} = r^k + t° v^{°i} {}^<B_k(v°)^{-2}$,	genogalinean boosts	(7.5.7c)
$r' = r < {}^<\Re(\theta_1, \theta_2, \theta_3)$,	genorotations	(7.5.7d)
$r' = r < {}^<\hat{\pi} = -r,\ t' = t <_t {}^<\hat{\tau}$,	genoinversions	(7.5.7e)

Lemma 7.5.1: *The Galilei-admissible group* ${}^<\hat{G}{}^>(3.1)$, *when written in its genospace, is locally isomorphic to the conventional Galilei group.*

Proof. The Galilei-admissible group ${}^<\hat{G}{}^>(3.1)$ is composed of two different Galilei-isotopic groups, one for action to the right $\hat{G}{}^>(3.1)$ and one for action to the left ${}^<\hat{G}(3.1)$, which are individually isomorphic to the conventional Galilei group. We therefore have the chain of the isomorphisms from transforms (7.5.5a)

$$\hat{G}{}^>(3.1) \approx G(3.1), \quad {}^<\hat{G}(3.1) \approx G(3.1), \quad {}^<\hat{G}{}^>(3.1) \approx G(3.1), \quad (7.5.8)$$

and the property of the lemma follows. **q.e.d.**

Lemma 7.5.1 also follows from the simple observation that the *genosphere* (7.5.4c), even though with a generally complex structure, coincides at the abstract level with the conventional sphere in each direction of time. In fact, in correspondence of the deformations of the semiaxes $1_k \rightarrow {}^<b^>_k{}^2$ we have the deformation of the corresponding component of the unit in the inverse amount $1_k \rightarrow {}^<b^>_k{}^{-2}$, thus preserving the original geometric structure.

The *Galilei-admissible algebra* ${}^<\hat{g}{}^>(3.1)$ can be characterized by the infinitesimal version of the Lie-admissible group (7.5.5a)

$$i\, dQ/dw_k = Q < X_k{}_{|\hat{\xi}} - X_k > Q_{|\xi>} = Q\, R\, X_k{}_{|\hat{\triangleleft}=R^{-1}} - X_k\, S\, Q_{|1>=S^{-1}}, \quad (7.5.9)$$

where, as indicated, each genoassociative product is computed in its own

envelope, that is, with respect to its own unit.

The Galilei–admissible algebra therefore admits the conventional, Galilean structure constants when computed in its appropriate genospace. In fact, the product of law (7.5.9) results to be different than the Lie product only when written in the original Lie space, that is, when both genomultiplications to the right and to the left are referred to the same unit I.

The *Galilei–admissible relativity* is a form–invariant description of systems under the genosymmetry $\hat{G}$(3.1). Unlike the conventional Galilei and Galilei–isotopic case, the primary objective of the Galilei–admissible relativity is the characterization of the *time–rate–of–variations of physical quantities, such as that of the energy*

$$H(t) = \hat{U}^>(t) > H(0) < ^<\hat{W}(t) = i H (R - S) H \neq 0, \qquad (7.5.10)$$

of which the conventional conservation is an evident particular case (see Vol. II, ref. [6] for the classical profile, and the subsequent studies for the operator counterpart).

The *isodual Galilei–admissible symmetry and relativities* are the image of the preceding ones under isodualities

$$^<\hat{1}_t^{>d} = - ^<\hat{1}_t^>, \qquad\qquad ^<\hat{1}^{>d} = - ^<\hat{1}^>. \qquad (7.5.11)$$

Their explicit construction is an instructive exercise for the interested reader.

To indicate the pohysical relevance of the Galilei–admissible relativity, we first introduce the following

Lemma 7.5.1 (Axiomatization of time arrows): *The Galilei–admissible relativity and its isodual provide an axiomatic, form–invariant characterization of all four possible time arrows of open–nonconservative processes.*

Consider a conventional conservative system under the conventional Galilei symmetry. A fundamental property is that *the underlying field characterizing time is unique*, as well known and given by the familiar structure $R(t,+,\times)$. Time-reversal is indeed admitted, but always preserving the base fields unchanged. In the transition to conventional treatments of dissipative systems (see, e.g., ref.s [11–13]) this fundamental property remains unchanged.

The characterization of open–nonconservative systems via the Galilei–admissible relativity implies a structural generalization in the characterization of time. In fact, it admits *four different generalization of the time field* $R(t,+,\times)$ *given by* $\hat{R}^>(t^>,+,>)$, $^<\hat{R}(^<t,+,<)$, $\hat{R}^{>d}(t^{>d},+,>^d)$ *and* $^{d<}\hat{R}(^{d<}t,+,^d<)$ *which characterize all four possible time arrows* (see Fig. 7.5.1 for details).

LIE-ADMISSIBLE AXIOMATIZATION OF THE FOUR TIME ARROWS

$$^<\hat{R}(^<\hat{t},+,<) \qquad\qquad \hat{R}^>(\hat{t}^>,+,>)$$

$$^{d<}\hat{R}(^{d<}\hat{t},+,d<) \qquad 0 \qquad \hat{R}^{>d}(\hat{t}^{>d},+,>d) \qquad \text{time}$$

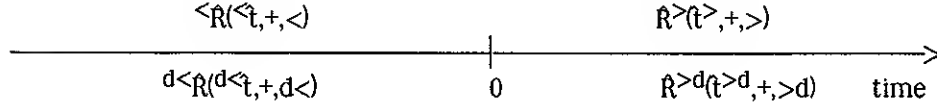FIGURE 7.5.1: A schematic view of the axiomatization of time via genofields introduced by the author in ref. [14]: the genofield $\hat{R}^>(\hat{t}^>,+,>)$ with genotime $\hat{t}^> = \hat{t}1^>$, where t is the ordinary time, and genounit $\hat{1}_t^> = (T_t^>)^{-1} \ne \hat{1}_t^>\hat{t}$ for *motion forward to future time* >; the genofield $^<\hat{R}(^<\hat{t},+,<)$ with genotime $^<\hat{t} = {}^<\hat{t}t$ and genounit $^<\hat{1}_t = (\hat{1}_t^>)^\dagger$ for *motion forward from past time* <; the isodual genofield $\hat{R}^{>d}(\hat{t}^{>d},+,>d)$ which is the image of $\hat{R}^>(\hat{t}^>,+,>)$ under isoduality for *motion backward from future time* $>^d = ->$; and the isodual genofield $^{d<}\hat{R}(^{d<}\hat{t},+,d<)$ which is the isodual image of $^<\hat{R}(^<\hat{t},+,<)$ for *motion backward in past time* $^d< = -<$. As one can see, the main element of the characterization is the generalization of the trivial unit +1 of the current description of time into a (nonsingular and) *nonhermitean* quantity $\hat{1}$. The conjugation under Hermiticity then allows the transition from motion forward to the future to motion forward from past time, while the conjugation under isoduality permits the representation of the remaining two directions. Note the fundamental point here: time t remains the *conventional real time* as ordinarily measured, and only its *unit* is generalized into a nonhermitean quantity. The simplest and most effective realization of the above notion of genotime is that by Jannussis [15] in which the genounit is a *complex quantity* $\hat{1} = n + im$, n, m $\in$ R(n,+,×). We therefore call *Jannussis complex times* the following four quantities $\hat{t}^> = \hat{t}1^> = t(n + im)$, $^<\hat{t} = {}^<\hat{t}t = (n - im)t$, $\hat{t}^{>d} = t(-n - im)$, $^d\hat{t} = {}^d\hat{t}t = (-n + im)t$, where, again, t is the ordinary real time. Intriguingly, Jannussis' notion of complex time is a quantitative mathematical representation of the notion of absolute time in ancient Greek philosophy.

The above axiomatic characterization of time then implies the following

Lemma 7.5.3 (Origin of irreversibility): *The Galilei-admissible relativity and its isodual, when restricted to the conjugation* R = R†, *provide an axiomatic, form-invariant reduction of macroscopic irreversibility to the ultimate level of the structure of matter, that of elementary particles in open-nonconservative conditions.*[87]

A comparison with the efforts by Prigogine [13] Hawking, Laflamme, Lyons

---

[87] The condition R = S† is necessary to have a consistent conjugation from motion forward to future time to that forward from past time, $\hat{t}^> = (^<\hat{t})^\dagger$ (see Fig. 7.5.1).

[16] and others (see, e.g., ref. [17]) at the reconciliation of macroscopic irreversibility with elementary systems is instructive. In these studies quantum mechanics is assumed as being *exactly valid* at the ultimate elementary level of matter, thus leaving the problem of the origin of irreversibility essentially unsettled.

On the contrary, we assume a structural generalization of quantum mechanics for the characterization of elementary particles in interior, open-irreversible conditions and a corresponding fully equivalent classical image [6] interconnected by genoquantization (Sect. II.4. In particular, the operator and classical Lie–admissible mechanics are *intrinsically irreversible,* that is, they are irreversible even for reversible Hamiltonians H. it then follows that *irreversibility originates at the ultimate level of the structure of matter, that of elementary particles in open–interior conditions. Classical macroscopic irreversibility is a mere aggregate of such irreducible elementary constituents under the same abstract axioms.*

The achievement of the above quantitative, form–invariant, axiomatization of the origin of irreversibility was one of the central reasons which motivated the construction of hadronic mechanics. For early studies on conventional Hilbert spaces and fields, see ref. [18]. The axiomatization on isospaces over isofields has been introduced in this section apparently for the first time. A Lie–admissible reformulation of conventional irreversible statistics is presented in Sect. II.7.9.

7.6: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF WIGNER'S THEOREM ON QUANTUM SYMMETRIES

After having identified the nonrelativistic dynamical symmetry, the next logical step is to identify the conditions for invariance in hadronic mechanics. Among the many different possibilities, we select the isotopies, genotopies and isodualities of Wigner's theorem on quantum symmetry as presented, e.g., in ref. [19], Sect. 5.1. The isotopies of Wigner's theorem were studied in ref. [20] of 1983. No additional reference is on record in the field at this time (early 1994) to our best knowledge.

As it is well known, an important notion in the statistical interpretation of quantum mechanics is the *transition probability*

$$\mathscr{P}(\psi, \phi) = \mathscr{P}(\phi, \psi) = |<\psi|\phi>|^2, \tag{7.6.1}$$

between two normalized vectors ψ and ϕ in a Hilbert space $\in \mathcal{H}$ with inner product $<\psi|\phi>$ over the complex field $C(c,+,\times)$. In order to reach a deeper connection with quantum symmetries, the states ψ and ϕ are interpreted as *unit rays* in $\mathcal{H}$, that is as an equivalence class of vectors of unit length in $\mathcal{H}$. The

correspondence between unit rays and quantum states is then no longer one–to–one because of unimodular phase factors exp(ia), where α is real, and/or because of superselection rules.

A *quantum mechanical symmetry* G [loc. cit] is a map from the set of unit rays in ∈ 3C onto the set of unit rays on a second Hilbert space 3C' (not necessarily different than 3C) which: preserves the transition probability

$$\mathscr{P}(g\psi, g\phi) \equiv \mathscr{P}(\psi, \phi), \quad g \in G, \tag{7.6.2}$$

where ψ and φ are unit rays in 3C and ψ' = Gψ, φ' = Gφ are unit rays in 3C'; the map is defined for every unit rays in 3C; and the map between unit rays is one–to–one and onto.

Theorem 7.6.1 (Wigner theorem on quantum symmetries [loc. cit.]):
Let G be a symmetry mapping unit rays in a Hilbert space 3C onto unit rays in a second Hilbert space 3C'. Then, the mapping of unit rays can be replaced by a mapping of vector which is either unitary or antiunitary, that is, there exists an onto mapping U: 3C → 3C', given by ψ ∈ 3C → ψ' = Uψ ∈ 3C', for each ψ ∈ 3C, such that

$$U(\psi + \phi) = U\psi + U\phi, \quad U(a\psi) = aU\psi \quad or \quad = \overline{a}U\psi, \tag{7.6.3a}$$

$$<U\psi | U\phi> = <\psi | \phi> \quad or \quad = \overline{<\psi | \phi>}. \tag{7.6.3b}$$

Consider now the Lie–isotopic branch of hadronic mechanics of Class I with all isotopic quantities characterized by the same element T > 0. From the studies of preceding sections it is evident that each quantum notion admits a consistent and unique isotopic generalization which is such to coincide with the quantum formulation at the abstract, realization–free level. This is evidently the same for the context of Wigner's theorem.

Let $\hat{3C}$ be a isohilbert space with isoinner product $< \hat{\psi} \hat{|} \hat{\phi} > = < \hat{\psi} | T | \hat{\phi} > \hat{1} \in \hat{C}(\hat{c},+,*)$. The quantities $\hat{\psi}$ and $\hat{\phi}$ can then be reinterpreted as *isounit rays* in $\hat{3C}$ evidently referred to $\hat{1}$. Since $\hat{1}$ is positive–definite as the ordinary unit +1, it is clear that Hilbert and isohilbert spaces as well as unit rays and isounit rays coincide at the abstract, realization–free level.

Definition 7.6.1 [20]: *Let $\hat{3C}$ and $\hat{3C}'$ be two (not necessarily different) isohilbert spaces of Class I with a unique isotopic element T. Then a "hadronic symmetry" $\hat{G}$ is a map from the set of isounit rays in $\hat{3C}$ onto the set of isounit rays into $\hat{3C}'$ which:*
1) preserves the transition isoprobability

$$\hat{\mathscr{P}}(\hat{g} * \hat{\psi}, \hat{g} * \hat{\phi}) \equiv \hat{\mathscr{P}}(\hat{\psi}, \hat{\phi}) = |<\hat{\psi}\hat{|}\hat{\phi}>|^2 = |<\hat{\psi}\hat{|}\hat{\phi}>|^2\hat{1}, \quad \hat{g} \in \hat{G}, \tag{7.6.4}$$

where $\hat{\psi}$ and $\hat{\phi}$ are isounit rays in $\mathfrak{H}$ and $\hat{\psi}' = \hat{g}*\hat{\psi}$, $\hat{\phi}' = \hat{g}*\hat{\phi}$ are isounit rays in $\mathfrak{H}'$;

2) is defined for every isounit ray in $\mathfrak{H}$; and

3) the mapping between isounit rays is one-to-one and onto.

Recall that an operator $\hat{U}$ is *isounitary* when it verifies the properties $\hat{U}*\hat{U}^\dagger = \hat{U}^\dagger*U = \hat{1}$. An operator $\hat{\mathcal{E}}$ is *anti-isounitary* when it has the structure $\hat{\mathcal{E}} = \hat{U}*\hat{C} = \hat{U}C$, where $\hat{U}$ is unitary and C is the operation of complex conjugation (the latter one occurs, e.g., for time isoinversions). The step-by-step repetition of the proof of Winger's theorem (ref. [9], Sect. 5.2) then leads to the following:

Theorem 7.6.2 (Operator isosymmetries [20]): *Let $\hat{G}$ be a Class I isosymmetry of a system of hadronic particles over the isofield of isocomplex numbers $\hat{C}(\hat{c},+,*)$ characterized by the same isotopic element T, which maps isounit rays in an isohilbert space $\mathfrak{H}$ onto isounit rays in a second (not necessarily different) isohilbert space $\mathfrak{H}'$. Then, the map of isounit rays can be replaced by a map of vectors which is either isounitary or antiisounitary, that is, there exists an onto map $\hat{U}: \mathfrak{H} \to \mathfrak{H}'$, given by $\hat{\psi} \in \mathfrak{H} \to \hat{\psi}' = \hat{U}*\psi \in \mathfrak{H}'$, for each $\hat{\psi} \in \mathfrak{H}$, such that for $\hat{a} \in \hat{C}(\hat{c},+,*)$*

$$\hat{U}*(\hat{\psi}+\hat{\phi}) = \hat{U}*\psi + \hat{U}*\phi, \quad \hat{U}*(\hat{a}*\psi) = \hat{a}*\hat{U}*\psi \quad \text{or} \ = \bar{\hat{a}}*\hat{U}*\psi, \quad (7.6.5a)$$

$$<\hat{U}*\hat{\psi}|\hat{U}*\hat{\phi}> = <\hat{\psi}|\hat{\phi}> \quad \text{or} \ = \overline{<\hat{\psi}|\hat{\phi}>}. \quad (7.6.5b)$$

Note the restriction of the formulation on an isofield of isocomplex numbers. This is due to the unifying power of isofields (Theorem I.2.7.1) according to which the absence of such condition would imply the formulation of the theorem over an isofield of isoquaternions.

Note also the restriction of the formulation to an isohilbert space with the same isotopic element of isoalgebras and isofields. This condition has been introduced for the automatic preservation of Hermiticity under isotopies (Sect. I.6.3).

The generalization of Theorem 7.6.2 to arbitrary fields and to different isotopic elements for the Hilbert space and the enveloping algebra is intriguing, but it is left to the interested reader for brevity.

We consider now the branch of hadronioc mechanics of Class II for antiparticles, also characterized by the same isotopic element T^d. Note that an operator $\hat{U}^d$ is *isodual isounitary* when it verifies the conditions $\hat{U}^d*^d\hat{U}^{d\dagger} = \hat{U}^{d\dagger}*^d\hat{U}^d = \hat{1}^d = -\hat{1}$. An operator $\hat{\mathcal{E}}^d$ is *isodual antiisounitary* when it admits the decomposition $\hat{\mathcal{E}}^d = \hat{U}^d*^d\hat{C}^d = \hat{U}^dC = -\hat{U}C$ where $\hat{U}^d$ ($\hat{U}$) is an isodual isounitary (isounitary) operator and C is the operation of complex conjugation. The repetition of the same iterim of Theorem 7.6.2 then leads to the following

Theorem 7.6.3 (Operator isodual isosymmetries): *Let* $\hat{G}^d$ *be a Class II isodual isosymmetry of a system of hadronic antiparticles over the isodual isofield of isocomplex numbers* $\hat{C}^d(\hat{c}^d,+,*^d)$ *characterized by the same isotopic element* T^d, *which maps isodual isounit rays in an isodual isohilbert space* $\mathcal{JC}^d$ *onto isodual isounit rays in a second (not necessarily different) isodual isohilbert space* $\mathcal{JC}^{d'}$. *Then, the map of isodual isounit rays can be replaced by a map of isodual vectors which is either isodual isounitary or isodual antiisounitary, that is, there exists an onto map* 0^d: $\mathcal{JC}^d \to \mathcal{JC}^{d'}$ *given by* $\hat{\psi}^d \in \mathcal{JC}^d \to \psi^{d'} = \hat{\psi}^d *^d 0^d \in \mathcal{JC}$; *for each* $\hat{\psi}^d \in \mathcal{JC}^d$, *such that*

$$(\hat{\psi}^d + \phi^d) *^d 0^d = \hat{\psi}^d *^d 0^d + \hat{\phi}^d *^d 0 , \tag{7.5.6a}$$

$$(\hat{\psi}^d *^d \hat{a}^d) *^d 0^d = \hat{\psi}^d *^d 0^d *^d \hat{a}^d \qquad or = \hat{\psi}^d *^d 0^d *^d \overline{\hat{a}}^d , \tag{7.5.6b}$$

$$< \hat{\phi}^d *^d 0^d | \hat{\psi}^d *^d 0^d >^d = < \hat{\phi}^d | \hat{\psi}^d >^d \qquad or = \overline{< \hat{\phi} | \hat{\psi} >}^d . \tag{7.5.6c}$$

The genotopies of Winger's theorem and their isoduals are left for brevity as an exercise for the interested reader.

7.7: NONRELATIVISTIC PARTICLES AND ANTIPARTICLES OF HADRONIC MECHANICS

Nonrelativistic quantum mechanics is based on only one symmetry, the Galilean symmetry, thus resulting in only one notion of particles and antiparticles. Nonrelativistic hadronic mechanics is based instead on a hierarchy of covering symmetries with progressive complexity to represent a progressive complexity of physical conditions. This results in the following hierarchy of particles and antiparticles of hadronic mechanics:

7.7.A: Nonrelativistic particles and antiparticles. The quantum notion of *nonrelativistic particles* as characterized by the Galilean symmetry $G(3.1)$ (see, e.g., ref. [3] and quoted references) is adopted in its entirety in our studies of the *exterior* problem.

Their primary physical characteristic is that of being *point-like*. In fact, the Galilean notion of "massive point" [1] persists in its entirety after first quantization. Equivalently, we can say that the Galilean particle possesses perennial and immutable intrinsic characteristics of mass, spin, charge, etc.

On mathematical grounds, Galilean particles are characterized in hadronic mechanics by their fundamental units, the trivial unit of time $I_t = +1$ and of space $1 = $ diag. (1, 1, 1). Statically, the Galilean particle is therefore equivalent to the *perfect sphere* of unit radius in Euclidean space $E(r,\delta,R)$.

The quantum notion of antiparticle *is not* adopted in hadronic mechanics

because characterized by the same symmetry $G(3.1)$ over the same field. The *nonrelativistic antiparticles* of hadronic mechanics for exterior problems are instead characterized by the isodual Galilean symmetry $G^d(3.1)$ and related methodology.

Physically, this implies that that antiparticles are still *point-like*, with perennial and immutable intrinsic characteristics, although such characteristics are now negative-definite. In fact, the latter notion of antiparticles *requires* negative-definite energy and all other physical characteristics, while moving backward in time.[88]

Mathematically, nonrelativistic antiparticles are characterized by the isodual units $I_t = -1$, $I^d = -$ diag. $(1, 1, 1)$. Statically, they can be conceived as the *isodual sphere* in isodual Euclidean space $E^d(r, \delta^d, R^d)$.

7.7.B: Nonrelativistic isoparticles and antiisoparticles. The *nonrelativistic isoparticles* are those characterized by the isogalilean symmetry $\hat{G}(3.1)$, while the *nonrelativistic antiisoparticles* are those characterized by the isodual isogalilean symmetry $\hat{G}^d(3.1)$. They apply for the interior dynamical problem of matter and antimatter, respectively, under the general[89] condition that the orbits are reversible.

The first and most visible difference with quantum particles is that *isoparticles are extended in space with a generally nonspherical shape* characterized by the isounit $\hat{1} =$ diag. $(b^\circ_1{}^{-2}, b^\circ_2{}^{-2}, b^\circ_3{}^{-2})$, while *antiisoparticles are equally extended and nonspherical* as characterized by the isodual isounit $\hat{1}^d = -$ diag. $(b^\circ_1{}^{-2}, b^\circ_2{}^{-2}, b^\circ_3{}^{-2})$.

The second equally visible difference with quantum mechanics is that *the intrinsic characteristics of isoparticles are local quantities thus changing with the changing of the local density, temperature, etc.* As familiar from the preceding chapter, the isorotational symmetry is a theory of *deformable* bodies. The isoparticles can therefore experience a deformation of their nonspherical shape of the type $\hat{1} =$ diag. $(b^\circ_1{}^{-2}, b^\circ_2{}^{-2}, b^\circ_3{}^{-2}) \rightarrow \hat{1}' =$ diag. $(b_1{}^{-2}, b_2{}^{-2}, b_3{}^{-2})$ (see Fig. 7.7.1 for more details). In turn, such a deformation implies a consequential alteration of the intrinsic magnetic moment. The alteration of the remaining characteristics is consequential, as studied in more details in the next chapter via the isoreps of the isotopic Poincaré symmetry.

---

[88]To avoid misrepresentations, one should keep in mind that *these negative-definite quantities are referred to negative-definite units, thus resulting to be equivalent to positive-definite quantities referred to positive-definite units.* As we shall see in Ch. 11.10, isoduality is equivalent to charge conjugation although with deeper implications.

[89] The structural reversibility of isoparticles originates from the Hermiticity of their isounits which are generally assumed to be time-independent. However, there are cases of Hermitean isounits which are not time-reversal invariant, such as the scalar functions T = exp(γt). These latter cases, strictly speaking, overlap with the subsequent class of genoparticles.

EXTENDED –DEFORMABLE ISOGALILEAN NUCLEONS
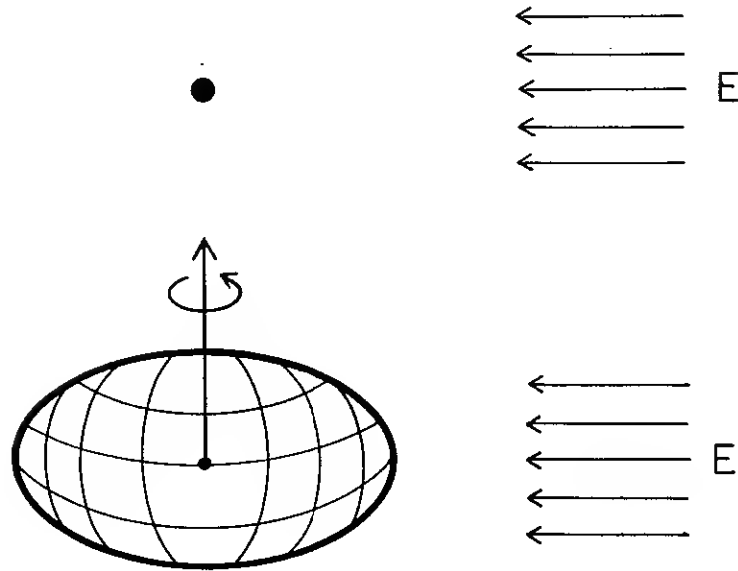
FIGURE 7.7.1: For quantum mechanics, protons and neutrons are points with perennial intrinsic characteristics. Hadronic mechanics represents instead the physical reality of their extended charge distributions of about 1 fm in radius. Their deformations under sufficient external fields then imply necessary *alterations of their intrinsic magnetic moments*, as requested by classical electrodynamics for the deformation of spinning charge distributions (see Ch. II.10). When members of a nuclear structure and represented as isoparticles (that is, obeying hadronic mechanics), protons and neutrons can have intrinsic magnetic moments different than the conventionally known values in vacuum. As studied in detail in Vol. III, this possibility permits the achievement of a numerical representation of the total magnetic moments of few body nuclei, such as the deuteron, which is lacking in quantum mechanical descriptions, despite relativistic and other corrections attempted for over half a century. These introductory lines are sufficient alone, prior to the study of the hadronic bound states, to indicate the characterization by hadronic mechanics of a novel structure model of nuclei.

The third visible difference between particles and isoparticles is that the

former only admit action–at–a–distance, potential interactions, while the latter admit additional contact nonpotential interactions.

Mathematically, *isoparticles (antiisoparticles)* are *characterized in hadronic mechanics by their fundamental units,* the time isounit $\hat{1}_t$ and the space isounit $\hat{1}$ and time isounit $\hat{1}_t$. As we shall see in Vol. III, in practical applications such isounits are generally factorized into a term representing the shape of the particle time terms $\hat{\delta}$ representing interactions,

$$\hat{1} = \text{diag.} (b_1^{-2}, b_2^{-2}, b_3^{-2}) \, \hat{\delta}(\hat{r}, \dot{r}..) , \quad \hat{1}_t = b_4^{-2} \hat{\delta}_4 . \tag{7.7.1}$$

We can also say that, statically, *isoparticles are represented by the isosphere* in Isoeuclidean spaces $\hat{E}(r,\hat{\delta},\hat{R})$ of Class I, while *antiisoparticles are represented by the isodual isosphere* in the isodual isoeuclidean spaces $\hat{E}^d(r,\hat{\delta}^d,\hat{R}^d)$ of Class II (Sect. I.5.2). As a result, isoparticles (antiisoparticles) can have all infinitely possible shapes unified by the isosphere (isodual isosphere).

Various simple cases of isoparticles have been presented in Sect. II.5.3. Other cases will be presented in the subsequent analysis of this and of the following volume.

7.7.C: Nonrelativistic genoparticles and antigenoparticles. The *nonrelativistic genoparticles* and *antigenoparticles* are those invariant under the Galilei–admissible symmetry $<\hat{G}>(3.1)$ and its isodual $<\hat{G}>^d(3.1)$, respectively. They include all features of isoparticles and antiisoparticles, with the additional property that they are in *irreversible conditions.* This identifies, apparently for the first time, the origin of macroscopic irreversibility in the ultimate *elementary* structure of matter, as pointed out in Sect. II..7.5.

Examples of genoparticles are the constituents of stars in highly dynamical internal conditions. Mathematically, genoparticles (antigenoparticles) are characterized by their fundamental units, the time genounit $<\hat{1}_f>$ and space genounit $<\hat{1}>$ (the isodual time genounit $<\hat{1}_t$ and isodual space genounit $<\hat{1}>^d$). Statically, *genoparticles (antigenoparticles) are characterized by the genosphere (isodual genosphere),* thus acquiring a complex geometry (Sect. I.7.7).

7.8: NONRELATIVISTIC BOUND STATES OF HADRONIC MECHANICS

The most significant nonrelativistic bound states of particles and antiparticles in hadronic mechanics are the following:

7.8.A: Nonrelativistic bound states of isoparticles. We shall now study the simplest possible bound state within the context of the Lie–isotopic branch of hadronic mechanics of Class I, that of two isoparticles.

We are particularly interested to show that *hadronic mechanics permits the verification of total conservation laws when the two particles admit different Planck's constants* (different isounits, see Sect. 1.1).

This clarification is due because of a contrary statement of note [21] claiming the violation of total conservation laws and related symmetries for particles with different Planck's constants. It should be noted that the notion of *generalized units for individual particles* had been introduced by this author in ref. [22] and the validity of total conservation laws for bound states of particles with different units proved in memoir [7] and in other reverences on hadronic mechanics preceding note [11] (see also the treatment of ref. [8]). The extension of the study to the case of isoparticles with spin and magnetic moments will be done later on.

TWO–BODY HADRONIC BOUND STATE OF ISOPARTICLES
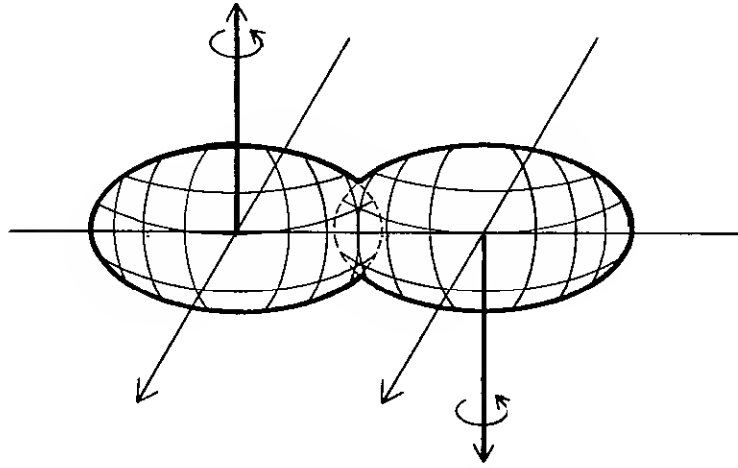
FIGURE 7.8.1: A conceptual view of the two–body bound states characterized by hadronic mechanics in which the constituents have *nonspherical charge distributions* in condition of partial or total mutual penetration resulting in nonlinear–nonlocal–nonhamiltonian internal interactions.

Our basic assumption is therefore that the two particles are characterized by the conventional Planck's constant when decoupled, and acquire generalized forms of Planck's constant when coupled under condition of mutual penetration.

The model can be realized by assuming that the particles are *perfectly spherical* when decoupled (i.e., they are ordinary particles as per Sect. 11.7.7), while they are deformed into *spheroidal ellipsoids* when coupled under strong interactions (i.e., they become isoparticles as per Sect. 11.7.7), and we shall write

$$\hat{1}_{a|\,free} = 1_a = \text{diag.}\,(\,1,\,1,\,1\,),\quad a = 1,\,2, \qquad (7.8.1a)$$

$$1°_{a}|_{coupled} = 1°_{a} = T°_{a}{}^{-1} = diag.(b°_{a1}{}^{-2}, b°_{a2}{}^{-2}, b°_{a3}{}^{-2}) = const. > 0, \quad (7.8.1b)$$

under the condition $1_{1} \neq 1°_{2}$, with corresponding different isoenvelopes $\hat{\xi}_{a}$, isofields $\hat{F}_{a}$, isohilbert spaces $\mathcal{JC}_{a}$, isostates $|\hat{\psi}_{a}>$, linear momenta p_{a}, etc. We shall also assume that the two isoparticles have conventional time isounits $1_{t1} = 1_{t2} = 1$ and the same mass $m_{1} = m_{2} = m \neq 0$, the extension of the results to different time isounits and masses being simple.

Recall that the isoeigenvalue equation of the linear momenta has the isotopic structure

$$P_{ak}*|\psi_{a}> = p_{ak}T_{a}|\psi_{a}> = -i 1°_{ak}\frac{\partial}{\partial r_{ak}}|\psi_{a}>, \quad (7.8..2)$$

where there is no sum on the repeated indices hereon unless specifically indicated.

The fundamental isocommutation rules are then given by

$$[r_{a}{}^{i} \hat{,} r_{b}{}^{j}]*|\hat{\psi}_{a}> = [p_{ai} \hat{,} p_{bj}]*|\hat{\psi}_{a}> \equiv 0, \quad (7.8.3a)$$

$$[p_{ai} \hat{,} r_{a}{}^{j}]*|\hat{\psi}_{a}> = -i b°_{ia}{}^{-2}\delta_{i}{}^{j}*|\hat{\psi}_{a}>, \quad i, j = 1, 2, 3, \quad a, b = 1, 2, \quad (7.8.3b)$$

where the isocommutator is

$$[A \hat{,} B]_{a} = [A \hat{,} B]_{\hat{\xi}_{a}} = A*B - B*A = AT°_{a}B - BT°_{a}A, \quad (7.8.4)$$

When the two isoparticles are disjoint they have the individual Hamiltonian in isospace

$$H_{a} = p_{a}{}^{2}/2m = p_{a}{}^{k} T°_{a} p_{ak} / 2m, \quad a = 1, 2, \quad (7.8.5)$$

with individual isoschrödinger equations

$$i \partial_{t}|\hat{\psi}_{a}> = H_{a}*|\hat{\psi}_{a}> = H_{a} T°_{a}|\hat{\psi}_{a}> = E_{a}|\hat{\psi}_{a}>. \quad (7.8.6)$$

Conventional interactions are easily accommodated via the addition of a potential $V(r_{ab})$ to the kinetic energy, with the only care now that r_{ab} is the isodistance (see below).

The nonpotential interactions must be represented with *anything except the Hamiltonian*, trivially, because they are nonhamiltonian by conception. In hadronic mechanics they are represented with the isounits. This implies that each isounit $1°_{a}$ = constant representing only the shape of the particles must be generalized into full isounits with nontrivial functional dependence, e.g., of the type $1_{a}(t, \dot{r}, \ddot{r}, ...) = 1°_{a}\hat{\xi}_{a}(t, \dot{r}, \ddot{r}, ...)$.

But each particle is the hadronic medium of the other or, equivalently, the contact–nonpotential interaction exercised by each particle on the other must be equal in magnitude. This implies $\hat{\varepsilon}_1 \equiv \hat{\varepsilon}_2 = \hat{\varepsilon},^{90}$ and we can write

$$\hat{T}^\circ_a \to \hat{1}_a = \hat{T}^\circ_a \, \hat{\varepsilon}_a(t, \dot{r}, \ddot{r}, \ldots), \qquad \hat{\varepsilon}_1 \equiv \hat{\varepsilon}_2 \, \hat{\varepsilon}, \tag{7.8.7a}$$

$$\hat{T}^\circ_1 = \text{diag.}\,(\, b^\circ_{11}{}^{-2}, b^\circ_{12}{}^{-2}, b^\circ_{13}{}^{-2}\,) \neq \hat{T}^\circ_2 = \text{diag.}\,(\, b^\circ_{21}{}^{-2}, b^\circ_{22}{}^{-2}, b^\circ_{23}{}^{-2}\,). \tag{7.8.7b}$$

In order to represent the system, the first and most fundamental point is the identification of the *total isounit* and *total isotopic element* which, for necessary condition of consistency (even for conventional quantum theories [3]), are given by the tensorial products

$$\hat{1}_{tot} = \hat{1}_1 \times \hat{1}_2 = T_{tot}{}^{-1}, \quad T_{tot} = T_1 \times T_2. \tag{7.8.8}$$

This essentially implies that, according to hadronic mechanics, the bound state here considered is based on the total isoassociative algebra of operators (Sect. I.4.3)

$$\hat{\xi}_{tot}: \qquad A * B = A T_{tot} B, \tag{7.8.9}$$

total isofield of real or complex numbers (Set. I.2.5)

$$\hat{F}_{tot}: \hat{F}_1 \times \hat{F}_2. \tag{7.8.10}$$

and total isohilbert space with related states (Sect. I.6.3)

$$\mathcal{H}_{tot}: \quad \mathcal{H}_1 \times \mathcal{H}_2: \, |\hat{\psi}_{tot}> \, = \, |\hat{\psi}_1> \times |\hat{\psi}_2>. \tag{7.8.11}$$

The total quantities of the system must then be properly defined in the total isospaces by following essentially the same rules as those for ordinary quantum mechanics [3]. In particular, the coordinates $r_a{}^k$ and momenta p_{ak} of the individual particles now become

---

$$\hat{r}_1{}^\kappa = r_1{}^\kappa \times \hat{1}_2, \qquad \hat{r}_2{}^\kappa = \hat{1}_1 \times r_2{}^\kappa, \tag{7.8.12a}$$

$$\hat{p}_{1k} = p_{1k} \times \hat{1}_2, \qquad \hat{p}_{2k} = \hat{1}_1 \times p_{2k}. \tag{7.8.12b}$$

while the total quantities become

$$P = \hat{p}_1 + \hat{p}_2 = p_1 \times \hat{1}_2 + \hat{1}_1 \times p_2, \tag{7.8.13a}$$

$$R = \tfrac{1}{2}(\hat{r}_1 + \hat{r}_2) = \tfrac{1}{2}(r_1 \times \hat{1}_2 + \hat{1}_1 \times r_2) \qquad M = 2m, \tag{7.8.13b}$$

with relative expressions

$$r = \hat{r}_1 - \hat{r}_2 = r_1 \times \hat{1}_2 - \hat{1}_1 \times r_2, \tag{7.8.14a}$$

$$p = \tfrac{1}{2}(\hat{p}_1 - \hat{p}_2) = \tfrac{1}{2}(p_1 \times \hat{1}_2 - \hat{1}_1 \times p_2), \qquad \mu = \tfrac{1}{2}m. \tag{7.8.14b}$$

The total Hamiltonian for the two interacting particles can then be written

$$H_{tot} = p_1{}^2/2m + p_2{}^2/2m + V(\hat{r}_1 - \hat{r}_2) \tag{7.8.15}$$

and reduced to the form from Eq.s (11.5.4.6)

$$H_{tot} = P^2/2M + p^2/2\mu + \hat{V}(r) =$$

$$= (\hat{\delta}_{tot}{}^{ij} P_i T_{tot} P_j)/2M + (\hat{\delta}_{tot}{}^{ij} p_i T_{tot} p_j)/2\mu + V(r)\hat{1}_{tot}. \tag{7.8.16}$$

The system is then described by the isoschrödinger equation

$$i\, \partial_t |\hat{\psi}_{tot}> = H_{tot} * |\hat{\psi}_{tot}> = H_{tot} T_{tot} |\hat{\psi}_{tot}> = E_{tot} |\hat{\psi}_{tot}>, \tag{7.8.17}$$

where one recognizes the presence of conventional local–potential–Hamiltonian forces described by $V(\hat{r})$, as well as the additional nonlocal nonhamiltonian interactions represented by the common factor $\hat{\delta}(t, \dot{r}, \ddot{r}, ...)$ in $\hat{1}_{tot}$ or, equivalently, by the common factor $\hat{\delta}^{-1}(t, \dot{r}, \ddot{r}, ...)$ in T_{tot}. In fact, the isounit now enters into the structural equation of the system, thus acquiring a direct dynamical role of representing interactions without potential energy.

The validity of the total conservation laws then follows in a unique and unambiguous way [7,8]. In fact, one has the independence of the total linear momentum from the relative coordinates, say, for particles 1 and 2 in one space-dimension [7,8]

$$[P, r]_{\hat{\xi}_{tot}} * _{tot} |\hat{\psi}_{tot}> =$$

$$= [(p_1 \times \hat{1}_2 + \hat{1}_1 \times p_2), (r_1 \times \hat{1}_2 - \hat{1}_1 \times r_2)]_{\hat{\xi}} T_{tot} |\hat{\psi}_{tot}> =$$

$$= [p_1 \times \hat{1}_2, r_1 \times \hat{1}_2]_{\hat{\xi}_{tot}} T_{tot} |\hat{\psi}_{tot}> - [\hat{1}_1 \times p_2, \hat{1}_1 \times r_2]_{\hat{\xi}_t} T_{tot} |\hat{\psi}_{tot}> =$$

$$= [p_1, r_1]_{\hat{A}_1} T_1 |\hat{\psi}_{tot}> - [p_2, r_2]_{\hat{A}_2} T_2 |\hat{\psi}_{tot}> \equiv 0, \tag{7.8.18}$$

with similar generalizations in arbitrary dimensions.

The above results establish the following properties

$$[H_{tot}, \hat{} P]_{\hat{\xi}_{tot}} = [V(r), \hat{} P]_{\hat{\xi}_{tot}} \equiv 0, \qquad (7.8.19)$$

where the isocommutator is in $\hat{\xi}_{tot}$, thus implying following

Lemma 7.8.1 [7,8]: *Hadronic bound states of isoparticles with different Planck's constants (different isounits) verify all ten total Galilean conservation laws when properly treated via total isofields, total isospaces and total isoalgebras whose isounit is the tensorial product of the individual isounits .*

By recalling that the basic isogalilean invariance for a system of particles must necessarily have a tensorial structure with the total isounit given by trhe tensorial product of the individual isounits, we have the following

Corollary 7.8.1A: *The results of Lemma 7.8.1 equally follow by imposing the isogalilean invariance.*

In fact, the isotopies of Mackey's imprimitivity theorem (Sect. II.7.5) uniquely imply structure (7.8.15) of the Hamiltonian. Total conservation laws are then implicit in the isotopies of the Galilean symmetry (Sect. II.7.4).

Note the restriction of the generalizations to *interior dynamical effects*, e.g., particles passing from a perfectly spherical to an ellipsoidical shape, the acquisition of contact–nonpotential interactions, the alteration of Planck's constant, etc. The only behaviour visible from the outside is the conventional stability of the system, as it is the case for all isolated systems.

Thus, the validity of conventional total conservation laws, by no means, implies the unique validity of quantum, mechanics for the interior structure. In fact, the same total conservation laws are admitted by a structural generalization of quantum mechanics for the interior problem only.

As we shall see in Ch. 1 of Volume III, the hadronic bound state presented in this section permits the first exact representation of the total magnetic moment of the deuteron which has not been achieved by quantum mechanics despite all possible relativistic, tensorial and other corrections.

The origin of this exact representation is precisely the deformation of shape of the charge distributions of the proton a = 1 and the neutron a = 2 at the formation of their bound state

$$I_a = \text{diag.} (1, 1, 1)_{| \text{free}} \rightarrow \hat{1}_a = \text{diag.} (b^{\circ}{}_{1a}{}^{-2}, b^{\circ}{}_{2a}{}^{-2}, b^{\circ}{}_{3a}{}^{-2})_{| \text{coupled}}. \qquad (7.8.20)$$

This (generally small) deformation implies a consequential *necessary* (also

generally small) alteration of their intrinsic magnetic moments

$$\mu_{|\,free} \quad \rightarrow \quad \hat{\mu}_{|\,coupled} \neq \hat{\mu}, \qquad (7.8.21)$$

which then permits the exact, numerical representation of the experimental evidence. As a matter of fact, the experimental value of the total magnetic moment of the deuteron permits the calculation of the numerical amount of deformation of the charge distribution of the proton and neutrons when coupled in the deuteron. In short, the experimental value of the deuteron magnetic moment permits the *numerical* calculation of the isounits, as we shall see.

We now study the impact of the isotopy in the final eigenvalues. Suppose for simplicity that the hadronic bound state is at rest. Then the isoeigenvalue equation for operator (7.5.16) reduces to

$$[\,(\,\delta_{tot}{}^{ij}\,p_i T_{tot}\,p_j\,)\,/\,2\mu\;+\;V(r)1_{tot}\;]\,T_{tot}|\,\hat{\psi}_{tot} > \;=\; E\,|\,\hat{\psi}_{tot} > . \quad (7.8.22)$$

But the trajectories of all two-body bound states are in a plane, here assumed to be the (x, y)-plane. Suppose that the polar axis of the spheroidal ellipsoids is the z-axis, and that we have equal spheroids for simplicity. Under these simplified conditions, the isotopy can be factorized and we can write T_{tot} = $b^{o2}\delta_t$. But, as now familiar from preceding studies, isounits must be averaged to constants when studying the system from the exterior. This implies that, for outside measurements, we have the constant factorization T_{tot} = B° I_{tot}, where I_{tot} is the 2×2 unit matrix.

After computing the isoeigenvalues k_i of the relative momenta p_i, Eq. (7.8.22) reduces to (ħ = 1)

$$[\,\frac{1}{2\tilde{\mu}}\,\delta^{ij}\,k_i\,k_j\;+\;V(r)\,]\,|\,\hat{\psi}_{tot} > \;=\; E\,|\,\hat{\psi}_{tot} >, \quad \tilde{\mu} = \mu\,B°, \qquad (7.8.23)$$

where δ^{ij} is now the conventional Euclidean metric. As one can see, we have an *isotopic renormalization of the reduced mass,* also called *isorenormalization,* $\mu \rightarrow \tilde{\mu} = \mu B°$.

This occurrence will be studied in more details at the relativistic level. It has quite intriguing implications because it permits the construction of bound states that would be inconsistent for quantum mechanics, e.g., because they would otherwise require a "positive binding energy" (see Vol. III).

The reader should be aware that the above isorenormalization of the reduced mass appears first at the primitive *Newtonian* treatment of interior problems (see App. III.A, Vol. II, ref. [6]), and then persists at the operator nonrelativistic level, to acquire its full meaning under relativistic treatments.

For completeness we indicate that note [21] studied bound states of particles with different Planck constants $\hbar_1 = 1°_1$ and $\hbar_2 = 1°_2$ (which are precisely our isounits) by means of the following total and relative quantities

$$P = p_1 + p_2, \qquad r = r_1 - r_2, \qquad \text{etc.} \qquad (7.8.24)$$

which are defined over a *conventional* Hilbert space $\mathcal{H}$ *without* tensorial products. Then the following property holds

$$[\, P_i \,, r_j \,]_\xi \; = \; -i\,(1_1 - 1_2)\,\delta_{ij}\,, \qquad (7.8.25)$$

which implies the presumed lack of conservation of the total energy [21]

$$[\, H_{tot} \,, P \,]_\xi \; \neq \; 0\,, \qquad (7.8.26)$$

with consequential violation of the conservation law of the total linear momentum and other physical quantities.

The basic flaw of note [21] is *the lack of use of the tensorial product of the individual states,* which is necessary even for conventional quantum mechanics [3] in order not to violate linearity, transitivity and other basic axioms.

It should be also indicated that note [21] claims violation of total conservation laws for *constants* generalizations $\hbar^\circ_a$, while total conservation laws are proved above for *integro–differential* generalizations $\hat{\hbar}_a(t, \dot{r}, \ddot{r}, ...) = 1$.

It is unfortunately that note [21] of 1991 did not quote ref. [22] of 1983 on the introduction of different units for different particles, ref. [7] on the proof of total conservation laws for bound systems of particles with different units, or other papers on the rather vast literature on the integro–differential generalizations of Planck's constant studied by hadronic mechanics.[91]

7.8.B: Nonrelativistic bound states of antiisoparticles. The next illustrative model is the minimal possible hadronic bound states of two antiisoparticles. It is essentially given by the isodual image of the preceding one with total isosymmetry $\hat{G}^d(3.1){\times}\hat{G}^d(3.1)$, total isounit $1_{tot} = 1_1{}^d{\times}1_2{}^d$, total isodual isohilbert spaces $\mathcal{H}_{tot}{}^d = \mathcal{H}_1{}^d{\times}\mathcal{H}_2{}^d$ with states $| \,\hat{\psi}_{tot} >^d = | \,\hat{\psi}_2 >^d{\times}| \,\hat{\psi}_1 >^d, | \,\hat{\psi}_k >^d = - < \hat{\psi}_k |,$ etc.

The end result is the reversal of the sign of the eigenvalues of the preceding case, with the understanding that they are now computed in isodual isospaces.

7.8.C: Isoselfdual hadronic bound states. Some of the most interesting

---

[91] A clarification of these aspects was submitted by G. F. Weiss in a note entitled "Comments on a recent note on multiple Planck's constants" to the journal of note [21], *Phys. Rev. Lett.* but the clarification was rejected by the editors. The same clarification was rejected by other journals, such as *Phys. Rev D, Modern Phys. Lett.* and others. These rejections are indicated here as one illustration (among many) of the current editorial condition of theoretical physics lamented in the preface of this volume.

and novel bound states of hadronic mechanics are the *isoselfdual bound states,* that is, bound states which coincide with their isodual image. The simplest case is given by the bound state of one particle and its antiparticle.

The condition of isoselfduality requires that the symmetry is given by

$$G_{tot} = \hat{G}(3.1) \times \hat{G}^d(3.1) \equiv G_{tot}{}^d , \qquad (7.8.29)$$

the *total space unit* is

$$\hat{1}_{tot} = \hat{1} \times \hat{1}^d \equiv \hat{1}_{tot}{}^d , \qquad (7.8.30)$$

the total Hilbert space and related states are

$$\mathfrak{H}_{tot} = \mathfrak{H} \times \mathfrak{H}^d \equiv \mathfrak{H}_{tot}{}^d : \ |\hat{\psi}_{tot}> = |\hat{\psi}> \times |\hat{\psi}>^d \equiv |\psi_{tot}>^d , \quad (7.8.31),$$

and so on.

The isoselfdual states therefore exists in both the isospace of particles and that of antiparticles. The computation of the total energy must therefore be done either in $\mathfrak{H}$, in which case it is positive, or in $\mathfrak{H}^d$, in which case it is negative. In particular, for the former case we have the isoschrödinger equation

$$i \frac{\partial}{\partial t} |\hat{\psi}> = \frac{1}{2m} \delta^{ij} p_i T p_i T |\hat{\psi}> + \frac{1}{2m^d} \delta^{dij} p_i T^d p_j T^d |\hat{\psi}> + V(r) |\hat{\psi}> =$$

$$= [2 \frac{1}{2m} \delta^{ij} p_i T p_j + \hat{V}(r)] T |\hat{\psi}> = E |\hat{\psi}> , \ E > 0, \qquad (7.8.32)$$

namely, the isoeigenvalue of an antiparticle are positive when computed on the state of a particle (because $|>^d = -<|$). The model follows the same lines as those of the preceding hadronic bound states.

A quite intriguing aspect is time because *the total time of isoselfdual bound states is evidently null,* $t_{tot} = t + t^d \equiv 0$, although the total time isounit $\hat{1}_t = \hat{1}_{1t} \times \hat{1}_{2t}{}^d$ is not null because the two units operate in different spaces. This property is important because, an *isoselfdual bound state acquires the time of the field in which it is immersed* and *it is attrracted by both matter and antimatter.* For instance, an isoselfdual particle in the gravitational field of Earth acquires the time here on Earth. This notion requires a relativistic approach for its proper treatment and its study is deferred to the next chapters.

As we shall see, isoselfdual bound states permit the prediction of novel states which are not predicted by quantum mechanics. The best illustration is given by the bound states of one electron and one positrons. The only bound state between these particles predicted by quantum mechanics is is the *positronium atom.* The latter, is at large mutual distances of the constituents, in which case quantum mechanics is indeed exactly valid.

The novelty here is that *hadronic mechanics predicts a fundamentally novel bound state of one electron and one positron at mutual distances of the*

order of 1 fm, with truly intriguing applications in various fields studied in Vol. III.

7.9: LIE–ADMISSIBLE REFORMULATION OF THE IRREVERSIBLE STATISTICS OF THE BRUXELLES SCHOOL

The Lie–admissible time evolution was introduced by this author in classical, operator and statistical mechanics back in 1978 [4] (see also ref.s [10]) and, for the case of a density matrix ρ, can be written ($\hbar = 1$)

$$i \partial_t \rho = (H_0, \rho) = H_0 R \rho - \rho S H_0 = H_0 < \rho - \rho > H_0 , \qquad (7.9.1)$$

where H_0 is a Hermitean Hamiltonian of the conventional Liouville equation

$$i \partial_t \rho_0 = [H_0, \rho_0] = H_0 \rho_0 - \rho_0 H_0 . \qquad (7.9.1)$$

As now familiar, the statistical mechanics characterized by Eq.s (7.9.1) is *structurally irreversible*, i.e., irreversible even for a reversible Hamiltonian. A study of the classical irreversible statistics of Lie–admissible type was done in ref. [23], a general review up to 1984 can be found in ref. [18] and subsequent studies in ref. [24].

An irreversible operator statistics has been proposed by I. Prigogine and his school in Bruxelles in various contributions (see ref.s [25] and literature quoted therein), here referred to as *Bruxelles school (BS) statistics, which is* based on the following modification of the original density $\rho_0(t)$ via a *nonunitary* operator $\Lambda(t)$,

$$\rho(t) = \Lambda(t) \rho_0(t), \quad \Lambda \Lambda^\dagger \neq 1 . \qquad (7.9.3)$$

The Lie–admissible structure of the above statistics has been established by Jannussis and Mignani [26]. Their argument can be summarized as follows. Suppose $H = H_0 + \lambda V$ is the generator of the time evolution of Λ and H_0 that of ρ_0. It is then easy to show that

$$i \partial_t \Lambda = [H_0, \Lambda] + \lambda (V \Lambda - \Lambda \rho_0 V \rho_0^{-1}) . \qquad (7.9.4)$$

The latter equation is a conventional Liouville equation with an external term which can always be identically reformulated in terms of Eq. (7.9.1) (Sect. I.7.2). This yields the Lie–admissible structure of the time evolution of the Λ-operator [26]

$$i \, \partial_t \Lambda = (H_o, \Lambda) = H_o R \Lambda - \Lambda S H_o, \qquad (7.9.5a)$$

$$R = 1 + \lambda H_o^{-1} V, \qquad S = 1 + \lambda \rho_o V p_o^{-1} K_o^{-1}. \qquad (7.9.5b)$$

The time evolution of the density p can then be readily written in Lie–admissible form [loc. cit.]

$$i \, \partial_t p = (H_o, p) = H_o R' p - \rho S' H_o, \qquad (7.9.6a)$$

$$R' = 1 + \lambda \, H_o^{-1} V, \quad S' = 1 + \lambda H_o^{-1}. \qquad (7.9.6b)$$

It si easy to see that, by using the genoquantization of Sect. 11.2.4, the above Lie–admissible statistics is in unique and unambiguous operator counterpart of the classical statistics of ref. [4].

As a final comment, one should note that a direct nonunitary transform of the conventional Liouville equation (7.9.2) would yield the *Lie–isotopic*, rather than the Lie–admissible form, as one can see, e.g., from a time–independent nonunitary transform

$$i \, \partial_t (U \, p_o \, U^\dagger) = i \, \partial_t \rho_o' = U [H_o, \rho_o] U^\dagger = H_o' T R_o' - \rho_o' T H_o', \quad (7.97a)$$

$$\rho_o' = U \rho_o U^\dagger, \quad H_o' = U H_o U^\dagger, \quad U U^\dagger \neq 1, \quad T = (U U^\dagger)^{-1}. \qquad (7.9.7b)$$

This would result in a statistics which is *structurally reversible*, that is, reversible for a reversible Hamiltonian and T–operator much along conventional quantum mechanics. A significance of the BS statistics is that of bypassing the above limitation via rule (7.9.3) by therefore achieving a structurally irreversible statistics.

We finally note that, to avoid the problematic aspects studies earlier (lack of invariance under its own time evolution, lack of preservation of Hermiticity–observability at all times, lack of measurement theory, etc.) Eq.s (7.9.5) and (7.9.6) must be formulated according to the rule of the Lie–admissible branch of hadronic mechanics, that is, on the now familiar genohilbert spaces over genofields.

APPENDIX 7.A: WILHELM'S GALILEI –COVARIANT ELECTRODYNAMICS

Physics is a discipline in which truly fundamental issues are never put to rest and at times are discussed for centuries. This is the case for the historical debate on the *ether* (also called *space* or *substratum*) which raged during the second half of the past century, somewhat abated during the first part of this century because of the successes of the special relativity, and has recently returned to be

the subject of novel studies (see, e.g., the various papers in ref. [27]).

In his first paper written in 1956 [28] (when a first year undergraduate student in physics), this author discussed the following topics:

1) Since no wave can exist and propagate without a medium, the ether must be a medium with well specified characteristics filling up the entire Universe as a necessary condition for light to exist and propagate;

2) Since light is a purely *transversal* wave, the space (or ether or substratum) must be *rigid* and, in particular, must be an incompressible, homogeneous and isotropic medium;

3) The old issue of the "ethereal wind" (the expected resistance encountered by matter when moving in a space assumed to be a medium) has no scientific basis, because elementary particles themselves such as electrons, to be constituted by a characteristic frequency, must be local vibrations of the ether itself. The motion of a body through the ether therefore implies the motions of said characteristic oscillations from one region of the ether to another without any possible "wind".

4) The assumption of space as a physical medium implies the existence of a privileged inertial reference frame *for the electromagnetic waves.*

5) The ether cannot necessarily provide a privileged inertial reference frame *for matter* because no observer can detect it. As a result, the assumption of space as a physical medium can eventually result to be fully compatible with the special relativity of particles.

According to the above view, special relativity remains *exactly valid* for the characterization of *elementary particles* in exterior conditions when space is assumed to be a physical medium, with revisions expected only for the characterization of electromagnetic waves.

Comprehensive studies on the implications for electrodynamics have been conducted by H. E. Wilhelm [29] under the following assumptions: 1) space is a homogeneous and isotropic medium with *permittivity* $\epsilon_0 = 10^{-9}/36\pi$ As/Vm and *permeability* $\mu_0 = 4\pi \times 10^{-7}$ Vs/Am (to yield the speed $c_0 = 1/\sqrt{\epsilon_0 \mu_0} = 3 \times 10^8$ m/sec), and 2) space and time are independent. These studies have resulted in the so-called *Galilei covariant electrodynamics in vacuum* which we regret to be unable to review for brevity (see [loc. cit.]).

We merely mention that the above theory leaves the conventional alectrodynamical equations unchanged, and merely studies their *Galilean* covariance when written in the familiar form

$$\nabla \times E = -\partial_t B, \ \nabla \times H = \partial_t D + J, \nabla \cdot D = \rho, \nabla \cdot B = 0, \qquad (7.A.1)$$

where $D = \epsilon_0 E$, $B = \mu_0 H$.

It appears that the Galilei–covariant electrodynamics in vacuum is supported by available experiments, including those by Sagnac, Aharon–Bohm, Wilson, Ives–Stilwell, and others. The theory has also received a resurgence of interest from

the discovery in 1965 by Penzias and Wilson of the cosmic 2.7 K° microwave background because its isotropic frame can be precisely assumed as the privileged inertial frame of electromagnetic waves. For all these aspects we refer the interested reader to ref. [29] and literature quoted therein).

In closing we mention that the above comments refer to electrodynamics *in vacuum*. A structural revision of Maxwell's electrodynamics is apparently requested *in metals* along the line of the *Ampere–Neumann electrodynamics* as studied by P. Graneau [30] which will be considered in the applications of Vol. III.

REFERENCES

1. GALILEO GALILEI, *Dialogus de Systemate Mundi* (1638), translated and reprinted by Mc. Millan, New York (1916)
2. E. C. G. SUDARSHAN and N. MUKUNDA, *Classical Dynamics: A Modern Perspective*, Wiley, New York (1974)
3. G. W. MACKEY, *Unitary Group Representations in Physics, Probability and Number Theory* , Benjamin, New York (1978)
4. R. M. SANTILLI, Hadronic J. $\underline{1}$, 228, 574 and 1279 (1978);
5. R. M. SANTILLI, *Foundations of Theoretical Mechanics* , Vol. 1 (1978, Vol. II (1983), Springer–Verlag, Heidelberg–New York
6. R. M. SANTILLI, *Isotopic Generalization of Galilei's and Einstein's Relativities*, Vol. 1 and II, first edition(1991), second edition, Ukraine Academy of Sciences, Kiev 1(994)
7. R. M. SANTILLI, Hadronic J. Suppl. $\underline{4B}$, issue no. 2 (1989)
8. R. M. SANTILLI, Hadronic J. $\underline{15,}$ 1 (1992)
9. R. M. SANTILLI, Hadronic J. $\underline{17,}$ 311(1994)
10. R. M. SANTILLI, *Lie–Admissible Approach to the Hadronic Structure*, Vol. 1 (1978), Vol. II (1982), Hadronic Press, Palm Harbor, FL
11. E. DAVIES, *Quantum Theory of Open Systems*, Academic Press, New York (1976)
12. K.. BLUM, *Density matrix and Applications*, Plenum, New York (1980)
13. I. PRIGOGINE, *Nonequilibrium Statistical Mechanics*, Wiley, New York (1962)
14. R. M. SANTILLI, Algebras, Groups and Geometries $\underline{10}$, 273 (1993)
15. A. JANNUSSIS, Hadronic J. $\underline{13,}$ 399 and 415 (1990); and in *Proceedings on the Third International Workshop on Symmetry Methods in Physics*,G. Pogosyan et al., Editors, JINR, Dubna, Russia (1994);
16. S. W. HAWKING, R. LAFLAMME and G. W. LYONS, Phys. Rev. D $\underline{47}$, 5342 (1993)
17. H. C. BAKER, Phys. Rev. A $\underline{30,}$ 773 (1984)
18. A. SCHOEBER, Editor, *Irreversibility and Nonpotentiality in Statistical Mechanics*, Hadronic Press, Palm Harbor, FL (1984)
19. L. C. BIEDENHARN and J. D. LOUCK, *The Racah–Wigner Algebra in Quantum Theory*, Addison–Wesley, Reading, MA (1981)

20. R. M. SANTILLI, Lett. Nuovo Cimento **38**, 509 (1983)

21. E. FISHBACH, G. L. GREENE, and R. J. HUGHES, Phys. Rev. Lett. **66**, 256 (1991)

22. R. M. SANTILLI, Lettere Nuovo Cimento **37**, 545 (1983)

23. J. FRONTEAU, A. TELLES-ARENAS and R. M. Santilli, Hadronic J. **3**, 130 (1979)

24. J. P. CONSTANTOPOULOS and C. N. KTORIDES, J. Phys. A 17,L29 (1984)

25. C. GEORGE, F. HENIN, F. MAYNE, AND I PRIGOGINE, Hadronic J. 1, 520 (1978); B. MISRA, I. PRIGOGINE and M. COURBAGE, Physica A 98, 1 (1979); I. PRIGOGINE, in XXVI Nobel Conference (1990)

26. A. JANNUSSIS and R. MIGNANI, Physica A **187**, 575 (1992)

27. F. SELLERI and M. BARONE, Editors, *Frontiers of Fundamental Physics*, Plenum, New York (1994)

28. R. M. SANTILLI, *Why space is rigid*, Il Pungolo Verde, Campobasso, Italy (1956)

29. H. E. WILHELM, Intern. J. Math ~ Math. Sciences **8**, 589 (1985) and **14**, 769 (1991); Radio Sci. **29**, 1006 (1985); Fusion Techn. **6**, 174 (1984); Z. Naturforsch. **45a**, 736 and 749 (1990); Apeiron **13**, 17 (1992), and **15**, 1 (1993); Physics Essays **6**, 420 (1993); Hadronic J. Suppl. **8**, 441 (1993); and in *Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Plenum, New York (1994)

30. P. GRANEAU, *Ampere-Neumann Electrodynamics of Metals*, Second Edition, Hadronic Press, Palm Harbor, FL (1994)

8: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF THE SPECIAL RELATIVITY

8.1: STATEMENT OF THE PROBLEM

The rather widespread notion of "universal constancy of the speed of light" c_0 is a philosophical abstraction because in the physical reality the speed of electromagnetic waves is a function of the local physical characteristics of the medium in which they propagate.

As an example, the speed of light in our atmosphere varies with the density, and then assumes different values depending on whether propagating in water, glass, etc.

The body of knowledge due to Lorentz [1], Poincaré [2], Einstein [3], Minkowski [4] and others (for historical accounts see, e.g., ref.s [5]) and today known as the *special relativity*, is *exactly* valid under the conditions in which the speed of light is the constant value c_0, i.e., for propagation in vacuum (*exterior relativistic problem*). However, the same discipline is *inapplicable*[92] for the more general physical conditions of light propagating within (transparent) physical media with a locally varying speed (*interior relativistic problem*) for a variety of reasons reviewed in more details later on, such as: loss of the light cone; massive physical particles propagating at speeds higher than the local speed of light; loss of the relativistic composition of speeds; and others.

Moreover, as recalled a number of times, empty space is *homogeneous and isotropic,* while physical media such as our atmosphere are generally *inhomogeneous and anisotropic.* It is evident that a discipline built for the former conditions is inapplicable for the representation of the latter in an exact form.

Our first task is therefore the construction of a generalization–covering of the special relativity capable of representing light propagating in generally inhomogeneous and anisotropic media with a locally varying speed c.

---

[92] The use of the term "violated" would be inappropriate because the special relativity was not conceived for a variable speed of light. Needless to say, the *approximate* character of the special relativity for interior conditions should be kept in mind

Note that the reduction of the above classical problem to second quantization would eliminate the very characteristics to be represented. To begin, the propagation of electromagnetic waves in atmosphere (such as radio waves with a wavelength of the order of the meter) is a purely *classical* problem whose necessary reduction to photons has no scientific value (or credibility). Also, the reduction of the problem to photons scattering through the molecules of our atmosphere would eliminate the inhomogeneity, anisotropy and locally varying speed to be represented, evidently because the problem would be reduced to photons propagating in vacuum with speed c_o.

The problem considered therefore requires first a purely classical representation and then the identification of its appropriate operator counterpart.

In the transition to classical extended particles in interior relativistic conditions, the inapplicability of the special relativity and the need for a covering relativity become more compelling because, in addition to the general loss of homogeneity and isotropy of the medium, we have the loss of the Hamiltonian character of the system due to the contact–resistive forces, as studied in detail by this author in monographs [6,7]. In turn, such nonhamiltonian character implies the lack of applicability of the fundamental symmetries underlying the special relativity, the *Lorentz and Poincaré symmetries*, e.g., because individual particles have generally unstable trajectories with continuously varying angular momenta even though the system as a whole can be stable (*closed variationally nonselfadjoint systems* [7]).

Finally, in the transition to extended particles in operator relativistic interior conditions, such as for a proton propagating in the core of a collapsing star, we have the loss of the local–differential character of the interactions in favor of nonlocal–integral forms due to the mutual overlapping of the wavepacket/wavelength/charge–distribution of the particle considered within the now familiar *hadronic medium*. Note that motion in interior conditions is generally *nonuniform* and the reference frames are generally *noninertial* as occurring in the physical reality, although admitting of inertial frames when studied from the exterior (see later on for specifications).

In summary, our objective is the construction of a generalization–covering of the special relativity for the characterization of interior relativistic problems for generally nonuniform motion referred to generally noninertial frames. The covering relativity should: 1) be constructed first at the purely classical level; 2) admit a unique operator version; 3) recover the conventional special relativity whenever motion returns to be in vacuum; 4) admit the isogalilean relativity of the preceding chapter at the nonrelativistic limit, and 5) be admitted in the tangent space of the isogeneral relativity of the next chapter.

From the viewpoint of the fundamental space–time symmetries, our objective is the construction of generalizations–coverings of the Lorentz and Poincaré symmetries which: 1) yield the invariance of arbitrarily nonlinear, nonlocal and nonhamiltonian systems; 2) admit the conventional symmetries as

particular cases; and 3) are such to coincide with the conventional symmetries at the abstract, realization–free level. The generalized Poincaré symmetry must also admit the isogalilean symmetry of the preceding chapter under a suitable group contraction.

As we shall see in this chapter, the above conditions are verified by the *isotopies* and *genotopies of the special relativity*, also called *isospecial* and *genospecial relativities*, respectively, representing particles, and the *isodual isospecial and genospecial relativities* representing antiparticles.

The isospecial relativity (its isodual) is conceived for closed–isolated systems of particles (antiparticles) with nonlinear–nonlocal–nonhamiltonian internal effects, while the genospecial relativity and its isodual are conceived for open–nonconservative systems. Also, the former admit reversible center–of–mass trajectories, while the latter are ideally suited for an invariant relativistic characterization of irreversibility in its four possible directions of time.

The fundamental carrier spaces are the *isotopies* and *genotopies of the Minkowski space*, called *isominkowskian* and *genominkowskian spaces*, respectively, as studied in Vol. 1, while the basic symmetries are the *isotopies* and *genotopies of the Poincaré symmetry*, also called *isopoincaré* and *genopoincaré symmetries*, respectively, the first possessing a Lie–isotopic structure and the second the more general Lie–admissible structure.

In particular, the isospecial relativity can be defined as a description of systems of particles on isominkowski spaces which is invariant under the isopoincaré symmetry, with the antiautomorphic conjugation under isoduality describing systems of antiparticles.

The classical isospecial relativity was first proposed by this author in 1983 [8], jointly with the first isotopies on record of the Lorentz symmetry. The isospecial relativity was then studied at the classical level in memoir [9] and in monographs [10,11]. The operator version of the isospecial relativity was first outlined in ref. [12] and then studied in more details in ref. [13]. A recent presentation of the isotopies of the Minkowski geometry has appeared in ref. [14]. A recent presentation of the isotopies of the special relativity and of the underlying Poincaré symmetry can be found in ref. [15]. This chapter contains the first presentation of the operator isospecial relativity with a number of novel advances, such as the initiation of the representation theory of the isopoincare symmetry and others. Independent reviews can be found in monographs [16–18].

The foundations of the genospecial relativity first appeared in the original proposal to build a Lie–admissible generalization of Lie's theory [19]. Its classical version first appeared in monographs [20,21] and then in ref.s [10,11]. The operator genospecial relativity was briefly considered in memoir [12]. Our current mathematical knowledge on the Lie–admissible theory was outlined by this author at the 1994 International Congress of Mathematicians [22]. The operator genospecial relativity presented in this chapter is mostly new.

A considerable number of applications of the isospecial relativity by

independent researchers have appeared in the literature (although no independent study on its structure has appeared at this writing – summer 1994 – to our knowledge). Some of them will be considered in this volume, and other in Vol. III.

From a mathematical viewpoint, the content of this chapter could be reduced to the presentation of the genospecial relativity because it contains as a particular case the isospecial one. However, for clarify we shall first present the Lie–isotopic theory and then pass to the more general Lie–admissible formulation.

8.2: ISOMINKOWSKIAN GEOMETRIZATION OF PHYSICAL MEDIA AND ITS ISODUAL

8.2.A: Isominkowski spaces and their isoduals. Consider an electromagnetic wave as in Fig. 8.2.1 which propagates first in empty space (exterior relativistic problem), then throughout our atmosphere (interior relativistic problem), and then returns to propagation in vacuum. As well known, the Minkowski space [4]

$$M(x,\eta,R): \quad x = \{r, x^4\}, x^4 = c_0 t, \quad \eta = \text{diag.} (1, 1, 1, -1), \quad r \in E(r,\delta,R), \quad (8.2.1a)$$

$$x^2 = x^\mu \, \eta_{\mu\nu} x^\nu = x^1 \, x^1 + x^2 \, x^2 + x^3 \, x^3 - x^4 \, x^4 \, \in \, R(n,+,\times), \quad \mu, \nu = 1, 2, 3, 4, \quad (8.2.1b)$$

where $E(r,\delta,R)$ is the three–dimensional Euclidean space, provides a *relativistic geometrization of the homogeneity and isotropy of empty space*. As such, it is exactly valid for exterior relativistic conditions.

The *isominkowski spaces* (see Sect. I–5.3) were submitted in ref. [8] to provide a *relativistic geometrization of the inhomogeneity and anisotropy of interior physical media*, such as light propagating in the interior conditions of Fig. 8.2.1. As indicated earlier, Earth's atmosphere is inhomogeneous because of the radial variation of the density and anisotropic because of the intrinsic angular momentum of Earth.

In their Class 1 version (that with isounits which are nowhere singular, bounded, Hermitean and positive–definite), the isominkowski spaces can always (although not necessarily[93]) be written in the diagonal form

$$\hat{M}(x,\hat{\eta},\hat{R}): \quad x = \{r, x^4\} = \{r, c_0 t\}, \quad r \in \hat{E}(r,\hat{\delta},\hat{R}), \quad \hat{\eta} = T \eta, \quad \hat{1} = T^{-1}, \quad (8.2.2a)$$

$$x^2 = (x^\mu \, \hat{\eta}_{\mu\nu} \, x^\nu) \hat{1} = (x^1 \, b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_3^2 x^3 - x^4 b_4^2 x^4) \hat{1} =$$

$$= (x^1 \frac{1}{n_1^2} x^1 + x^2 \frac{1}{n_2^2} x^2 + x^3 \frac{1}{n_3^2} x^3 - t \frac{c_0^2}{n_4^2} t) \hat{1} \, \in \, \hat{R}(\hat{n},+,*), \quad (8.2.2b)$$

$$T = \text{diag.} (b_1^2, b_2^2, b_3^2, b_4^2) > 0, \quad b_\mu > 0, \mu = 1, 2, 3, 4, \quad (8.2.2c)$$

where the quantities b_μ are assumed to be positive–definite and expressed in the

---

[93] We shall study in Chapter II–10 isominkowskian spaces of Class I with a *nondiagonal isometric* $\hat{\eta}$ which emerge rather naturally in the isotopies of the Dirac equation.

equivalent form $b_\mu = 1/n_\mu$ for reasons explained below.

Note that isospaces $\hat{M}(x,\hat{\eta},\hat{R})$ have the most general possible nonlinear–nonlocal–nonlagrangian structure (when projected in the original space) because the functional dependence of $\hat{\eta}$ remains completely unrestricted under isotopies. The isometric $\hat{\eta}$ can therefore depend on local coordinates x, velocities $\dot{x}$, accelerations $\ddot{x}$, density μ, temperature τ and any needed additional quantity, $\hat{\eta} = \hat{\eta}(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$.

ISOMINKOWSKIAN GEOMETRIZATION OF PHYSICAL MEDIA
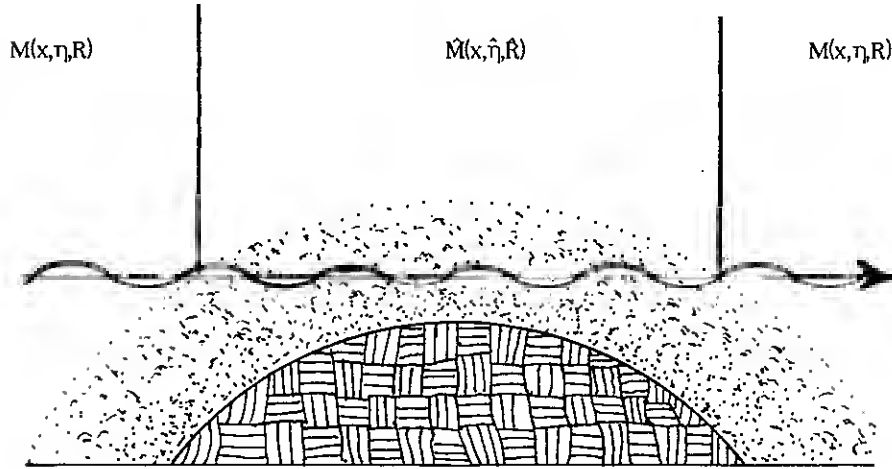
FIGURE 8.2.1. A schematic view of the geometrical foundations of the isospecial relativity, the isominkowskian representation of the propagation of electromagnetic waves through inhomogeneous and anisotropic media such as Earth's atmosphere. A central condition is the preservation of the same geometric axioms for both exterior and interior conditions, and the characterization of different physical conditions via *different realizations* of the same axioms. As pointed out in Sect. 1.8.1, this main feature is necessary in order to broaden the physical conditions admitted by the special relativity without abandoning its axioms, thus permitting a physical and mathematical unity of thought in the transition from exterior to interior conditions. Note that this geometric unity is not permitted by other approaches, such as the deformations of the Minkowski spaces without the inverse deformation of the unit, because such deformed spaces are no longer locally isomorphic to the original space. At any rate, the latter deformations permit a rather limited class of generalizations, while the isotopies are "directly universal", that is, they permit all infinitely possible generalizations of the original space (universality) directly in the x-frame of the observer (direct universality).

We also recall from Sect. 1.5.3 that, despite evident structural differences

between $M(x,\eta,R)$ and $\hat{M}(x,\hat{\eta},\hat{R})$, the joint liftings $\eta \to \hat{\eta} = T\eta$ and $I \to \hat{1} = T^{-1}$ imply that *all infinitely possible isominkowskian spaces are locally isomorphic to Minkowskian space*, $\hat{M}(x,\hat{\eta},\hat{R}) \approx M(x,\eta,R)$ [8,10]. Moreover, owing to the positive-definiteness of the isotopic element T, it is easy to see that $\hat{M}(x,\hat{\eta},\hat{R})$ and $M(x,\eta,R)$ coincide at the abstract level.

Note that the quantity

$$\hat{1} = \{ b_\mu^{-2} \} = \{ \hat{1}_s, \hat{1}_t \}, \quad \hat{1}_s = \text{diag.}(b_1^{-2}, b_2^{-2}, b_3^{-2}), \quad \hat{1}_t = b_4^{-2}, \quad (8.2.3)$$

is the *isounit of the isoseparation* x^2 *in isominkowskian spaces*. This implies the existence of four generally different isounits $\hat{1}_\mu = b_\mu^{-2}$ for each of the corresponding axes, resulting in the *space isounit* $\hat{1}_s$ and the *time isounit* $\hat{1}_t$.

When $\hat{M}(x,\hat{\eta},\hat{R})$ is projected in our space-time, we have the separation $x^2 = x^\mu \hat{\eta}_{\mu\nu} x^\nu$ which can be identically written as a *conventional* separation $\bar{x}^2$ in a Minkowski space $M(\bar{x},\eta,R)$ with local coordinates $\bar{x} = T^{\frac{1}{2}}x = (b_\mu x^\mu)$ (no sum)

$$x^2 = x^t \hat{\eta} x = x^t T \eta x = (T^{\frac{1}{2}} x)^t \eta (T^{\frac{1}{2}} x) = \bar{x}^t \eta x . \qquad (8.2.4)$$

The re-interpretation of $\bar{x}$ in our space-time can therefore be expressed via a change of the conventional units $I_\mu = 1$. We reach in this way the *isounits of the projection of the local coordinates in our space-time* (first identified in ref. [51])

$$\hat{1} = \{ \hat{1}_\mu \} = \{ b_\mu^{-1} \} = \{ \hat{1}_s, \hat{1}_t \}, \quad \hat{1}_s = \text{diag.} \{ b_1^{-1}, b_2^{-1}, b_3^{-1} \}, \quad \hat{1}_t = b_4^{-1}.$$
$$(8.2.5)$$

From now on, whenever considering the isounit $\hat{1}_\mu = b_\mu^{-2}$ we shall tacitly imply that we are in isospace $\hat{M}(x,\hat{\eta},\hat{R})$, while the use of the isounits $\hat{1}_\mu = b_\mu^{-1}$ will imply that we are working in the conventional space $M(x,\eta,R)$. As we shall see in the next chapter, the latter isounits have rather deep gravitational implications.

In our studies we shall also need the *isodual isominkowskian spaces* which, in their Class II version (that with the same topological properties of Class I except that the isounits are negative-definite), can be written in the diagonal form

$$\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d) : \quad x = \{r, x^4\} = \{r, c_o t\}, \quad r \in \hat{E}^d(r,\delta^d,\hat{R}^d), \quad \hat{\eta}^d = -\hat{\eta}, \quad \hat{1}^d = -\hat{1}, \quad (8.2.6a)$$

$$x^{2d} = (x^\mu \hat{\eta}^d_{\mu\nu} x^\nu) \hat{1}^d = (-x^1 b_1^2 x^1 - x^2 b_2^2 x^2 - x^3 b_3^2 x^3 + x^4 b_4^2 x^4) \hat{1}^d =$$

$$= (-x^1 \frac{1}{n_1^2} x^1 - x^2 \frac{1}{n_2^2} x^2 - x^3 \frac{1}{n_3^2} x^3 + t \frac{c_o^2}{n_4^2} t) \hat{1}^d \in \hat{R}^d(\hat{n}^d,+,\times^d), \quad (8.2.6b)$$

$$T = \text{diag.}(-b_1^2, -b_2^2, -b_3^2, -b_4^2) > 0, \, b_\mu^d = -b_\mu < 0, \quad \mu = 1, 2, 3, 4. \quad (8.2.6c)$$

Let us finally recall from Sect. I.5.3 that the isodual separation verifies the property

$$x^{2d} = (- x^{\mu} \, \hat{\eta}_{\mu\nu} \, x^{\nu}) \, (-1) \; \equiv \; x^{2} \; = \; (\, x^{\mu} \, \hat{\eta}_{\mu\nu} \, x^{\nu} \,) \, 1 \, , \qquad (8.2.7)$$

which, as we shall see, is at the basis of the isodual invariance of relativistic interior laws in a way fully parallel to the nonrelativistic case of the preceding chapter.

The difference between space and time inversions and isoduality should be noted. In the former case we have the inversion of the signs of the coordinates x = {r,x^4} → −x = {−r, −x^4} *by preserving the base field* R(n,+,×), that is, by preserving the basic units l_{μ} = +1. In the latter case, we preserve instead the coordinates, x = {r, x^4} → x = {r, x^4}, *but change the base field into its isodual* R^d(n^d,+,×^{d}), that is, by changing the sign of the units $l_{\mu}{}^d$ = −1.

As a result, space and time inversions occur within the same Minkowski space M(x,η,R), while isoduality implies the map into a new space, the isodual Minkowski space M^d(x,η^d,R^d). As we shall see in Ch. II.10, the map M(x,η,R) → M^d(x,η^d,R^d) in operator realization is equivalent to charge conjugation.

. In summary, the spaces important for our isotopic studies of systems with *reversible* center-of-mass trajectories are the following four: **conventional Minkowski spaces** M(x,η,R) for the exterior relativistic problem of particles; **isodual Minkowski spaces** M^d(x,η^d,R^d) for the exterior relativistic problem of antiparticles; **isominkowski spaces** M̂(x,η̂,R̂) for the interior relativistic problem of particles; and **isodual isominkowski spaces** M̂^{d}(x,η̂^{d},R̂^{d}) for the interior relativistic problem of antiparticles.

Note that the isodual Minkowski spaces do not exist in the literature on the special relativity [1–5,23–25]. They are structurally new because they require a generalization of the unit for their very definition.

The additional spaces needed for genotopic studies (i.e., those for systems in *irreversible* conditions) will be studied in Sect. II.8.6.

8.2.B: Characteristic quantities of physical media.

The Minkowski space provides a "direct geometrization" of the constancy of the speed of light c_0 in vacuum as well as of the homogeneity and isotropy of the vacuum. that is, a geometrization directly via the metric or, equivalently, via the line element

$$ds^2 = dr^k \, dr^k - dx^4 \, dx^4 = dr^k \, dr^k - dt \, c_0{}^2 \, dt. \qquad (8.2.8)$$

The quantities representing such geometrization, called *characteristic quantities of the vacuum*, are therefore the values I_{μ} = I for all space and time directions.

The isominkowskian spaces provide a direct geometrization of the locally varying speed of light as well as of the inhomogeneity and anisotropy of physical media, that is, a geometrization directly via the isometric. This is achieved via the quantities b_{μ} (or, equivalently, η_{μ}) which are called the *characteristic functions of the physical medium.*

In fact, the locally varying character of the speed of light is also directly

expressed by the isometric $\hat{\eta}$ in diagonal form with line isoelement

$$ds^2 = (= dr^k \, b_k^2 \, dr^k - dt \, c^2 \, dt \,)\hat{1} = (\, dr^k \, \frac{I}{n_k^2} \, dr^k - dt \, \frac{c_o^2}{n_4^2} \, dt \,)\hat{1}, \qquad (8.2.9)$$

The quantity n_4 can therefore be interpreted as the conventional index of refraction of light which is known to have a rather complex local dependence on the local density μ, temperature τ, etc., $n_4 = n_4(\mu, \tau, ...)$.

The inhomogeneity of physical media can be represented, e.g., via an explicit dependence of the characteristic functions on the local density μ, while their anisotropy can be represented via different values among the b's, the factorization of a preferred direction of the medium as in Finsler's geometry, and other means.

A first intuitive understanding of the isominkowski spaces can be reached by noting that the functions n_μ essentially extend the local index of refraction n_4 to all space–time components. This type of "space–time symmetrization" of the index of refraction can also be reached by subjecting the coordinates $\bar{x} = \{r, x^4/n_4\}$ to ordinary Lorentz transformations, although their appropriate derivation is via the isolorentz transforms (see below).

When the local behaviour is needed at one given interior point, one must use the full nonlinear–nonlocal dependence of the b's. This is illustrated by the local speed of light at one given point of our atmosphere $c = c(\mu, \tau, ...) = c_0/n_4(\mu, \tau, ...)$.

When the global behaviour throughout a given physical medium is requested, the characteristic functions can be averaged into constants, b_μ° = Aver. (b_μ), or n_μ° = Aver. (n_μ), $\mu = 1, 2, 3, 4$, in which case they are called *characteristic constants of the medium.* This is evidently the case for the a*verage speed of light* throughout our atmosphere $c = c_0/n_4^\circ$, in which case n_4° is the a*verage index of refraction.*

As we shall see in Vol. III, the available experimental verifications of the isominkowskian geometrization of physical media are mostly external, that is, they measure a given effect throughout the entire medium considered, in which case the characteristics constants b_μ° are applicable.

By recalling that physical media are generally opaque to light, the isotopies $M(x,\eta,R) \rightarrow \hat{M}(x,\hat{\eta},\hat{R})$ have the additional meaning of *extending to all physical media the geometric structure of light in vacuum.* In this latter case, the characteristic constant b_4° geometrizes the density of a given medium, while the constants b_k° geometrize the internal nonlinear–nonlocal effects.

It is evident that different physical media necessarily require different isounits $\hat{1}$ because of their different sizes, densities, temperature, chemical composition, etc. This occurrence is similar to the need of infinitely possible Riemannian spaces in general relativity in order to represent the infinitely possible astrophysical masses.

By no means the above outline exhaust all possible physical meanings of the characteristic quantities. Another application of the isominkowskian geometry particularly important for these volumes is the geometrization of the physical medium inside a hadron as depicted in Fig. 8.2.2.

The above outline refers, specifically, to physical media made up of matter. By recalling that all physical quantities change sign under isoduality, the *characteristic quantities of physical media composed of antimatter* in isodual isominkowskian representation are the quantities $b^d_k = - b_k$ for space and $b_4{}^d = - b_4$ for time.

8.2.C: Isominkowskian geometry and its isodual.
The isominkowskian and isodual isominkowskian geometries are the geometries of isospaces $\hat{M}(x,\hat{\eta},\hat{R})$ and $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$, respectively.

These new geometries coincide at the abstract level with the conventional Minkowskian geometry and its isodual. However, they admit novel characteristics in local realization which are absent in the conventional geometries.

ISOMINKOWSKIAN GEOMETRIZATION OF HADRONS
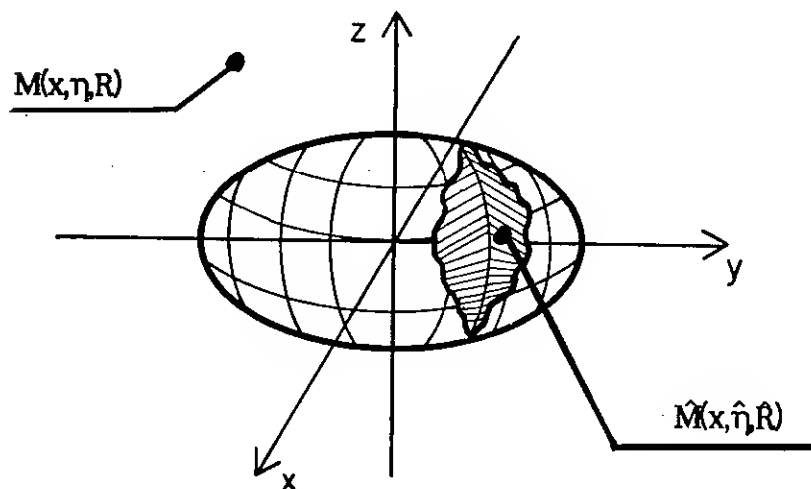
FIGURE 8.2.2. A schematic view of a fundamental application of the isominkowskian geometry for strong interactions, the geometrization of the physical media inside hadrons, namely, the hyperdense superposition of the wavepackets of the constituents called hadronic media [47], which are also manifestly inhomogeneous and anisotropic much along Earth's atmosphere. The lifting $M(x,\eta,R) \rightarrow \hat{M}(x,\hat{\eta},\hat{R})$ therefore represents the *departure* from empty space in the interior of hadrons. Relativistic hadronic mechanics on $\hat{M}(x,\hat{\eta},\hat{R})$ is constructed to represents the *departures* from motion in empty space experienced by a constituent when moving within a hyperdense hadronic medium. In this case the

time characteristic constant b^o_4 provides a geometrization of the *density* of the hadron considered, while the space characteristic constants b^o_k provide a representation of the *actual nonspherical shape* of the hadron considered, as well as other characteristics studied in Vol. III (such as an average of the internal, nonlinear, velocity–dependent effects). Recall that all hadrons essentially have the same charge radius of about 1fm but different masses and, thus, different densities. Different hadrons are therefore expected to have different isounits.[94] As we shall see in Vol. III, the above isominkowskian characterization of hadrons appears to be confirmed by all available experimental evidence and permits truly novel predictions, such as the prediction of a subnuclear energy called "hadronic energy" [53] because originating in the structure of the individual hadrons, rather than nuclei, atoms or molecules.

A basic property of the isominkowski geometry is that it is *isoflat*, that is, it verifies the axiom of flatness in isospace $\hat{M}(x,\hat{\eta},\hat{R})$. However, when projected in the original space $M(x,\eta,R)$, the isominkowskian geometry admits a curvature much more general than that of the Riemannian geometry, evidently because the metric g of the latter depends only on local coordinates, $g = g(x)$, while that of the former has an arbitrary, integro–differential dependence $\hat{\eta} = \hat{\eta}(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$.

The above occurrence is essentially due to the fact that any deviation from flatness (curvature) is compensated by an inverse deviation of the related unit, thus preserving the original geometric characteristics of flatness.

The space component of the isominkowskian geometry is the isoeuclidean geometry of the preceding chapter. In the following we shall therefore limit ourselves to an illustration of the main features of the isominkowskian geometry in the isohyperbolic (3–4)–plane with isoinvariant

$$x^2 = [x^3 b_3^2(x, \dot{x}, \ddot{x}, \mu, \tau, ..)x^3 - x^4 b_4^2(x, \dot{x}, \ddot{x}, \mu, \tau, ...)x^4]\hat{1} = \text{inv}. \quad (8.2.10)$$

The isotopic image $\tilde{v}$ of a hyperbolic angle (speed) v is given by $\tilde{v} = vb_3b_4$, as provable via the use of the isorepresentations of $S\hat{O}(1.1)$[95], with corresponding *isohyperbolic functions* and related properties

$$\text{isosinh } \tilde{v} = b_4^{-1} \sinh(v\, b_3\, b_4), \quad \text{isocosh } \tilde{v} = b_3^{-1} \cosh(v\, b_3\, b_4), \quad (8.2.11a)$$

---

[94] This point illustrates the rather drastic departures of hadronic mechanics from other trends, such as seeking "universal constants". In fact, as nicely illustrated by Makhaldiani in the Foreword of Vol. I, the conventional constants of quantum mechanics are turned by hadronic mechanics into *variables with infinitely possible values*.

[95] This can be easily proved by turning the isotopic element $T = \text{Diag.} (b_1^2, b_2^2)$ of $S\hat{O}(2)$ in Sect. 11.6.5 into the noncompact form $T = \text{Diag.} (b_1^2, -b_2^2)$, in which case $S\hat{O}(2)$ is mapped precisely into $S\hat{O}(1.1)$, including the mapping of isotrigonometric into isohyperbolic functions. Note finally that $S\hat{O}(2)$ and $S\hat{O}(1.1)$, as well as isotrigonometric and isohyperbolic functions are unified by the isotopies of Class III.

$$b_3{}^2 \, \text{isocosh}^2 \, \tilde{v} \; - \; b_4{}^2 \, \text{isosinh}^2 \, \tilde{v} \; = \; \cosh^2 \tilde{v} \; - \; \sinh^2 \tilde{v} \; = \; 1. \quad (8.2.11b)$$

The above structures illustrates that the correct treatment of the isominkowskian geometry requires the *isospecial functions* of Ch. I.6. In particular, the isoeuclidean subcase requires the use of the *isotrigonometry* (App. I.6.A), the isospherical coordinates (Sect. II.5.5), the isospherical harmonics (Sect. II.6.6), etc., while the space–time component of the isogeometry requires the corresponding isohyperbolic extension. The isodual geometries then require the isodual images of the above special isofunctions.

8.2.D: The isolight cone and its isodual. As recalled in Sect. I.8.1, one of the reasons for the inapplicability of the special relativity for interior conditions is the loss of the conventional *light* cone in vacuum. A most important property of the isominkowskian geometry is the reconstruction in isospace of the exact light cone, called *isolight* cone, for light propagating within transparent media with a locally varying speed.

Consider expression (8.2.10) specialized to the form $x^2 = 0$. It is easy to see that, when projected in the original space $M(x,\eta,R)$, this expression is not a cone, e.g., because of the general loss of straight generating lines. However, when written in isospace $\hat{M}(x,\hat{\eta},\hat{R})$, the expression $x^2 = 0$ is indeed a perfect light cone for reasons similar to the fact the the isosphere is a perfect sphere in isospace. In fact, the deformations of the characteristic quantities of the original cone, $1_\mu = 1$ $\rightarrow 1_\mu = b_\mu{}^2$ are compensated by the inverse deformations of the related units $1_\mu = 1 \rightarrow \hat{1}_\mu = b_\mu{}^{-2}$.

To compute the characteristic angle of the isolight cone, we use the isotrigonometry yielding

$$\Delta x \; = \; \hat{D} \, b_4 \sin\tilde{\alpha}, \qquad \Delta t \; = \; \hat{D} \, b_3 \sin\tilde{\alpha}, \qquad\qquad (8.2.12a)$$

$$\Delta x \, / \, \Delta t \; = \; D \, b_4 \sin\tilde{\alpha} \, / \, D \, b_3 \sin\tilde{\alpha} \; = \; (b_4 \, / \, b_3) \, c_0, \quad \hat{\alpha} \; = \; \alpha \, b_3 \, b_4, \quad (8.2.12b)$$

where $\hat{D}$ is the isohypothenuse, from which tang $\tilde{\alpha} = c_0 = $ const., namely, *the characteristic angles of the conventional light cone and its isotopic generalization coincide.*

We see in this way that the isominkowskian geometry has the remarkable capability of reconstructing not only the exact character of the light cone for variable speeds, but also the conventional characteristic angle in vacuum.

This occurrence illustrates the overall unity of physical and mathematical thought achieved by the isominkowskian geometry because it allows the use of the *same* light cone for motion in vacuum with constant speed c_0 and motion in interior conditions with variable speed $c = c_0 b_4 = c(x, \mu. \tau, \ldots)$.

As in the conventional case, we have the three possibilities $x^2 < 0$, $x^2 = 0$ and $x^2 > 0$ which characterize *isotime–like, isolight–like and isospace–like vectors,* that is, vectors contained inside the *isolight cone,* on its surface or

outside it, respectively.

The *isodual isolight cone* is characterized by the change of the sign of all quantities, including their units. In fact, under isoduality, the isospace $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$ preserves the original local variables x^μ but changes the sign of their isounits 1_μ → $1_\mu^d = -b_\mu^{-2}$.

LIGHT CONE IN EMPTY SPACE	LIGHT "CONE" IN PHYSICAL MEDIA	ISOLIGHT CONE IN PHYSICAL MEDIA
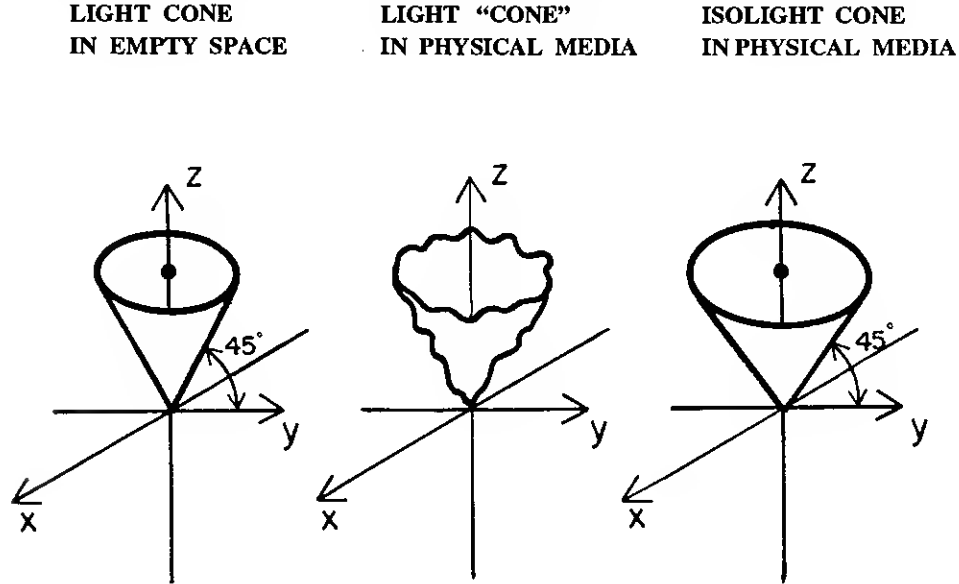
FIGURE 8.2.3: A rather natural expectation is that a locally varying speed of light implies the loss of the axioms of the special relativity. The isominkowskian geometry disproves this expectation because it reconstructs in isospace all essential features of the Minkowskian geometrization in vacuum, including the light cone and its characteristic angle this figure, thus permitting the preservation of the axioms of the special relativity at the abstract level.

8.2.E: Relativistic isoplane waves. Recall that a characterization of electromagnetic waves in the homogeneous and isotropic vacuum is given by the the familiar plane-waves

$$\psi(x) = N\, e^{i(k^\mu \eta_{\mu\nu} x^\mu)} = e^{i(k\,r - Et)} = e^{ik^1 r^1} e^{ik^2 r^2} e^{ik^3 r^3} e^{-iEt}. \quad (8.2.13)$$

The isominkowskian geometrization of plane waves propagating within physical media is given by the so-called *isoplane waves*

$$\hat{\psi}(x) = \hat{N}\, \hat{e}^{i(k^\mu \eta_{\mu\nu} x^\mu)} = \hat{N}\, e^{i(k^\mu \hat{\eta}_{\mu\nu} x^\mu)} = \hat{N}\, e^{i(kT_s r - E T_t t)}, \quad (8.2.14)$$

characterized by the lifting of the exponentiation of Eq.s (8.2.13) into the *isoexponentiation* (Sect. I.4.3).

In fact, by recalling the expression $\hat{e}^X = (e^{XT})\hat{1}$, $\hat{1} = T^{-1}$ and that each component of the isominkowskian coordinates has its own isounit $\hat{1}_\mu = b_\mu^{-2}$, expression (8.2.14) originates from the property

$$\hat{e}^{\,i(\,k^\mu\,\eta_{\mu\nu}\,x^\nu)} = \{\hat{e}_1^{\,ik^1x^1}\}*_1\{\hat{e}_2^{\,ik^2x^2}\}*_2\{\hat{e}_2^{\,ik^3x^3}\}*_3\{\hat{e}_4^{\,-ik^4x^4}\} =$$

$$= \{e^{ik^1b_1^2x^1}\,e^{ik^2b_2^2x^2}\,e^{ik^3b_3^2x^3}\,e^{-ik^4b_4^2x^4}\}\,b_4^{-2} = \{e^{ik^\mu\,\hat{\eta}_{\mu\nu}\,x^\nu}\}\,b_4^{-2}, \qquad (8.2.15)$$

where the factor b_4^{-2} is absorbed in the isonormalization coefficient $\tilde{N}$. As we shall see in the next section, the same results are obtained in the matrix representation of translations

The isoplanewaves therefore represent the inhomogeneity and anisotropy of the medium directly with their structure. Note the uniqueness of form (8.2.14) which is admitted as solutions by all isorelativistic equations, as we shall see in Ch. 11–10.

We finally note that *the isoplane waves are isoselfdual*, that is, the exponent in Eq. (8.2.14) is invariant under isoduality

$$k^\mu\,\hat{\eta}_{\mu\nu}\,x^\mu \;\rightarrow\; k^{\mu d}\,\hat{\eta}^d_{\mu\nu}\,x^\mu \equiv k^\mu\,\hat{\eta}_{\mu\nu}\,x^\mu\,. \qquad (8.2.16)$$

where we have preserved the local coordinates as requested by the isospace $\hat{M}(x,\hat{\eta}^d,\hat{R}^d)$.

The above property essentially represents the known fact that *electromagnetic waves are identically emitted by matter and antimatter.*

8.2.F: Nonrelativistic limits of isominkowskian spaces.
Consider the Minkowskian separation in the form $x^2 = r^k r^k - tc_0^2 t = D^2$. It is then well known that at nonrelativistic limit we have the reduction

$$\mathrm{Lim}_{D/c_0^2 \to 0}\, M(x,\eta,R) \;=\; E(t,R_t) \times E(r,\delta,R)\,. \qquad (8.2.17)$$

In particular, the four–dimensional unit $1 = $ diag. (1, 1, 1, 1) decouple into the space unit $1_3 = $ diag. (1, 1, 1) and the time unit 1.

In Ch. VI of ref. [11] we have studied the corresponding case for the isoseparation $dr^k b_k^2 dr^k - tc_0^2 b_4^2 t = D^2$ and shown that at the nonrelativistic limit we have the isotopic reduction

$$\mathrm{Lim}_{D/c_0^2 \to 0}\, \hat{M}(x,\hat{\eta},\hat{R}) \;=\; \hat{E}(t,\hat{R}_t) \times \hat{E}(r,\delta,\hat{R})\,. \qquad (8.2.18)$$

In particular, the four-dimensional isounit $\hat{1} = \{\hat{1}_s, \hat{1}_t\}$ becomes decoupled into the space-isounit $\hat{1}_s$ = diag. $(b_1{}^2, b_2{}^2, b_3{}^2)$ and the time isounit $\hat{1}_t = b_4{}^{-2}$.

The above reduction is important for the mutual compatibility of the various branches of hadronic mechanics, as we shall see.

8.3: ISOLORENTZ SYMMETRY, ISOPOINCARÉ SYMMETRY AND THEIR ISODUALS

We now study the fundamental symmetries of the isospecial relativity, the isolorentz and isopoincaré symmetries of Kadeisvili Class 1, first from an "abstract" viewpoint here intended without any reference whether representing classical or operator realizations. The classical realizations have been studied in detail in monograph [11]. The operator realizations will be studied in the next section.

8.3.A: Abstract isolorentz symmetry and its isodual. Consider the separation in Minkowski space

$$x^2 = x^t\, \eta\, x = x^\mu\, \eta_{\mu\nu}\, x^\nu = \text{inv.}\,,\quad x = \{\,r, x^4\,\},\quad \eta = \text{diag.}\,(1, 1, 1, -1)\,. \quad (8.3.1)$$

Its simple invariance group, the six-dimensional *Lorentz group* L = L(3.1) in (3+1)-dimension [1][96] is defined by the conditions

$$x' = \Lambda\, x\,,\qquad \Lambda^t\, \eta\, \Lambda = \eta\,,\quad \text{Det}\,\Lambda = \pm 1\,, \qquad (8.3.2)$$

and can be constructed via: the (ordered set of) parameters $w = \{w_k\} = \{\theta, v\}$, $k = 1, 2, ..., 6$, given by the Euler's angles θ and boosts parameter v; the (ordered set of) generators $X = \{X_k\} = \{M_{\mu\nu}\}$, $\mu, \nu = 1, 2, 3, 4$, in their fundamental representation

$$M_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{31} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$(8.3.3a)$$

$$M_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad M_{20} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad M_{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$(8.3.3h)$$

---

[96] The contemporary literature on the Lorentz and Poincaré symmetries is so vast to discourage even a partial outline. One may consult, e.g., ref.s [23–25].

plus the space–time inversions $\pi x = \{-r, x^4\}$ and $\tau x = \{r, -x^4\}$.

From condition (8.3.2) we have

$$\eta_{\mu 4} \, \Lambda^\mu_4 \, \Lambda^\mu_4 = \eta_{44} , \qquad (8.3.4)$$

Thus,

$$(\Lambda^4_4)^2 = 1 + \sum_{k=1,2,3} (\Lambda^k_k)^2 \geq 1 , \qquad (8.3.5)$$

from which we have the relations

$$\Lambda^4_4 \geq +1 \quad \text{or} \quad \Lambda^4_4 \leq -1 . \qquad (8.3.6)$$

This establishes the known property that the Lorentz group is not connected because it admits the following four disjoint components: $L_+^\uparrow$: det $\Lambda = +1$, $\Lambda^4_4 \geq +1$; $L_-^\uparrow$: Det $\Lambda = -1$, $\Lambda^4_4 \geq 1$; $L_-^\downarrow$: Det $\Lambda = -1$, $\Lambda^4_4 \leq -1$; $L_+^\downarrow$: Det $\Lambda = 1$, $\Lambda^4_4 \leq -1$.

The primary groups which can be constructed from the above components are the *proper orthochronous Lorentz group* $L_+^\uparrow \approx SO(3.1)$, the *orthochronous Lorentz group* $L^\uparrow = L_+^\uparrow \cup L_-^\uparrow$; and the *proper Lorentz group* $L_0 = L_+^\uparrow \cup L_+^\downarrow$ (see ref. [23–25] for more details).

The explicit transforms of $L_+^\uparrow$ are reducible to the Lorentz boosts (or rotations in the (3, 4)–Minkowski plane)

$$\begin{aligned}
& x'^1 = x^1, \quad x'^2 = x^2, \\
& x'^3 = x^3 \cosh v - x^4 \sinh v = \gamma (x^3 - \beta x^4), \\
& x'^4 = - x^3 \sinh v + x^4 \cosh v = \gamma (x^4 - \beta x^3),
\end{aligned} \qquad (8.3.7)$$

where

$$\cosh v = \gamma = (1 - \beta^2)^{-\frac{1}{2}} , \sinh v = \beta \gamma , \quad \beta = v / c_0 , \qquad (8.3.8)$$

plus the conventional rotations in the (1, 2, 3)–space.

Consider now the broadest possible Hermitean, real–valued, nonsingular and signature–preserving (Class I) generalizations of the Minkowski separation due to inhomogeneity, anisotropy, curvature, or any other physical reason (see Vol. III)

$$x^\mu \, \eta_{\mu\nu} \, x_\nu \rightarrow x^\mu \, \hat\eta_{\mu\nu}(x, \dot x, \ddot x, \mu, \tau, ...) \, x^\nu = \text{inv.}, \quad \hat\eta = \hat\eta^\dagger , \text{sig.} \, \hat\eta = \text{sig} \, \eta. \quad (8.3.9)$$

Their symmetries, first derived by this author in ref. [8], can be constructed via one of the most important results of the isotopies, the *Fundamental Theorem on Isosymmetries* of Sect. I.4.6, according to the following steps:

Step 1: Identification of the isotopic element T via the factorization $\hat\eta = T\eta$ which, under the assumed conditions, can always be diagonalized in the form

$$T = \text{diag.} (b_1^2, b_2^2, b_3^2, b_4^2) = T^\dagger > 0, \quad b_\mu > 0 . \qquad (8.3.10)$$

The fundamental isounit of the theory is therefore

$$ \hat{1} = T^{-1} = \text{diag.}\, (b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}) \qquad \qquad)8.3.11) $$

Step 2: Lifting of the conventional fields into the isofields $\hat{R}(\hat{n},+,*)$ with isonumbers $\hat{n} = n\hat{1}$ and isounit $\hat{1}$. This implies first in the isoexponentiation the lifting of the parameters $w \rightarrow \hat{w} = w\hat{1}$, although their isoproduct with any quantity Q remains the conventional one, $\hat{w}*Q = wT^{-1}TQ = wQ$. Second, we have the generalization of the parameters in the explicit form of the isotransforms $w \rightarrow \tilde{w} = \{\tilde{\theta}, \tilde{v}\}$, where $\tilde{\theta}_1 = \theta_1 b_2 b_3$, $\tilde{v} = v b_3 b_4$, etc.

Step 3: Construction of the isospaces with isometric $\hat{\eta}$, which are given by the isominkowskian spaces of Class 1 $\hat{M}(x,\hat{\eta},\hat{R})$. This implies the use of the isospecial functions with isounit $\hat{1}$ as recalled earlier.

Step 4: Explicit construction of the isotopies $\hat{L}$ of L, called *isolorentz symmetry*. They are given by the isotransformations

$$ x' = \hat{\Lambda}(\hat{w}) * x = \hat{\Lambda}(\hat{w})T x = \tilde{\Lambda}(w) x, \qquad \hat{\Lambda} = \tilde{\Lambda}\hat{1}, \qquad (8.3.12) $$

verifying the conditions

$$ \hat{\Lambda}^\dagger \hat{\eta} \hat{\Lambda} = \hat{\Lambda} \hat{\eta} \hat{\Lambda}^\dagger = \hat{1} \hat{\eta} \hat{1}, \quad \text{or} \quad \tilde{\Lambda}^\dagger \hat{\eta} \tilde{\Lambda} = \tilde{\Lambda} \hat{\eta} \tilde{\Lambda}^\dagger = \hat{\eta}, \quad (8.3.13a) $$

$$ \hat{\text{Det}}\, \hat{\Lambda} = [\text{Det}\,(\hat{\Lambda}\, T)]\hat{1} = \pm\hat{1}, \quad \text{or} \quad \text{Det}\, \tilde{\Lambda} = \pm 1, \qquad (8.3.13b) $$

(where we have used definition (1.6.3.19) for the isodeterminant). The property

$$ \hat{\eta}_{\mu\nu}\, \tilde{\Lambda}^\mu{}_4\, \tilde{\Lambda}^\nu{}_4 = \hat{\eta}_{44}, \qquad (8.3.14) $$

then implies the relation

$$ (\tilde{\Lambda}^4{}_4)^2 = 1 + b_4^{-2} \sum_{k=1,2,3} (\tilde{\Lambda}^k{}_k)^2 b_k^2 \geq 1, \qquad (8.3.15) $$

from which we have

$$ \tilde{\Lambda}^4{}_4 \geq +1, \text{ or } \tilde{\Lambda}^4{}_4 \leq -1. \qquad (8.3.16) $$

in full analogy with the conventional case

We reach in this way the important result that the isotopies (of Class 1) do not alter the topological structure of the Lorentz groups. In fact, $\hat{L}$ is not connected and has the following four disjoint components: $\hat{L}_+{}^\uparrow$: $\text{Det}\tilde{\Lambda} = +1$, $\tilde{\Lambda}^4{}_4 \geq +1$; $\hat{L}_-{}^\uparrow$: $\text{Det}\tilde{\Lambda} = -1$, $\tilde{\Lambda}^4{}_4 \geq +1$; $\hat{L}_-{}^\downarrow$: $\text{Det}\tilde{\Lambda} = -1$, $\tilde{\Lambda}^4{}_4 \leq -1$; $\hat{L}_+{}^\downarrow$: $\text{Det}\tilde{\Lambda} = 1$, $\tilde{\Lambda}^4{}_4 \leq -1$.

Some of the most important isogroups which can be constructed with the above isocomponents then are the *proper orthochronous isolorentz group* $\hat{L}_+{}^\uparrow \approx S\hat{O}(3.1)$, the *orthochronous isolorentz group* $\hat{L}^\uparrow = \hat{L}_+{}^\uparrow \cup \hat{L}_-{}^\uparrow$; and the *proper isolorentz group* $\hat{L}_0 = \hat{L}_+{}^\uparrow \cup \hat{L}_+{}^\downarrow$.

The connected component $\hat{L}_+ = S\hat{O}(3.1)$ can be expressed via the original parameters and generators in terms of isoexponentiations (Sect. 1.4.3)

$$\hat{\Lambda}(\hat{w}) = \prod{}^{*}_{k=1,2,...,6} \hat{e}^{\, i\, X_k * \hat{w}_k} = \{ \prod_{k=1,2,...,6} e^{\, i\, X_k\, T\, w_k} \} \hat{1}, \qquad (8.3.17a)$$

$$\tilde{\Lambda}(w) = \prod_{k=1,2,...,6} e^{\, i\, X_k\, T\, w_k}, \qquad (8.3.17b)$$

whose closure to a finite (and actually six-dimensional) structure is ensured by the *isotopic Baker–Campbell–Hausdorff theorem* I.4.5.1. Note that Eq.s (8.3.17) have no unknown because the isotopic element T is known from the deformed metric (8.3.9).

The discrete part is characterized by the *space–time isoinversions* (see Ch. II.6)

$$\hat{\pi} * x = \pi x = \{-\mathfrak{r}, x^4\}, \quad \hat{\tau} * x = \tau x = \{\mathfrak{r}, -x^4\}, \hat{\pi} = \pi\hat{1}, \hat{\tau} = \tau\hat{1} \quad (8.3.18)$$

In particular, the space isoinversion is contained in $\hat{L}_{-}{}^{\uparrow}$; the time isoinversion is contained in $\hat{L}_{-}{}^{\downarrow}$; and the total isoinversion is contained in $\hat{L}_{+}{}^{\downarrow}$.

Under sufficient continuity, boundedness and regularity conditions for T (which are all verified for Class I), the convergence of infinite series (8.3.17) is ensured by the original convergence, thus permitting the explicit calculation of the symmetry isotransforms in the needed explicit, finite form.

The space components, the isorotations, have been studied in Ch. II.6. The additional transforms, called *general isolorentz boosts*, were computed for the first time in ref. [8], and can be written $x' = \hat{\Lambda}(\tilde{w})*x = \Lambda(\tilde{w})x$ with explicit form

$$x'^1 = x^1, \quad x'^2 = x^2, \qquad (8.3.19a)$$

$$x'^3 = b_3^{-1}\{ x^3 b_3^2 \text{ isocosh } \tilde{v} - x^4 b_4^2 \text{ isosinh } \tilde{v} \} =$$

$$= x^3 \cosh(v\, b_3\, b_4) - x^4 b_3^{-1} b_4 \sinh(v\, b_3\, b_4) = \hat{\gamma}(x^3 - b_3^{-1} b_4 \hat{\beta} x^4), (8.3.19b)$$

$$x'^4 = b_3\{-x^3 \text{ isosinh } \tilde{v} + x^4 \text{ isocosh } \tilde{v}\} =$$

$$= -x^3 b_3 b_4^{-1} \sinh(v\, b_3\, b_4) + x^4 \cosh(v\, b_3\, b_4) = \hat{\gamma}(x^4 - b_3 b_4^{-1}\hat{\beta} x^3), (8.3.19c)$$

where

$$\beta^2 = v^k v^k / c_0^2, \quad \hat{\beta}^2 = v^k b_k^2 v^k / c_0 b_4^2 c_0 = v * v / c^2, \qquad (8.3.20a)$$

$$\cosh(v\, b_1\, b_2) = \hat{\gamma} = (1 - \hat{\beta}^2)^{-\frac{1}{2}}, \quad \sinh(v\, b_1\, b_2) = \hat{\beta}\,\hat{\gamma}. \qquad (8.3.20b)$$

The *restricted isolorentz transforms* are given by the isorotations and isoboosts for *constant* characteristic quantities b°_μ.

The verification of the invariance of generalized separation (8.3.9) by isotransforms (8.3.19) is instructive. Numerous applications to nuclear physics, particle physics, gravitation and other fields will be studied in the rest of these volumes.

Step 5: Construction of the isocommutation relations of the Lie–isotopic

algebra $\hat{L}_+^\uparrow \approx s\hat{o}(3.1)$ also computed for the first time in ref. [8]. They can be derived via the methods for the transition from a Lie–isotopic group to its corresponding isoalgebra (Ch. 11.4), and can be written

$$s\hat{o}(3.1): \quad [\,M_{\mu\nu}\,\hat{,}\,M_{\alpha\beta}\,] = M_{\mu\nu}\, T\, M_{\alpha\beta} - M_{\alpha\beta}\, T\, M_{\mu\nu} =$$

$$= \hat{\eta}_{\nu\alpha}\, M_{\beta\mu} - \hat{\eta}_{\mu\alpha}\, M_{\beta\nu} - \hat{\eta}_{\nu\beta}\, M_{\alpha\mu} + \hat{\eta}_{\mu\beta}\, M_{\alpha\nu} \,, \qquad (8.3.21)$$

thus establishing the local isomorphism, $s\hat{o}(3.1) \sim so(3.1)$ from the property of Class I: sign $\hat{\eta} \equiv$ sig η [8,11].

The isocenter of $s\hat{o}(3.1)$ is given by [loc. cit.]

$$C^{(0)} = 1, \qquad C^{(1)} = \tfrac{1}{2} M_{\mu\nu}\, T\, M^{\mu\nu} = M * M - N * N\,, \qquad (8.3.22a)$$

$$C^{(3)} = \tfrac{1}{2}\, \epsilon^{\mu\nu\rho\sigma}\, M_{\mu\nu}\, T\, M_{\rho\sigma} = - M * N, \qquad (8.3.22b)$$

$$M = \{\, M_{12}, M_{23}, M_{31}\,\}, \quad N = \{\, M_{01}, M_{02}, M_{03}\,\}\,. \qquad (8.3.22c)$$

The following properties should be kept in mind:

A) The isotopies leave completely unrestricted the functional dependence of the isometric $\eta(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$, i.e., of the characteristic functions $b_\mu(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$;

B) While the Lorentz group is linear, local and canonical, its isotopic covering is nonlinear, nonlocal and noncanonical in its general form (8.3.17), (8.3.19) when projected in the original space $M(x,\eta,R)$, and is isolinear, isolocal and isocanonical in isospace $\hat{M}(x,\hat{\eta},\hat{R})$.

C) For the restricted case in which the characteristic quantities are constants b°_μ, the isolorentz group becomes linear and local, but remains noncanonical.

D) The final results are remarkably simple. In fact, the explicit form of the symmetry isotransformations of generalized separation (8.3.9) are merely given by plotting the values b_μ in Eq.s (8.3.19) without any need of further calculation; and

E) The isolorentz symmetry is "directly universal", that is, it applies for all conceivable (signature preserving) deformations $\hat{\eta} = T\eta$ of the Minkowski metric η (universality), without any need to change the coordinates (direct universality).[97]

Note also the appearance of the modulus in the characterization of $\hat{\gamma}$, Eq.s (8.3.20) This is due to the fact that in the special relativity v^2/c_0^2 is always smaller than one, thus requiring no modulus, while in the isospecial relativity the

---

[97] The isolorentz symmetry of the test is restricted to Class I. We should here note that: the isosymmetry of Class III permits the removal of the signature–preserving restriction; that of Class IV permits the introduction of *singular* isotopic elements and isounits; while that of Class V allows the introduction of fundamentally novel realizations of the Lorentz group, such as those defined with respect to an isounit which is a lattice, or a discontinuous function, or a distribution.

quantity $v*v/c^2 = v_k b_k{}^2 v_k/c_0 b_4{}^2 c_0$ can be smaller as well as bigger than one, as we shall see shortly.

The nontriviality of the isotopies is first expressed by their nonlinear–nonlocal–noncanonical character which implies the expectation of numerically different predictions, as we shall see.

Note also that the quantity embedding all nonlinear–nonlocal–noncanonical terms, the isounit $\hat{1}$, is evidently invariant because it is the unit of the isoalgebra $s\hat{o}(3.1)$, and therefore we have for all possible generators $[\hat{1}, X_k] = \hat{1}*X_k - X_k*\hat{1} = X_k - X_k \equiv 0$.

The reader should finally note that the natural appearance of the isoparameters $\tilde{v} = v b_3 b_4$ from the isoexponentiations (8.3.17) in a way completely independent from the isohyperbolic calculus, thus confirming the overall unity of the isotopies.

As recalled earlier in this volume, the most general relationship between the Lorentz and isolorentz algebras (Ch. I.4) is characterized by *nonunitary transforms* which evidently imply a deformation of the weight structure of the original symmetry. A particular relationship between L and $\hat{L}$ preserving the original weights is given by the *Klimyk's rule* (Lemma I.4.7.5).

In this section we introduce a third relationship which is particularly useful for all Lie algebras of orthogonal groups and their isotopic images.

As noted in Sect. I.8.2.A, the isoseparation in $\hat{M}(x,\hat{\eta},\hat{R})$ can be identically written in the *conventional* space $M(\bar{x},\eta,R)$ although referred to the new coordinates $\bar{x} = (x^\mu b_\mu)$ (no sum). The *isotopic* transforms $x' = \hat{\Lambda}(\tilde{w})*x$ in $\hat{M}(x,\hat{\eta},\hat{R})$ can therefore be connected in a one–to–one way with the *conventional* transform in $M(\bar{x},\eta,R)$, $\bar{x}' = \Lambda(\tilde{w})x$, provided that the latter are referred to the isoparameters $\tilde{\theta} = \theta b_1 b_2$, $\tilde{v} = v b_3 b_4$, etc.,

In fact, we have the identities

$$\bar{x}'^t \eta \bar{x}' = [\Lambda(\tilde{w}) \bar{x}]^t \eta [\Lambda(\tilde{w}) \bar{x}] = \bar{x}'^t \eta \bar{x} =$$

$$x'^t \hat{\eta} x' = [\hat{\Lambda}(\tilde{w}) * x]^t \hat{\eta} [\hat{\Lambda}(\tilde{w}) * x] = x^t \hat{\eta} x, \qquad (8.3.23)$$

from which we derive the desired connection

$$\hat{\Lambda}(\tilde{w})^\mu{}_\nu = b_\mu{}^{-1} \Lambda(\tilde{w})^\mu{}_\nu b_\nu \quad \text{(no sum)}. \qquad (8.3.24)$$

It is an instructive exercise for the interested reader to prove that the preceding realization verifies the fundamental conditions (8.3.13) and yields the correct isolorentz boosts (8.3.19), as well as the isorotations of Ch. II.6.

The *isodual isolorentz group* $\hat{L}^d$ is the image of L under the antiautomorphic map $\hat{1} \to \hat{1}^d = -\hat{1}$. As such, it is defined over the isodual isofield $\hat{R}^d(\hat{n}^d,+,*^d)$ with *negative-definite isonorm* $\lceil \hat{n} \rceil^d = |n| \hat{1}^d = -|n| \hat{1} < 0$, $n \neq 0$. All parameters (representing angles, velocities, etc.) and generators (representing angular momentum, etc.) therefore change sign under isoduality.

The *proper orthochronous isodual isolorentz group* can be expressed in the form

$$\hat{\Lambda}^d(\hat{w}^d) = \prod^*_{k=1,2,...,6} \{\hat{e}^{\,i\,X_k\,*\,\hat{w}_k}\}^d = \{\prod_{k=1,2,...,6} e^{\,i^d\,X^d_k\,T^d\,w^d_k}\}\hat{1}^d =$$

$$= -\{\prod_{k=1,2,...,6} e^{\,i\,X_k\,T\,w_k}\}\hat{1} = -\hat{\Lambda}(\hat{w}) .$$ (8.3.25)

The isodual isolorentz algebra $\hat{L}_+^{\uparrow} \approx \textbf{sô}^d(3.1)$ then becomes

$$[\,M^d_{\mu\nu}\,\hat{,}\,M^d_{\alpha\beta}\,]^d = -\,M_{\mu\nu}\,T\,M_{\alpha\beta} + M_{\alpha\beta}\,T\,M_{\mu\nu} =$$

$$= -\,\eta^d_{\nu\alpha}\,M^d_{\beta\mu} + \eta^d_{\mu\alpha}\,M^d_{\beta\nu} + \eta^d_{\nu\beta}\,M^d_{\alpha\mu} - \eta^d_{\mu\beta}\,M^d_{\alpha\nu} ,$$ (8.3.26)

while the isocasimirs change sign

$$C^{(0)d} = \hat{1}^d = -\hat{1}, \qquad C^{(1)d} = \tfrac{1}{2}M^d_{\mu\nu}\,T^d\,M^{d\mu\nu} = -M*M + N*N ,$$ (8.3.27a)

$$C^{(3)d} = \tfrac{1}{4}\epsilon^{\mu\nu\rho\sigma}\,M^d_{\mu\nu}\,T^d\,M^d_{\rho\sigma} = +M*N,$$ (8.3.27b)

The explicit transforms can then be computed accordingly.

8.3.B: Abstract isopoincare´ symmetry and its isodual.

The extension of the above results to the isotopies of the conventional Poincaré symmetry [2] is straighforward, as studied in ref.s [11,15] (see also review [26]). The separation among two points x and y in isominkowski spaces of Class I is given by

$$(x-y)^{\hat{2}} = [\,(x-y)^\mu\,\hat{\eta}_{\mu\nu}(x,\dot{x},\ddot{x},m,\tau,....)(x-y)^\nu\,]\hat{1} \in \hat{R}(\hat{n},+,*) .$$ (8.3.28)

Its largest possible isolinear, isolocal and isocanonical symmetry is given by the *isopoincaré symmetry* $\hat{P}(3.1) = \hat{L}(3.1)\times\hat{T}(3.1)$, where $\hat{T}(3.1)$ represents the *isotranslations*, i.e., translations in isospace.

A generic element of $\hat{P}(3.1)$ can be written

$$\hat{g} = (\hat{\Lambda},\hat{a}), \qquad \hat{\Lambda} \in \hat{L}(3.1) \qquad \hat{a} \in \hat{T}(3.1) ,$$ (8.3.29)

with isocompositions and isodecompositions (Sect. 1.4.5)

$$\hat{g}'*\hat{g} = (\hat{\Lambda}',\hat{a}')*(\hat{\Lambda},\hat{a}) = (\hat{\Lambda}*\hat{\Lambda}',\hat{a} + \hat{\Lambda}'*a') ,$$ (8.3.30a)

$$(\hat{\Lambda},a) = (\hat{\Lambda},0)*(\hat{1},a) = (\hat{\Lambda}^{-1}a,\hat{1})*(\hat{\Lambda},0) .$$ (8.3.30b)

The parameters of $\hat{P}(3.1)$ coincides with those of $P(3.1)$, $w = \{w_k\} = (\theta, v, a)$, $k = 1, 2, ..., 10$, where θ represents the Euler's angles, v represents the Lorentz parameters, and a characterizes conventional space–time translations. The generators of $\hat{P}(3.1)$ are also the conventional ones $X = \{X_k\} = \{M_{\mu\nu}, P_\mu\}$, where P_μ are the generators of translations, this time in the regular 5×5-dimensional representation (see, e.g., [23–25]. In this case the conventional four–dimensional structure of L is extended to five dimension to accommodate translations with x

$= (r, x^4, 1)$, thus requiring the five-dimensional isounit

$$\hat{1} = \text{diag.} (0b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2}, 1) . \qquad (8.3.31)$$

The *proper orthochronous isopoincaré group* $P_+^\uparrow = S\hat{O}(3.1) \times \hat{T}(3.1)$ admits the isoexponential form

$$x' = \hat{g} * x , \qquad \hat{g} = \prod_k^* \hat{e}^{\,iX_k * \hat{w}_k} = \{ \prod_k e^{\,iX_k T w_k} \} \hat{1} , \qquad (8.3.32)$$

The isolorentz component has been given in the preceding subsection. The isotranslations can be expressed in the form

$$\hat{T}(3.1) * x = \{ \hat{e}^{\,iP\,\eta\,a} \} * x = \{ e^{\,iPT\eta\,a} \}_x = e_\xi^{\,iP\,\hat{\eta}\,a} \} * x = x + aB^{-2} , \qquad (8.3.33a)$$

$$\hat{T}(3.1) * p \equiv 0. \qquad (8.3.33b)$$

The *general isopoincaré transformations* are given by

$x' = \Lambda(\tilde{w}) * x$	isolorentz transforms ,	(8.3.34a)
$x' = x + aB^{-2}$,	isotranslations ,	(8.3.34b)
$x' = \hat{\pi} * x = (-r, x^4)$,	space isoinversion	(8.3.34c)
$x' = \hat{\tau} * x = (r, -x^4)$,	time isoinversions ,	(8.3.34c)

where the functions $B_\mu = B_\mu(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$ are given by

$$B_\mu^{-2} = b_\mu^{-2} + a^\alpha [b_\mu^{-2} \hat{,} P_\alpha] / 1! + a^\alpha a^\beta [[b_\mu^{-2} \hat{,} P_\alpha] \hat{,} P_\beta] / 2! + ... (8.3.35)$$

General isotransforms (8.3.34) are evidently nonlinear–nonlocal–noncanonical in $M(x, \eta, R)$. The *restricted isopoincaré transforms* occur for constants characteristic quantities b°_μ and can be written

$x' = \tilde{\Lambda}^\circ(\tilde{w}^\circ) * x$	isolorentz transforms ,	(8.3.36a)
$x' = x + ab^{\circ -2}$,	isotranslations ,	(8.3.36b)
$x' = \hat{\pi} * x = (-r, x^4)$,	space isoinversion	(8.3.36c)
$x' = \hat{\tau} * x = (r, -x^4)$,	time isoinversions ,	(8.3.36c)

in which case they recover linearity and locality but not canonicity.

The isocommutation rules of of the isoalgebra $\hat{p}_+^\uparrow(3.1)$ are given by

$$[M_{\mu\nu} \hat{,} M_{\alpha\beta}] = \hat{\eta}_{\nu\alpha} M_{\beta\mu} - \hat{\eta}_{\mu\alpha} M_{\beta\nu} - \hat{\eta}_{\nu\beta} M_{\alpha\mu} + \hat{\eta}_{\mu\beta} M_{\alpha\nu} , \qquad (8.3.37a)$$

$$[M_{\mu\nu} \hat{,} P_\alpha] = \hat{\eta}_{\mu\alpha} P_\nu - \hat{\eta}_{\nu\alpha} P_\mu), \quad [P_\mu \hat{,} P_\nu] = 0 , \qquad (8.3.37b)$$

Which prove the isomorphism $\hat{p}_+^\uparrow(3.1) \approx p_+^\uparrow(3.1)$. The isocenter is characterized by the isocasimirs

$$C^{(0)} = 1, \quad C^{(1)} = P^2 = P T P = P_\mu \, \hat{\eta}^{\mu\nu} P_\nu = P_\mu \, P^\mu, \tag{8.3.38a}$$

$$C^{(2)} = \hat{W}^2 = \hat{W}_\mu \, \hat{\eta}^{\mu\nu} \, \hat{W}_\nu = W_\mu \, W^\mu, \quad \hat{W}_\mu = \tfrac{1}{2} \, \epsilon_{\mu\alpha\beta\rho} M^{\alpha\beta} * P^\rho. \tag{8.3.38b}$$

As one can see, the isoplane waves of Sect. II.8.2.E are a direct manifestation of the isopoincaré group. In fact, the matrix isoexponentiation (8.3.34a) is equivalent to isoform (8.2.14).

The *isodual isopoincaré group* $\hat{P}^d(3.1) = \hat{L}^d(3.1) \times \hat{T}^d(3.1)$ is the antiautomorphic image of $\hat{P}(3.1)$ under the map $1 \to 1^d = -1$. It is therefore defined over the isodual isofield $\hat{R}^d(\hat{n}^d, +, *^d)$ and can be expression via the isodual isotransforms

$$x'^d = \hat{\Lambda}^d(\tilde{w}^d) *^d x = \Lambda(\tilde{w}) * x, \quad x'^d = x + a^d B^{-2d} = x + a B^{-2}, \tag{8.3.39b}$$

$$x'^d = \pi^d *^d x = (-r, x^4), \quad x'^d = \tau^d *^d x = (r, -x^4), \tag{8.3.39c}$$

As such, they formally coincide with the original transforms, with the understanding that the coordinates x are now referred to a negative–definite isounit.

8.3.C: Universal isospinorial covering $S\hat{L}(2.\hat{C})$ and its isodual.

As recalled earlier, the Lorentz group L is not connected. Its universal, spinorial, simply connected covering $\mathcal{L}$ is the group SL(2.C) of all complex 2×2–dimensional matrices A over the field C(c,+,×) verifying the unimodularity condition DetA = 1. In particular, we have the isomorphism $L_+^\uparrow \approx SL(2.C)/Z_2$, where Z_2 is the center of SL(2.C) consisting of the two elements ±I, where I = diag. (I, I) is the unit (see, e.g., ref. [24,25] for details).

It is easy to see that the above lines admit a simple yet intriguing isotopic generalization. In this subsection we shall study the isogroup aspect. The isospinorial aspect will be studied in more detail in Ch. II.10.

Consider the isotopic $S\hat{L}(2.\hat{C})$ group which consists of all complex 2×2–dimensional matrices $\hat{A}$ over the isofield $\hat{C}(c,+,*)$ of isocomplex numbers $\hat{c} = c\hat{1}_2$, with isotopic element T_2 (diag. $(g_{11}, g_{22}) > 0$ and isounit $\hat{1}_2 = $ diag. $(g_{11}^{-1}, g_{22}^{-1})$, $g_{kk} > 0$, verifying the *isounimodularity condition* $\hat{Det}\hat{A} = [Det(\hat{A}T_2)]\hat{1}_2 = \hat{1}_2$, or $Det(\hat{A}T_2) = 1$.

The *universal isospinorial covering of the isolorentz group* $\hat{L}$ is given by $\hat{\mathcal{L}} = S\hat{L}(2.\hat{C})$. For this purpose, we introduce the following new realization of the *isopauli matrices* besides those of Sect. II.6.8

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & (b_1 g_{11}^{-1}) \\ (b_1 g_{11}^{-1}) & 0 \end{pmatrix} = b_1 \, \sigma_1 \hat{1}_2, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -ib_2 g_{22}^{-1}) \\ i(b_2 g_{11}^{-1}) & 0 \end{pmatrix} = b_2 \, \sigma_2 \hat{1}_2, \tag{8.3.40}$$

$$\hat{\sigma}_3 = \begin{pmatrix} (b_3 g_{11}^{-1}) & 0 \\ 0 & -(b_3 g_{22}^{-1}) \end{pmatrix} = b_3 \, \sigma_3 \hat{1}_2, \quad \hat{\sigma}_4 = \begin{pmatrix} (b_4 g_{11}^{-1}) & 0 \\ 0 & (b_4 g_{22}^{-1}) \end{pmatrix} = b_4 \, \sigma_4 \hat{1}_2, \tag{8.3.40}$$

where the b's are the characteristic functions of the isominkowski space, $\sigma_4 = I_2 =$ diag. (1, 1) and σ_k represents the conventional Pauli matrices.

The isocommutation rules and isoeigenvalues are given by

$$[\hat{\sigma}_i \hat{,} \hat{\sigma}_j] = \hat{\sigma}_i T_2 \hat{\sigma}_j - \hat{\sigma}_j T_2 \hat{\sigma}_i = i \epsilon_{ijk} b_i b_j b_k^{-1} \hat{\sigma}_k, \qquad (8.3.41a)$$

$$\hat{\sigma}^2 *_2 |> = \hat{\sigma}_1 T_2 \hat{\sigma}_1 + \hat{\sigma}_2 T_2 \hat{\sigma}_2 + \hat{\sigma}_3 T_2 \hat{\sigma}_3) T_2 |> = (b_1^2 + b_2^2 + b_3^2)|>, \quad (8.3.41b)$$

$$\hat{\sigma}_3 *_2 |> = b_3 \sigma_3 1 |> = \pm b_3 |>. \qquad (8.3.41c)$$

They characterize an *isospinorial realization* of $S\hat{U}(2)$ isomorphic to the conventional SU(2) (from the positive-definiteness of the b_k). Despite this isomorphism, *the eigenvalues of the conventional and isopauli matrices are different*, Eq.s (8.3.41b) and (8.3.41c). This alteration of conventional quantum eigenvalues should be expected from the underlying nonlinear–nonlocal–nonhamiltonian effects. Note also that the above isopauli matrices are different than those studied in Ch. 11.6 because they mix the elements g_{kk} of the two-dimensional unit 1_2 of $\hat{E}(z,\delta,\hat{R})$ with the elements b_μ of the four-dimensional isounit 1_4 of $\hat{M}(x,\hat{\eta},\hat{R})$.

An arbitrary element $\hat{A} \in S\hat{L}(2.\hat{C})$ can be written

$$\hat{A} = -\sum_{k=1,2,3} c_k \hat{\sigma}_k - c_4 \hat{\sigma}_4 = \{-\sum_{k=1,2,3} c_k b_k \sigma_k - c_4 b_4 \sigma_4\} 1_2, \quad (8.3.42)$$

where the c's are ordinary complex numbers.

Any point x of the isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ over $\hat{R}(\hat{n},+,*)$ can be represented with the Hermitean 2×2-dimensional matrix $\tilde{x}$ over $\hat{C}(\hat{c},+,*)$ according to the rule

$$x \rightarrow \tilde{x} = -\sum_{k=1,2,3} x^k \hat{\sigma}_k - x^4 \hat{\sigma}_4 = \{-\sum_{k=1,2,3} x^k b_k \sigma_k - x^4 b_4 \sigma_4\} 1_2. \quad (8.3.43)$$

In fact, we have the identities on a field with isounit 1_2[98]

$$x^\mu 1_2 = \tfrac{1}{2} b_\mu^{-2} \hat{Tr}(\tilde{x} * \hat{\sigma}_\mu) = \tfrac{1}{2} b_\mu^{-2} Tr(\tilde{x} * \hat{\sigma}_\mu) 1_2, \qquad (8.3.44a)$$

$$\hat{Det}\, \tilde{x} = [Det(\tilde{x} T_2)] 1_2 = (x^k b_k^2 x^k - x^4 b_4^2 x^2) 1_2. \qquad (8.3.44b)$$

The transformations in $S\hat{L}(2.C)$, $\tilde{x}' = \hat{A}*\tilde{x}*\hat{A}^\dagger = \tilde{A}\tilde{x}\tilde{A}^\dagger$, can then be connected with the isolorentz transforms $x' = \hat{A}*x = \tilde{A}x$. In fact, by ignoring the 1_2 factor, isostructures (8.3.43) can be reinterpreted as a conventional form in the space $M(\bar{x},\eta,R)$ with $\bar{x} = \{x^\mu b_\mu\}$ (no sum), yielding the expressions

$$x'^\mu = \tilde{\Lambda}(c)^\mu_\nu x^\nu = \tfrac{1}{2} b_\mu^{-2} Tr(\tilde{x}' * \hat{\sigma}_\mu) = \tfrac{1}{2} b_\mu^{-2} Tr(\tilde{A} b_\nu \sigma_\nu \tilde{A}^\dagger b_\mu \sigma_\mu) x^\nu. \quad (8.3.45a)$$

---

[98] Note that expressions (8.3.44a) are defined over the isofield with the *two-dimensional* isounit 1_2, rather than the four-dimensional one of the Minkowski space. This difference can be easily removed via suitable redefinitions and it is ignored.

By recalling realization (8.3.24) of the isotransform $\hat{\Lambda}$, simple algebra yields the representation of the isolorentz transforms in terms of the complex elements c_μ of $\hat{SL}(2.\hat{C})$

$$\hat{\Lambda}^4{}_4 = \sum_{k=1,2,3} |c_k|^2 + |c_4|^2, \tag{8.3.46a}$$

$$\hat{\Lambda}^k{}_4 = b_k^{-1} b_4 \left(c_4 \bar{c}_k + \bar{c}_4 c_k + i \epsilon^{kmn} c_m \bar{c}_n \right), \tag{8.3.46b}$$

$$\hat{\Lambda}^4{}_k = b_4^{-1} b_k \left(c_4 \bar{c}_k + \bar{c}_4 c_k - i \epsilon^{kmn} c_m \bar{c}_n \right), \tag{8.3.46c}$$

$$\hat{\Lambda}^m{}_n = b_m^{-1} b_n \left[\delta^m{}_n \left(-\sum_{k=1,2,3} |c_k|^2 + |c_4|^2 \right) + \right.$$
$$\left. + c_m \bar{c}_n + \bar{c}_m c_n + i \epsilon^{mnj} \left(\bar{c}_4 c_j + - c_4 \bar{c}_j \right) \right]. \tag{8.3.46d}$$

This establishes the isospinorial character of the covering. The simply connected character of $\hat{SL}(2.\hat{C})$ is evident because isotopies of Kadeisvili Class I do not alter the original connectedness.[99] The proof of the isomorphism $\hat{L}_+^\uparrow \approx \hat{SL}(2.\hat{C})/\hat{Z}_2$ is left to the interested reader, where $\hat{Z}_2$ is the isocenter of $\hat{SL}(2.\hat{C})$ given by $\pm \hat{1}_2$.

The *isodual universal covering of the isolorentz group* $\hat{SL}^d(2.\hat{C}^d)$ is the group of Hermitean 2×2 matrices over the isodual isofield $\hat{C}^d(\hat{c}^d,+,*^d)$ with isodual isounit $\hat{1}_2^d = - \text{Diag} (g_{11}, g_{22}) < 0$ verifying the condition of *anti-isounimodularity condition* $\hat{\text{Det}}\hat{A}^d = [\text{Det}(\hat{A}^d \text{T}^d)]\hat{1}_2^d = - \hat{\text{Det}}\hat{A}$.

To characterize $\hat{SL}^d(2.\hat{C}^d)$ we introduce the following *isodual isopauli matrices*

$$\hat{\sigma}_\mu^d = b_\mu^d \sigma_\mu^d \hat{1}_2^d = - \hat{\sigma}_\mu, \tag{8.3.47}$$

with explicit forms

$$\hat{\sigma}_1^d = \begin{pmatrix} 0 & -(b_1 g_{11}^{-1}) \\ -b_1 g_{11}^{-1} & 0 \end{pmatrix}, \qquad \hat{\sigma}_2^d = \begin{pmatrix} 0 & +i b_2 g_{22}^{-1} \\ -i(b_2 g_{11}^{-1}) & 0 \end{pmatrix}, \tag{8.3.48a}$$

$$\hat{\sigma}_3^d = \begin{pmatrix} -(b_3 g_{11}^{-1}) & 0 \\ 0 & +(b_3 g_{22}^{-1}) \end{pmatrix}, \qquad \hat{\sigma}_4^d = \begin{pmatrix} -(b_4 g_{11}^{-1}) & 0 \\ 0 & -(b_4 g_{22}^{-1}) \end{pmatrix}, \tag{8.3.48b}$$

where $\sigma_4^d = \hat{1}_2^d = -\text{diag.} (1, 1)$ and $\sigma_k^d = -\sigma_k$ are the isodual Pauli matrices.

The isocommutation rules and related isodual isoeigenvalues are given by

$$[\hat{\sigma}_i^d \, \hat{,} \, \hat{\sigma}_j^d]^d = - \hat{\sigma}_i T_2 \hat{\sigma}_j + \hat{\sigma}_j T_2 \hat{\sigma}_i = -i \epsilon_{ijk} b_i^d b_j^d b_k^{-1d} \hat{\sigma}_k^d, \tag{8.3.49a}$$

$$<|^d *^d_2 \hat{\sigma}^{2d} = <|(- b_1^2 - b_2^2 - b_3^2), \qquad <|^d *^d_2 \hat{\sigma}^d_3 = <|[-(\pm b_3)], \tag{8.3.49b}$$

and evidently characterize a realization of the $\hat{SU}^d(2)$ algebra locally isomorphic to $SU^d(2)$.

An arbitrary element $\hat{A}^d \in \hat{SL}^d(2.\hat{C}^d)$ can be written

---

$$\hat{A}^d = -\sum_{k=1,2,3} c_k{}^d \, \hat{\sigma}_k{}^d - c_4{}^d \, \hat{\sigma}_4{}^d \,, \tag{8.3.50}$$

where the isoduality of complex numbers should be kept in mind (Ch. 1.2).

Any point x of the isominkowskian space $\hat{M}^d(x,\eta^d,\hat{R}^d)$ over $\hat{R}^d(\hat{n}^d,+,*^d)$ can be represented with the 2×2-dimensional matrix $\tilde{x}$ over $\hat{C}^d(\hat{c}^d,+,*^d)$ according to the rule

$$x \;\rightarrow\; \tilde{x}^d = -\sum_{k=1,2,3} x^k \hat{\sigma}_k{}^d - x^4 \, \hat{\sigma}_4{}^d = \{ \,+ \sum_{k=1,2,3} x^k \, b_k \, \sigma_k + x^4 \, b_4 \, \sigma_4 \, \} \, \hat{1}_2{}^d \,. \tag{8.3.51}$$

and isodual rules

$$x^\mu \hat{1}_2{}^d \;=\; \tfrac{1}{2} \, b_\mu{}^{-2d} \, \hat{\mathrm{Tr}} \, (\, \tilde{x}^d \ast^d \hat{\sigma}_\mu{}^d \,) = \; \tfrac{1}{2} \, b_\mu{}^{-2} \mathrm{Tr} \, (\, \tilde{x} \ast \hat{\sigma}_\mu \,) \hat{1}_2 \,, \tag{8.3.52a}$$

$$\hat{\mathrm{Det}} \, \tilde{x}^d \;=\; [\, \mathrm{Det} \, (\, \tilde{x}^d \, T_2{}^d \,) \,] \hat{1}_2{}^d \;=\; (\, -x^k \, b_k{}^2 \, x^k + x^4 \, b_4{}^2 \, x^2 \,) \hat{1}_2{}^d \equiv (\, x^\mu \hat{\eta}_{\mu\nu} \, x^\nu \,) \hat{1}_2 \tag{8.3.52b}$$

thus recovering the conventional separation on $\hat{R}(\hat{n},+,*)$. The isoduality of the remaining aspects then follows.

As we shall see in Ch. 11.10, the realizations of $S\hat{L}(2.\hat{C})$ and $S\hat{L}^d(2.\hat{C}^d)$ of this section permit a fundamentally novel interpretation of the *conventional* Dirac equation with consequential novel interpretation of antiparticles via isoduality.

8.4: RELATIVISTIC HADRONIC MECHANICS AND ITS ISODUAL

In this section we study the foundations of the *isotopies of relativistic quantum mechanics* also called *relativistic hadronic mechanics*. The central methodological tool is the operator realization of the isopoincaré symmetry. A knowledge of its classical realization as studied in ref.s [11] is requested for the understanding that the former is a unique image of the latter.

8.4.A: Isodifferentials and isoderivatives on isominkowski spaces. A prerequisite for the operator realization of the isopoincaré symmetry is a knowledge of the isocalculus on isomanifolds whose study has been initiated in Ch. 1.6.

Let $\hat{M}(x,\hat{\eta},\hat{R})$ be an isominkowski space of topological Class 1 with: isometric in the covariant form $\hat{\eta}_{\mu\nu} = T_\mu{}^\alpha \, \eta_{\alpha\nu}$, η = diag. (1, 1, 1, −1), μ, ν, α = 1, 2, 3, 4, and contravariant form $\hat{\eta}^{\mu\nu}$, $\hat{\eta}^{\mu\alpha}\hat{\eta}_{\alpha\nu} = \delta^\mu{}_\nu$; local coordinates in their contravariant form $x = \{x^\mu\} = \{r, x^4\}$, $x^4 = c_0 t$, (c_0 is light speed in vacuum), and covariant form $x_\mu = \hat{\eta}_{\mu\nu} x^\nu$; isotopic element T and isounit $\hat{1} = T^{-1}$ in the diagonal form with the most general possible dependence, including that on the local coordinates,

$$T = T(x, ...) = \mathrm{Diag.} \, (\, b_1{}^2, b_2{}^2, b_3{}^2, b_4{}^2) \;=\; (\, T^\mu{}_\nu \,) \;=\; (\, T_\mu{}^\nu \,) > 0, \, b_\mu = b_\mu(x, ...) > 0,$$

$$\hat{1} = [\, T(x, ...) \,]^{-1} \;=\; (\, \hat{1}^\mu{}_\nu \,) \;=\; (\, \hat{1}_\mu{}^\nu \,) > 0 \,. \tag{8.4.1}$$

The *isodifferentials* on $\hat{M}(x,\hat{\eta},\hat{R})$ are given by

$$\hat{d}x^{\mu} = T^{\mu}{}_{\nu}(x, ...) \, dx^{\nu} = \hat{b}_{\mu}{}^2 \, dx^{\mu} \quad \text{(no sum)}, \quad \mu = 1, 2, 3, 4 . \tag{8.4.2}$$

A definition of the *isoderivative* of a *function* f(x) on $\hat{M}(x,\hat{\eta},\hat{R})$ which preserves all conventional axioms of the derivative, plus the axiom of preserving the original unit,[100] is given by

$$\frac{\hat{\partial} f(x)}{\hat{\partial} x^{\mu}} = \hat{1}_{\mu}{}^{\nu}(x, ...) \, \{ \operatorname*{Lim}_{x' \to x} \frac{f(x') - f(x)}{x'^{\nu} - x^{\nu}} \} = \hat{1}^{\mu}{}_{\nu} \, (x, ...) \frac{\partial f(x)}{\partial x^{\nu}} = b_{\mu}{}^{-2} \frac{\partial f(x)}{\partial x^{\mu}}, \tag{8.4.3}$$

where $\partial f / \partial x$ is the conventional derivative.

The above rules then imply the expressions

$$\frac{\hat{\partial} \, x^{\mu}}{\hat{\partial} x^{\nu}} = \hat{\delta}^{\mu}{}_{\nu} = \hat{1}^{\mu}{}_{\nu} = b_{\mu}{}^{-2} \delta^{\mu}{}_{\nu} , \tag{8.4.4}$$

which permit a first definition of the operator isopoincaré symmetry. The realization most important for physical application is however based on the expression (for diagonal isounits)

$$\frac{\hat{\partial} \, x_{\mu}}{\hat{\partial} \, x^{\nu}} = \hat{\eta}_{\nu\rho} \quad \frac{\hat{\partial} \, x_{\mu}}{\hat{\eta}_{\rho\sigma} \hat{\partial} \, x^{\sigma}} = \hat{\eta}_{\nu\rho} \frac{\hat{\partial} \, x_{\mu}}{\hat{\partial} x_{\rho}} = \hat{1}^{\rho}{}_{\alpha} \, \hat{\eta}_{\nu\rho} \frac{\partial \, x_{\mu}}{\partial \, x_{\alpha}} = \hat{1}^{\rho}{}_{\alpha} \, \hat{\eta}_{\nu\rho} \, \delta^{\mu}{}_{\alpha} =$$

$$= \hat{1}^{\rho}{}_{\mu} \, T_{\nu}{}^{\delta} \eta_{\delta\rho} = \eta_{\mu\nu} , \tag{8.4.5}$$

where η is the *conventional* Minkowskian metric.

From expression (8.4.2) we derive the *invariant isomeasure* on *isominkowski spaces*

$$\hat{d}x = \hat{d}x^1 \, \hat{d}x^2 \, \hat{d}x^3 \, \hat{d}x^4 = T^1{}_{\mu} T^2{}_{\nu} T^3{}_{\alpha} T^4{}_{\beta} \, dx^{\mu} \, dx^{\nu} \, dx^{\alpha} \, dx^{\beta} = \tag{8.4.6}$$

$$b_1{}^2 \, b_2{}^2 \, b_3{}^2 \, b_4{}^2 dx^1 \, dx^2 \, dx^3 \, dx^4 = \text{Det T } dx^1 \, dx^2 \, dx^3 \, dx^4 = \text{Det T } \, dx .$$

The (indefinite) *isointegration* on *isominkowski spaces* must be defined as the inverse of the isodifferential (Sect. I.6.7), i.e., such that $\int_{(u)} \hat{d}x^{\mu} = x^{\mu}$, yielding

---

[100] This axiom is evidently absent in the conventional calculus (as well in the so-called q–calculus) because in both cases the unit is the trivial constant quantity +1 which, as such, is preserved under differentiations and derivations. The axiom of preservation of the unit is therefore a peculiarity of hadronic mechanics and leads in a unique way to definition (8.4.2) and (8.4.3). In fact, alternative definitions, such as $\hat{d}x = d[T(x, ...)x] = x \, dT(x, ...) + T(x, ...)dx$, would imply the unacceptable map from an original set of functions defined over a given isofield into a set of functions defined over a *different* isofield due to the change of the unit under differentiation. The author would like to thank the mathematicians G. T. Tsagas and D. S. Sourlas for consultations on this point. Note that definition (8.4.2) is indeed compatible with definition (8.4.3).

the realization

$$\int \hat{d}x = \int_{(1)} \int_{(2)} \int_{(3)} \int_{(4)} \hat{1}^1{}_\rho \hat{1}^2{}_\sigma \hat{1}^3{}_\gamma \hat{1}^4{}_\tau T^\rho{}_\mu T^\sigma{}_\nu T^\gamma{}_\alpha T^\tau{}_\beta \, dx^\mu \, dx^\nu \, dx^\alpha \, dx^\beta =$$

$$= \int_{(1)} \int_{(2)} \int_{(3)} \int_{(4)} \, dx^1 \, dx^2 \, dx^3 \, dx^4 = x^1 \, x^2 \, x^3 \, x^4 . \qquad (8.4.7)$$

The above notions are sufficient for our needs at this time. Additional studies are left for the specialized literature.

Note the simplicity of the above isocalculus for a diagonal isounit, its rather intriguing structure for nondiagonal isounits and its nontriviality because holding for isounits with an explicit dependence in the differentiation–integration variables.

8.4.B: Operator isopoincare´ symmetry and its isodual. The structural elements of the most general possible *relativistic hadronic mechanics* on isominkowski spaces of Class 1 $\hat{M}(x,\hat{\eta},\hat{R})$ are given by:

A) the lifting of the field of complex numbers $C(c,+,\times)$ with ordinary elements $c = a + ib$, a, b $\in R(n,+,\times)$ into the *isofield* of *isocomplex numbers* (Sect. I.2.6)

$$\hat{C}(\hat{c},+,*): \hat{c} = c\,\hat{1}, \quad \hat{c}_1 + \hat{c}_2) = (c_1 + c_2 \hat{1}, \quad \hat{c}_1 * \hat{c}_2 = \hat{c}_1 T \hat{c}_2 = (c_1 c_2)\hat{1}; \quad (8.4.8)$$

B) The lifting of the conventional Hilbert space $\mathcal{H}$ with states $|\psi>, |\phi>, ...,$ inner product $<\psi | \phi> \in C(c,+,\times)$ and normalization $<\psi | \psi> = 1$ into the *isohilbert space* (Sect. I.6.2)[101]

$$\hat{\mathcal{H}}: \; <\hat{\psi}\lceil\hat{\phi}> \; = \; <\hat{\psi}(x)| T(x, ...)|\hat{\phi}(x)>\hat{1}(x, ...) \in \hat{C}(\hat{c},+,*), \; <\hat{\psi}|T|\hat{\psi}> = 1; \quad (8.4.9)$$

C) The lifting of the enveloping associative algebra ξ of operators A, B, ... , with conventional associative product AB and unit I, IA = AI = A, $\forall$ A $\in \xi$, into the *enveloping isoassociative operator algebra* (or *isoenvelope* for short) $\hat{\xi}$ with the same elements A, B, ..., but isotopic product and isounit[102]

$$\hat{\xi}: \; A * B = A\,T\,B, \quad \hat{1} = T^{-1}, \, \hat{1} * A = A * \hat{1} = A, \quad \forall A \in \hat{\xi}. \qquad (8.4.10)$$

We are now sufficiently equipped to introduce the fundamental symmetry of isorelativistic hadronic mechanics, the *operator isopoincare symmetry* $\hat{P}(3.1) = \hat{L}(3.1)\times\hat{T}(3.1)$, for isounits of Class 1 but otherwise possessing the most general

---

[101] Note that in general $|\hat{\psi}> \neq T^{-1/2}|\psi>$. In fact, as we shall see in Ch. 11.10, $|\psi>$ is the solution of linear–local–Hamiltonian equations, while $|\hat{\psi}>$ is the solution of more general nonlinear–nonlocal–nonhamiltonian equations.

[102] The reader should recall that *all* products must be isotopic as a *necessary condition to preserve isolinearity*. For instance, operator products such as $AA^\dagger$ or modular actions such as $A|>$ can be easily proved to yield a host of inconsistencies and must be replaced with the correct isoexpressions $A*A^\dagger = ATA^\dagger$ and $A*|> = AT|>$.

possible functional dependence.

Consider a system of N elementary particles in relativistic conditions with space–time coordinates $x_a = \{x_a{}^\mu\}$ and momenta $p_a = \{p_{a\mu}\}$ $a = 1, 2, ..., N$, $\mu = 1, 2, 3,$ 4. Suppose that their mutual distances are sufficiently smaller than the size of their wavepackets ($< 1\text{fm} = 10^{-13}$ cm) to activate interactions which are nonlinear (in coordinates, wavefunctions and their derivatives $x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}$), nonlocal–integral (in all needed quantities) and nonpotential–nonhamiltonian.

Under these interior relativistic conditions the particles can be represented in $\hat{M}(x,\hat{\eta},\hat{R})$ by embedding all nonminkowskian interactions in the isotopic elements or isounits resulting in an arbitrary functional dependence of the type[103]

$$T = T^\dagger = T(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \mu, \tau, ...) > 0, \hat{1} = T^{-1} = \hat{1}(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, ...), \quad (8.4.11)$$

which can be assumed as common to all particles because representing their common hadronic medium.[104]

The isopoincaré symmetry can be defined as the largest possible isolinear, isolocal and isocanonical symmetry of the isoseparation (for $a < b$)

$$x^2{}_{ab} = [\,(x^\mu{}_a - x^\mu{}_b)\,\hat{\eta}_{\mu\nu}(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, ...)\,(x^\nu{}_a - x^\nu{}_b)\,]\hat{1} \in \hat{R}(\hat{n},+,*), \quad (8.4.12)$$

and can be constructed via the quantities $x^\mu{}_a$ and $p_{a\nu}$ under isotopic rules.

In coordinates representation the quantities $x^\mu{}_a$ are ordinary scalars (because coordinates are not lifted in $\hat{M}(x,\hat{\eta},\hat{R})$). The *relativistic four–momentum operator* is then given by

$$\hat{p}_{a\mu} * |\hat{\psi}> = -i\frac{\partial}{\partial x^\mu}|\hat{\psi}> = -i\hat{1}^\mu{}_\nu\frac{\partial}{\partial x^\nu}|\hat{\psi}> = -i\,b_\mu{}^{-2}\frac{\partial}{\partial x^\mu}|\hat{\psi}>. \quad (8.4.13)$$

By using Eq..s (8.4.5), the *relativistic fundamental isocommutation rules* are given by

$$[\,p_{a\mu} \,\hat{,}\, x_{b\nu}\,] * |\hat{\psi}> = [\,p_{a\mu}\,T(x, ...)\,x_{b\nu} - x_{b\nu}\,T(x, ...)\,p_{a\mu}\,]\,T(x, ...)\,|\hat{\psi}> =$$

$$= (\,p_{a\mu} * x_{b\nu} - x_{b\nu} * p_{a\mu}\,) * |\hat{\psi}> = -i\,\eta_{\mu\nu}\,\delta_{ab} * |\hat{\psi}>, \quad (8.4.14a)$$

$$[\,x_{a\mu} \,\hat{,}\, x_{b\nu}\,] * |\hat{\psi}> = [\,p_{a\mu}\,\hat{,}\,p_{b\nu}\,] * |\hat{\psi}> \equiv 0, \quad (8.4.14b)$$

---

[103] The joint dependence on $\hat{\psi}$ and $\hat{\psi}^\dagger$ is generally avoided in order to allow the separation between particles and antiparticles (see Ch. 11.10). Note that this may lead to *nonhermitean* isounits with consequential natural emergence of the *genorelativistic hadronic mechanics* of Sect. 11.8.6, i.e., that possessing the more general Lie–admissible structure.

[104] If this is not the case, the extension of $\hat{1}$ to the reducible form $\hat{1} = \prod^x{}_{k=1, 2, ...,N}\hat{1}_a$ is trivial.

namely, *the structure of the brackets is generalized, but the eigenvalues* – $i\eta_{\mu\nu}\delta_{ab}$ *are the same as those for the conventional relativistic quantum mechanics.* This illustrates again, this time at the relativistic level, the capability of the isotopic methods of quantitative treatments of nonlinear–nonlocal–nonhamiltonian interactions while preserving the axioms of conventional linear–local–Hamiltonian theories.

We also indicate for completeness the alternative version of rules (8.4.14)

$$[\, p_{\mu a} \hat{,} x^\nu_{\ b} \,] * | \hat{\Phi} > \; = \; - i \hat{1}_\mu^{\ \nu} \delta_{\widehat{ab}} * | \hat{\Phi} > \; = \; - i \; b_\mu^{-2} \delta_\mu^{\ \nu} \delta_{ab} * | \hat{\Phi} > , \quad (8.4.15)$$

which exhibit an explicit isotopic character.

The transition from rules (8.4.15) to (8.4.14) is geometrically significant inasmuch as it illustrates that the isotopic character is contained in the contracted coordinates themselves, $x_\mu = \hat{\eta}_{\mu\nu}x^\nu$, thus permitting the preservation of conventional canonical eigenvalues.

The isosymmetry $\hat{P}(3.1)$ is characterized by the conventional (ordered set of) parameters $w = \{w_k\} = \{\theta, u, a\}$, $k = 1, 2, ..., 10$, and generators $X = \{X_k\} = \{M_{\mu\nu}, P_\mu\}$, where $M_{\mu\nu}$ are the generators of the isolorentz group for a system of N particles and P_μ are the generators of the isotranslations

$$M_{\mu\nu} \; = \; \sum_a (x_{a\mu}P_{a\nu} - x_{a\nu}\,P_{\mu a}\,; \qquad P_\mu = \sum_a P_{\mu a}\,, \qquad (8.4.16)$$

but now defined on the isoenvelope $\hat{\xi}$ over the isohilbert space $\hat{\mathcal{H}}$.[105]

A fundamental property is that the above generators remain Hermitean, thus observable, under isotopy. More specifically, starting from the original Hermiticity $M_{\mu\nu}^\dagger = M_{\mu\nu}$ and $P_\mu^\dagger = P_\mu$ in a conventional Hilbert space $\mathcal{H}$, the *same* generators preserve such Hermiticity under the isotopic generalization of the associative and Lie product, *provided* that one jointly lifts the field $C(c,+,\times)$ and Hilbert space $\mathcal{H}$ into a form admitting of the same basic unit $\hat{1}$, in which case we have the isohermiticity properties

$$M_{\mu\nu}^{\hat{\dagger}} = \; M^\dagger_{\mu\nu} \; = \; M_{\mu\nu}\,, \qquad P_\mu^{\hat{\dagger}} \; = \; P^\dagger = P_\mu. \qquad (8.4.17)$$

Such Hermiticity-observability then persists under the time evolution of the theory (the isounitary transforms).[106]

---

[105] Recall that $x_{\mu a}$ are ordinary scalars as requested by the isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ in which only the metric and the unit are lifted. Their re-interpretation as isoscalar would leave $M_{\mu\nu}$ unchanged because $\hat{x}_{\mu a} * P_{\mu\nu} \equiv x_{\mu a} P_{\nu a}$

[106] This is a fundamental point of the isorelativistic hadronic mechanics which has forced this author to rule out the q-deformations as usually treated with unmodified unit and related conventional fields and Hilbert spaces (App. 1.A.7.1). In fact, the *Hermiticity at a fixed time* is lost for q-deformations whenever q is an operator. The q-deformations are therefore structurally insufficient for the desired nonlinear–nonlocal–nonhamiltonian

The *connected isopoincaré group* $\hat{P}_+\hat{T}(3.1) = S\hat{O}(3.1)\times\hat{T}(3.1)$ of Class I is also formally the same as the abstract one (Sect. 11.8.3.B),

$$\hat{g}' = \{\hat{\Lambda}, \hat{a}\} * x = [\prod_{k=1,...,10} \hat{e}^{\ i X_k * \hat{w}_k}] * x = \{[\prod_{k=1,...,10} e^{\ i X_k T w_k}_{\not{g}}] x,$$

$$(8.4.18)$$

Note the fixed character of the generators (and parameters), but the infinitely possible isotopic elements T. This is due to the existence of infinitely many possible physical media in which the same generators $M_{\mu\nu}$ and P_α can be defined.

The isoinversions remain the same as in the abstract case, Eq.s (8.3.18). The explicit form of the *general isopoincaré transforms* then remains the same as the abstract ones

$$x' = \hat{\Lambda}*x, \quad x' = x + a B^{-2}, \quad x' = \hat{\pi}*x = (-r, x^4), \quad x' = \hat{\tau}*x = (r, -x^4), (8.4.19)$$

where: the isolorentz transforms $\hat{\Lambda}$ are reducible to a combination of isorotations (Ch. 11.6) and isoboosts,

$$x'^1 = x^1, \quad x'^2 = x^2, \tag{8.4.20a}$$

$$x'^3 = x^3 \cosh(vb_3b_4) - x^4 b_4 b_3^{-1}\sinh(vb_3b_4) = \hat{\gamma}(x^3 - b_3^{-1}b_4 \beta x^4), \tag{8.4.20b}$$

$$x'^4 = -x^3 b_3 b_4^{-1}\sinh(vb_3b_4) + x^4\cosh(vb_3b_4) = \hat{\gamma}(x^4 - b_3b_4^{-1}\beta x^3), \tag{8.4.20c}$$

the β, $\hat{\beta}$ and $\hat{\gamma}$ quantities are given by

$$\beta^2 = vv/c_0^2, \quad \hat{\beta}^2 = v^k b_k^2 v^k / c_0 b_4^2 c_0 = v*v/c^2, \tag{8.4.21a}$$

$$\cosh(v b_1 b_2) = \hat{\gamma} = (1-\hat{\beta}^2)^{-\frac{1}{2}}, \quad \sinh(v b_1 b_2) = \hat{\beta}\,\hat{\gamma}. \tag{8.4.21b}$$

and the functions $B_\mu = B_\mu(x, \dot{x}, \ddot{x}, ...)$ are given by the expressions in terms of isocommutators on $\hat{\mathfrak{K}}$

$$B_\mu^{-2} = b_\mu^{-2} + a^\alpha [b_\mu^{-2}, \hat{P}_\alpha]/1! + a^\alpha a^\beta [[b_\mu^{-2}, \hat{P}_\alpha], \hat{P}_\beta]/2! + ... (8.4.22)$$

The *restricted isopoincaré transforms* then occur for the b-quantities averages into b° constants.

The *operator isopoincaré algebra* $\hat{p}_+\hat{T}(3.1)$ can be easily computed via rules (8.4.14) and it is given by [12,13]

$$[M_{\mu\nu}, \hat{M}_{\alpha\beta}]*|\psi> = i(\eta_{\nu\alpha}M_{\beta\mu} - \eta_{\mu\alpha}M_{\beta\nu} - \eta_{\nu\beta}M_{\alpha\mu} + \eta_{\mu\beta}M_{\alpha\nu})*|\psi> \quad (8.4.23a)$$

---

treatment of the strong interactions. If q is a scalar, Hermiticity at a fixed time is indeed verified, but *Hermiticity is lost under time evolution* according to *Lopez's lemma* (App. 11.3.C). The origin of these basic insufficiencies is precisely due to the lack in the q-deformations of the lifting of the basic unit, underlying fields and Hilbert spaces in the amount inverse of the deformation of the associative product, as assumed at the foundations of hadronic mechanics.

$$[M_{\mu\nu} \hat{,} \ P_\alpha] * | \psi > = i \, (\eta_{\mu\alpha} P_\nu - \eta_{\nu\alpha} P_\mu) * | \psi >, \quad [P_\mu \hat{,} \ P_\nu] * | \psi > = 0, \quad (8.4.23c)$$

where the η's in the r.h.s. is the *conventional* Minkowski metric. Note by comparison that the $\hat{\eta}$ appearing in the commutation rules (8.3.37) is the *isominkowskian* metric.

Thus, the structure constants of the operator $\hat{p}_+^\dagger(3.1)$ algebra formally *coincide* with those of the conventional Poincaré algebra $p_+^\dagger(3.1)$, by confirming not only the local isomorphism $\hat{p}_+^\dagger(3.1) \approx \hat{p}_+^\dagger(3.1)$, but also the identity at the abstract level of the conventional and isotopic symmetries. This implies *the abstract identity between relativistic quantum and hadronic mechanics.*

It should be indicated that exactly the same situation occurs at the classical level [11], thus confirming that the isopoincaré symmetry of this section is the unique and correct operator image of the classical realization.

These rather fundamental properties should be kept in mind because they are at the basis of the covering isospecial relativity of the next section.

For completeness we note that, the use of the generators $M^\mu_\nu = \sum_a (x^\mu_a P_{\nu a} - x^\nu_a P_\mu)$ and $P_\mu = \sum_a P_{a\mu}$ would imply the replacement in rules (8.4.23) of $\eta_{\mu\nu}$ with $\hat{1}^\mu_\nu$ thus showing explicitly the isotopic structure of the theory. The local isomorphism $\hat{p}_+^\dagger(3.1) \approx \hat{p}_+^\dagger(3.1)$ would nevertheless persists owing to the positive–definiteness of the $\hat{1}^\mu_\nu$ elements and their consequential topological equivalence to the corresponding conventional elements δ^μ_ν.

One can readily verify that the *operator isocasimir invariants* are structurally the same as the abstract ones (8.3.38) although now realized on $\mathcal{H}$ and can be written

$$\hat{C}^{(0)} * | \hat{\psi} > = \hat{1} * | \hat{\psi} > = | \hat{\psi} >, \quad (8.4.24a)$$

$$C^{(2)} * | \hat{\psi} > = P^2 * | \hat{\psi} > = \hat{\eta}^{\mu\nu} P_\mu * P_\nu * | \hat{\psi} > = P_\mu * P^\mu * | \hat{\psi} >, \quad (8.4.24b)$$

$$C^{(4)} * | \hat{\psi} > = \hat{W}^2 * | \hat{\psi} > = \hat{\eta}^{\mu\nu} \hat{W}_\mu * \hat{W}_\nu * | \hat{\psi} >, \quad \hat{W}_\mu = \tfrac{1}{2} \epsilon_{\mu\alpha\beta\rho} M^{\alpha\beta} * P^\rho, \quad (8.4.24c)$$

The mechanism of embedding the nonlinear–nonlocal–nonhamiltonian interactions should be kept in mind. It can be expressed via the isotopy of the conventional second–order Casimir invariant

$$P^2 | \psi > = \eta^{\mu\nu} P_\mu P_\nu | \psi > \to P^2 * | \hat{\psi} > = \hat{\eta}^{\mu\nu}(x, \dot{x}, ...) P_\mu T(x, \dot{x}, ...) P_\nu T(x, \dot{x}, ...) | \hat{\psi} >. \quad (8.4.25)$$

In ref. [11] we proved that the nonrelativistic limit of the *classical* isopoincaré symmetry is the (classical) isogalilean symmetry. Assume then again $x^2 = [x^\mu \hat{\eta}_b x^\nu] \hat{1} = D^2$ and consider the limit $D/c_0 \to 0$. The proof of the following property is instructive:

Lemma 8.4.1: *The nonrelativistic limit of the operator isopoincaré symmetry is the (operator) isogalilean symmetry,*

$$\text{Lim}_{D/c_o \to 0} \hat{P}(3.1) = \hat{G}(3.1) . \tag{8.4.26}$$

The *isodual operator isopoincaré symmetry* has the same structure of the classical one $\hat{P}^d(3.1) = \hat{L}^d(3.1) \times \hat{T}^d(3.1)$ and it is the largest isolinear–isolocal–isocanonical symmetries of the isodual Minkowski space $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$. Its explicit formulation is left to the interested reader. The operator *universal covering of the Poincaré symmetry*, $\mathscr{P}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ and its isodual $\mathscr{P}^d(3.1) = S\hat{L}^d(2.\hat{C}) \times \hat{T}^d(3.1)$ will be studied via the isotopies of Dirac's equations in Ch. II.10.

A few comments are now in order. A dominant aspect of the operator analysis of this section is the abstract unity between the Poincaré symmetry $P(3.1)$ and its isotopic covering $\hat{P}(3.1)$, as illustrated by the preservation of the conventional structure constants under isotopies and other aspects.

In particular, the preservation of the conventional total quantities (8.4.16) as generators of the isopoincaré symmetry implies the validity of conventional *total conservation laws*, thus resulting in the operator relativistic version of the closed nonhamiltonian systems of ref. [7].

These results are not merely formal, because they hold under the most general possible nonlinear–nonlocal–nonhamiltonian interactions. As such, they have rather deep physical implications simply beyond any technical possibility of the Poincaré symmetry.

As we shall see in Vol. III, the abstract unity between $P(3.1)$ and $\hat{P}(3.1)$ permits novel structure models of nuclei, hadrons and stars with nonlinear–nonlocal–nonhamiltonian internal effects, yet admitting conventional, total, quantum mechanical conservation laws when inspected from the outside. In turn, these new models permit the study of the novel physical applications, such as the interpretation of unstable hadrons as the chemical synthesis of ligther hadrons, the representation of the Bose–Einstein correlation from nonlocal effects in the interior of the p–$\bar{\text{p}}$ fireball, the prediction of a novel subnuclear source of energy and others.

To have a preliminary idea of these new possibilities, recall that a main objective of the isopoincaré symmetries is the quantitative characterization of the *differences* experienced by particles and electromagnetic waves in the transition from motion in vacuum to motion within physical media. This objective is realized via the *the deviation of the isocasimir invariants* (8.4.24) *from the conventional ones,* thus implying different representations and, in the final analysis, a different notion of the very concept of particle.

Moreover, in the next chapter we shall show that the abstract unity of $P(3.1)$ and $\hat{P}(3.1)$ permits a novel quantization of gravity without Hamiltonian. One should be aware that the inclusion of gravity has been the primary reason for our insistence in this chapter to admit isounits with an explicit dependence on the local coordinates.[107] The very *consistency* of the new quantum gravity will

---

[107] Recall that, in the absence of gravitation, internal nonlinear–nonlocal–nonhamiltonian effects can be well approximated as being *velocity dependent* because for null speeds

then be ensured by said unity, while permitting a geometric unification of the special and general relativities.

A further aspect in which said abstract unity has direct physical implications is in regard to *causality*. It has been long believed that nonlocal interactions violate causality. This is indeed the case when they are introduced within the context of conventional quantum mechanics.

However, the use of the isotopic techniques implies a structurally different approach to causality. In fact, we have shown in the preceding analysis that causality is fully verified at the nonrelativistic level under the isotopic treatment of nonlocal interactions, as guaranteed by the isounitary character of the time evolution, the recovering of the number one by the isoexpectation value of the isounit and other reasons. A fully similar result then holds at the relativistic level.

The mechanism for the preservation of causality is, again, the embedding of all nonlocal interactions in the isounit. In turn, this implies the characteristic *integro-differential topology* of hadronic mechanics according to which the theory is everywhere local-differential except at the unity.

This essentially implies that the equations of motion remain focused on the local-differential, center-of-mass trajectory, while the nonlocal effects are mere *corrections* of said local-differential, quantum mechanical trajectory.

8.4.C: Relativistic isokinematics and its isodual. After having identified the basic isosymmetry of relativistic hadronic mechanics, the next logical step is to identify the implied generalization of the conventional relativistic kinematics known under the name of *relativistic isokinematics* on $\hat{M}(x,\hat{\eta},\hat{R})$ (ref. [11], Sect. IV.7).

To begin, the *iso-four-velocity* $u^\mu = dx^\mu/ds$ can be defined by

$$u^2 = u^\mu \hat{\eta}_{\mu\nu} u^\nu = -1, \quad u^4 = \frac{dx^4}{ds} = \hat{\gamma} c = \hat{\gamma} c_0 b_4, \qquad (8.4.27a)$$

$$u^k = \frac{du^k}{ds} = \frac{dx^4}{ds} \frac{dx^k}{dx^4} = \hat{\gamma} \hat{c} v^k = \hat{\gamma} c_0 b_4 v^k, \qquad (8.4.27b)$$

$$\hat{\gamma} = (1 - \beta^2)^{-\frac{1}{2}}, \quad \beta^2 = (v^k b_k^2 v_k)/(c_0 b_4^2 c_0), \quad c = c_0 b_4, \quad (8.4.27c)$$

where v is the ordinary velocity. The extension to isoderivatives $\hat{\partial}/\hat{\partial}s = \hat{1} d/ds$ is implied whenever needed.

We now introduce the *iso-four-momentum*

$$p = \{p^\mu\} = \{m u^\mu\} = \{m_0 \hat{\gamma} \hat{c} v^k, m_0 \hat{\gamma} c\}, \quad m = m_0 \hat{\gamma}. \quad (8.4.29)$$

Then, the isocasimir P^2 in Eq.s (8.4.24) implies the following *fundamental isoinvariant of the isospecial relativity*

resistive forces are generally null.

$$p^2 * |\hat{\psi}> \quad = \hat{\eta}^{\mu\nu} \, p_\mu * p_\nu * |\hat{\psi}> = (b_k^2 \, p_k * p_k - c^2 \, p_4 * p_4) * |\hat{\psi}> \quad =$$

$$= (m_0^2 \, \hat{\gamma}^2 \, c^2 \, v^k \, b_k^2 \, v^k - m_0^2 \, \hat{\gamma}^2 \, c^4) |\hat{\psi}> \; = \; [-m_0^2 \hat{\gamma}^2 \, c^4 (1 - \beta^2)] |\hat{\psi}> \; =$$

$$= (-m_0^2 \, c_0^4 \, b_4^4) |\hat{\psi}> \; = \; (-m_0^2 \, c^4) |\hat{\psi}> . \tag{8.4.39}$$

The abstract identity of the above isoinvariant with the conventional form should be kept in mind for an understanding of the new relativity and of its applications.

Note that the action of P^2 on the isoplanewaves (8.3.14) yields

$$\hat{\eta}^{\mu\nu} \, p_\mu * p_\nu * \hat{\psi} \; = \; \hat{\eta}^{\mu\nu} \, \partial_\mu \, \partial_\nu \, e^{ik_\alpha x^\alpha} \; = \; \hat{\eta}^{\mu\nu} \, k_\mu \, k_\nu \, \hat{\psi} , \tag{8.4.30}$$

which is isopoincaré invariant as expected.

8.4.D: Axioms of relativistic hadronic mechanics. The above results can acquire a better perspective via their reformulation in terms of the following:

AXIOM I: Relativistic hadronic mechanics represents closed-isolated systems of particles at mutual distances smaller than the size of their wavepackets or charge distributions and internal nonlinear-nonlocal-nonhamiltonian interactions via the following mathematical structure: 1) the isoenveloping algebra $\hat{\xi}$ of operators A, B, .., equipped with the isotopic product $A*B = ATB$, $T = T(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, ...) = T^\dagger > 0$, and (left and right) isounit $\hat{1} = T^{-1}$; 2) the isofields $\hat{R}(\hat{n},+,*)$ and $\hat{C}(\hat{c},+,*)$ of isoreal number $\hat{n} = n\hat{1}$ and isocomplex numbers $\hat{c} = c\hat{1}$, respectively, with the same isounit $\hat{1}$; 3) the isohilbert space $\hat{\mathcal{H}}$ with isostates $|\hat{\psi}>, |\hat{\phi}>, ...,$ isocomposition $<\hat{\psi}|\hat{\phi}> = <\hat{\psi}|T|\hat{\phi}>\hat{1} \in \hat{C}(\hat{c},+,*)$ and isonormalization $<\hat{\psi}|T|\hat{\psi}> = 1$; 4) the isominkowski space $\hat{M}(x,\hat{\eta},\hat{R})$, $\hat{\eta} = T\eta$, η = Diag. (1, 1, 1, -1) as the fundamental carrier space; 5) The remaining isotopic methods consisting of functional isoanalysis, the Lie-isotopic theory, etc.;which are such to recover the mathematical foundations of relativistic quantum mechanics at sufficiently large mutual distances under which $\hat{1} = 1$.

AXIOM II: The operators of the theory are constructed in terms of coordinates x and momenta p on an isominkowski space verifying the fundamental isocommutation rules (8.3.14). The observables of the theory are given by isohermitean combinations of coordinates and momenta which, as such, admit real eigenvalues (Sect. I.6.3).

AXIOM III: The time evolution is characterized by the isohermiticity-preserving Lie-isotopic groups of isounitary transforms. The universal symmetry of the theory is the isopoincaré symmetry $\hat{P}(3.1)$ on $\hat{M}(x,\hat{\eta},\hat{R})$

constructed with respect to the considered isounit $\hat{1}$ which preserve conventional generators, thus verifying conventional total conservation laws.

AXIOM IV: The equations of motion are characterized by the second-order isocasimir of $\hat{P}(3.1)$ as per Eq.s (8.4.30), or by their isolinearized forms of Dirac and other types (Ch. II.10).

AXIOM V: The quantities expected in actual measurements of an observable $A = A\hat{\dagger}$ are given by the isoexpectation values $\hat{<}A\hat{>}$ = $< \hat{\psi} \mid T A T \mid \hat{\psi} > / < \hat{\psi} \mid T \mid \hat{\psi} >$. which are real and represent total quantities in the center-of-mass of the systems considered as isolated from the rest of the universe.

The isoaxioms of the *isodual relativistic hadronic mechanics* can be easily derived from the above ones via isoduality $\hat{1} \to \hat{1}^d = -\hat{1}$. Note that such map preserves the sign of the isoexpectation values. However, their referral to the underlying isodual isofield with a negative unit and norm implies the sign reversal of the "numbers" of the theory.

8.4.E: Isorepresentations of the isopoincare´ symmetry and its isodual. The isorepresentation theory of Lie–isotopic algebras is vastly unexplored at this writing. In this section we shall present, apparently for the first time, the essential elements of the isorepresentations of the isopoincaré group $\hat{P}(3.1)$ = $\hat{L}(3.1)\times\hat{T}(3.1)$ of Class 1 which are needed for the applications of Vol. III. More technical studies are deferred to specialized treatments.

Let us begin by introducing the notation

$$M = m_0\, c = {}_0 c_0 / n_4 = M(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, , \mu, \tau, ...) , \qquad (8.4.31)$$

which, for the considered case of Class 1, is positive–definite, null or imaginary depending on whether the mass m is positive–definite, null or imaginary, respectively (since n_4 positive–definite by assumption).[108]

The second–order isocasimir characterizes the following possible cases as for the conventional poincaré group (see, e.g., [23–25]),

$$P^2 = - M^2 < 0, \quad P^4 > 0 \qquad \text{forward isotime–like hyperboloids,} \qquad (8.4.32a)$$

$$P^2 = - M^2 > 0, \quad P_4 < 0, \quad \text{backward isotime–like hyperboloids,} \qquad (8.4.32b)$$

$$P^2 = 0, \quad P_4 > 0 \qquad \text{forward isolight cone,} \qquad (8.3.32c)$$

$$P^2 = 0, \quad P_4 > 0 \qquad \text{backward isolight cone,} \qquad (8.4.32d)$$

[108] This is no longer necessary the case for isotopies of higher class or for genotopies where n_4 can be imaginary , see Sect. II.8.6.

$$P^2 = -M^2 > 0 , \qquad \text{isospace–like hyperboloids.} \qquad (8.4.32e)$$

In this section we shall study only the forward isotime–like and isolight cases $M \geqq 0$ with positive–definite energy $P_4 > 0$.[109] The corresponding isoinvariant measures are given by

$$\hat{\partial}\mu(P) = c \,\hat{\delta}(P^2 + M^2) * \hat{\theta}(P_4) * \hat{\partial}P , \quad \hat{\partial}\mu(P) = c \,\hat{\delta}(P^2) * \hat{\theta}(P_4) * \hat{\partial}P , \quad (8.4.33)$$

where $\hat{\delta}$ is the *isodirac function* (Sect. I.6.4), $\hat{\theta}$ is a simple isotopy of the conventional θ–function, $\hat{\partial}P$ is the isodifferential of Sect. I.8.3 and one should remember that $c = c_0/n_4$.

An *isounitary representation* of $\hat{P}(3.1)$ is a homomorphism of $\hat{g} \in \hat{P}(3.1)$ into the set of isounitary operators $\hat{U}(\hat{g})$ on the isohilbert space $\hat{\mathcal{H}}$ defined with respect to the same isounit $\hat{1}$,

$$\hat{g} \to \hat{U}(\hat{g}), \quad \hat{U}(\hat{g}) * \hat{U}^\dagger(\hat{g}) = \hat{U}^\dagger(\hat{g}) * \hat{U}(\hat{g}) = \hat{1} , \hat{U}(\hat{g}) * \hat{U}(\hat{g}') = \hat{U}(\hat{g} + \hat{g}') , \hat{U}(0) = \hat{1} . \quad (8.4.34)$$

The *irreducible isounitary isoreps* are the same as the conventional ones. The same occurs for the decomposition of reducible into irreducible representations and other aspects.

Recall that no known physical event can change the signature of the Minkowski metric. The isotopies represent this occurrence via a signature–preserving deformation of the Minkowski space and of the related Poincaré group. This means that the topologies of $P(3.1)$ and $\hat{P}(3.1)$ are the same everywhere except at the origin.

The unitary irreducible isoreps of $\hat{P}(3.1)$ can therefore be constructed along lines similar to those of $P(3.1)$, that is, via the representations of the *little isogroup*, i.e., the group leaving invariant the considered hyperboloid.

Our first task is now the identification of the basis $| >$ of the unitary irreducible isoreps of the proper orthochronous isogroup $\hat{P}_+^\dagger(3.1) = S\hat{O}(3.1) \times \hat{T}(3.1)$. Consider the hyperboloid $P^2 = -M^2 < 0$. The operators P_μ are isohermitean and isocommuting. The isoeigenvalues k_μ of P_μ can therefore be assumed as a first set of labels of the basis,

$$P_\mu * | k_\mu , \dots > = -i \,\hat{\partial}_\mu | k_\mu, \dots > = i \,b_\mu^{-2} \partial_\mu | k_\mu, \dots > = k_\mu | k_\mu, \dots > . \quad (8.4.35)$$

Note that the quantities k_μ coincide with the conventional eigenvalues (but not the quantities $k^\mu = \hat{\eta}^{\mu\nu} k_\nu$) as one can see by using for isobasis the isoplane–wave $\exp i (k_\mu x^\mu) = \exp i (k_\mu b_\mu^2 \eta_{\mu\nu} x^\nu)$.

---

[109] Note that the isospace–like hyperboloids define a new notion of tachyons, called *isotachyons*, which will not be considered for brevity. We merely mention the fact that, since the maximal causal speed within physical media can be *lower* than the speed of light in vacuum c_0, *there may exist isotachyons with speed within physical media* $\leqq c_0$.

It is easily seen that the isounitary irreducible isoreps of the isotranslations $\hat{g} = (0, \hat{a}) \in \hat{T}(3.1)$ in $\hat{M}(x,\hat{\eta},\hat{R})$ are given by the isoexponentials

$$\hat{U}(0, a) = \hat{e}^{\,i\,P_\mu\,\eta^{\mu\nu}\,a_\nu} = \{\,e^{\,i\,(\,P^k\,b_k^{\,2}\,a^k\,-\,P^4\,b_4^{\,2}\,a^4\,)}\,\}\hat{1}\,. \qquad (8.4.36)$$

To identify the appropriate little isogroup for $M > 0$, consider the explicit form of the isovector $\hat{W}_\mu = \frac{1}{2}\,\epsilon_{\mu\alpha\beta\rho}\,M^{\alpha\beta}*P^\rho$ under the usual assumptions $J_k = \epsilon_{kij}\,M_{ij}$, $N_k = M_{4k}$

$$\hat{W}_k = P_4\,J_k\,, \qquad \hat{W}_4 = J_k*P^k - (N\,\hat{\times}\,P\,)_k\,, \qquad (8.4.\,37)$$

where $\hat{\times}$ is the *isovectorial product* in isoeuclidean space (Sect. 1.5.2). It is evident that the little isogroup for the hyperboloid $M>0$, $P_4>0$, is the isotopic $\hat{SU}(2)$ group occurring for the particular values $P = (0, 0, 0, P_4)$ with generators $\hat{W}_k = P_4\,J_k$ and $\hat{W}_4 = 0$.

To identify the particular realization of $\hat{SU}(2)$ occurring when it is the little isogroup of $\hat{P}(3.1)$, we decompose the generators J_k into an intrinsic and an orbital part, $J_k = S_k + L_k$ with related isounits $\hat{1}_2$ and $\hat{1}_4$. The $\hat{SU}(2)$ part of the isobasis $|$ k, ... $>$ is therefore the direct product of the S-basis $|$ S, ... $>_2$ and the L-basis $|$ L, ... $>_4$. The spin component of $\hat{SU}(2)$ must necessarily be that in the universal covering of $\hat{P}(3.1)$, i.e., realization (8.3.40), and we write for the 2-dimensional case

$$J_k = \tfrac{1}{2}\,b_k\,\sigma_k\,\hat{1}_2 \quad \text{(no sum)}\,, [\,J_i\,\hat{,}\,J_j\,] = i\,\epsilon_{ijk}\,b_i\,b_j\,b_k^{-1}\,J_k\,, \qquad (8.4.38a)$$

$$J^2\,*_2\,|\,S,\,...\,>_2 = [\,(\,b_1^{\,2} + b_2 + b_3^{\,2}\,)\,/\,4\,]\,|\,S,\,...>_2\,, \quad J_3\,*_2\,|\,S,\,...\,>_2 = \pm\,\tfrac{1}{2}\,b_3\,|\,\,S,\,...>_2\,. \qquad (8.4.38b)$$

As one can see, the conventional value $3/4 = \frac{1}{2}(\frac{1}{2}+1)$ is lost in favor of the expression $(b_1^{\,2} + b_2 + b_3^{\,2})/4$. The value $\frac{1}{2}$ alone cannot therefore be used for the correct labeling of the spinorial basis. We shall therefore write $|\,S,\,...>_2 = |S, b_k >_2$ with the understanding that the conventional eigenvalue $S = 0, \frac{1}{2}, 1, ...$ fixes the *dimension* the isorepresentation $N = 2s+1$, but no longer its weights.

The orbital angular momentum is now in momentum representation on isominkowskian space $\hat{M}(p,\hat{\eta},\hat{R})$ and can be written

$$L_k = \epsilon_{kij}(\,p_i\,r_j - p_j\,r_i\,)\,, \quad r_k\,*_4\,|\,L,\,...>_4 = -\,i\,b_k^{-2}(\,\partial\,/\,\partial p^k\,)\,|\,L,\,...>_4\,, (8.4.39)$$

with commutation rules similar to (II.6.4.5),

$$[\,L_i\,\hat{,}\,L_j\,]\,*_4\,|\,L,\,...>_4 = i\,\epsilon_{ijk}\,L_k\,*_4\,|\,L,\,...>_4\,. \qquad (8.3.40)$$

The isoreps of $\hat{SU}(2)$ have been studied in Sect. II.6.6. In order to compute them in a simple form for the regular three-dimensional case, we introduce the reformulation of the isohyperboloid in $\hat{M}(p,\hat{\eta},\hat{R})$ into a conventional hyperboloid in

$M(\bar{p},\hat{\eta},\hat{R})$, $\bar{p} = T^{1/2}p$, i.e.,

$$p_\mu \,\hat{\eta}^{\mu\nu}\, p_\nu \equiv \bar{p}_\mu \,\eta^{\mu\nu}\, \bar{p}_\nu \,, \quad p_\mu \;=\; b_\mu \,\bar{p}_\mu \,(\text{no sum}), \quad \partial/\partial p^k = b_k^{-1}\, \partial/\partial \bar{p}^k\,, \quad (8.4.41)$$

under which

$$L_1 \;=\; b_2^{-1}\, b_3^{-1}\, \bar{L}_1 \,, \quad L_2 \;=\; b_1^{-1}\, b_3^{-1}\, \bar{L}_2 \,,\, L_3 \;=\; b_1^{-1}\, b_2^{-1}\, \bar{L}_3 \,, \qquad (8.4.42)$$

where $\bar{L}_k$ are the conventional angular momentum operators in $M(\bar{p},\eta,R)$.

We therefore have the expressions

$$L^2 *_4 |\, L, \ldots >_4 \;=\; L_k *_4 L^k *_4 |\, L, \ldots >_4 \;=\; b_k^{-2}\, L_k *_4 L_k *_4 |\, L, \ldots >_4 \;=$$

$$=\; b_1^{-2}\, b_2^{-2}\, b_3^{-3}\, \bar{L}_k\, \bar{L}_k |\, L, \ldots >_4 \;=\; 2\; b_1^{-2}\, b_2^{-2}\, b_3^{-3}\, |\, L, \ldots >_4 \,, \quad (8.4.43)$$

$$L_3 *_4 |\, L, \ldots >_4 = b_1^{-1} b_2^{-1}\, \bar{L}_3 |\, L, \ldots >_4 = b_1^{-1} b_2^{-1} L_3 |\, L, \ldots >_4 \,,\, L_3 = +1, 0, -1. \quad (8.4.44)$$

Again, the conventional value $2 = 1(1+1)$ is lost under isotopies and the basis for the orbital angular momentum cannot be labeled anymore solely with $L = 0, 1, 2, \ldots$ We therefore select the symbol $|\quad L, \ldots >_4 = |\, L, b_k >$ where $L = 0, 1, 2, \ldots$ is essentially used for the identification of the dimensionality of the isorep $N = 2L+1$.

The basis and generators of the little group $S\hat{U}(2)$ can therefore be written

$$|\, S, b_\mu >_2 \times |\, L, b_\mu >_4 \,, \quad \hat{W}_k \;=\; P_4 J_k \;=\; M\{\, (J_k \times \hat{1}_4) + (\hat{1}_2 \times L_k)\,\}\,, \quad (8.4.45)$$

where we have replaced b_k with b_μ because b_4 is contained in $M = m_o c_o b_4$. The additions of spin and angular momentum follow the hadronic rules (Sect. II.6.12), that is, the total angular momentum must characterize an isorep of $S\hat{U}(2)$ thus restricting the admissible values. The total basis of the isorep can be written $|\, > = |\, m_o, k_\mu, S, L, b_\mu >$. The case of higher dimensions is left to the interested reader.

In conclusion, the main result for the case of the hyperboloid $P^2 = -M^2 < 0$, $P_4 > 0$, is that, despite the isomorphism $\hat{P}_+^\uparrow(3.1) \approx P_+^\uparrow(3.1)$, the isorepresentations of $\hat{P}_+^\uparrow(3.1)$ are structurally different than the representations of $P_+^\uparrow(3.1)$. While the latter can be labeled by the mass m_o of the particle and its total angular momentum $J = 0, \tfrac{1}{2}, 1, \ldots$, the former can be labeled with the mass m_o, the characteristic quantities b_μ and the total hadronic spin J which has *continuously varying values* depending on the original value s and on the b's, $J = \hat{J}(J, b)$.

We assume the reader is aware that the above analysis deals with the projection of the isoreps in our space-time.

The case $P^2 = 0$, $p_4 > 0$, also implies structural departures from the conventional one. In fact, the case can be reduced to isorotations in an isoplane, i.e., to the isogroup $S\hat{O}(2)$ acting on the two-dimensional isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$. This isogroup is isoabelian and admits the one-dimensional isorepresentations isoexp$(i\hat{\times}\hat{\theta})$ where $\hat{\theta} = \theta b_1 b_2$ is the isoangle of the isorotations

and $\tilde{\lambda}$ is a *continuously varying* quantity dependent on the characteristic functions, $\tilde{\lambda} = \tilde{\lambda}(b)$ (Sect. 11.6.5). The novelty is now evident. The conventional quantity λ, the familiar *helicity,* can only assume the values $0, \pm\frac{1}{2}, \pm 1, \dots$ On the contrary, its isotopic generalization $\tilde{\lambda}$, called *isohelicity,* can assume continuously varying values in our space–time.

The above results can be verified via direct calculations. The property $P^2 = 0$ implies, as in the conventional case, $\hat{W}^2 = 0$ and therefore the proportionality between $\hat{W}_\mu$ and P_μ

$$\hat{W}_\mu = P_4 J_k * P^k P_\mu = P_4 b_4^{-2} J_k * P_k P_\mu . \tag{8.4.46}$$

But for $P^2 = b_k^2 P_k P_k - b_4^2 P_4^2 = 0$ we have $L_k * P_k = 0$ and P_4 can be written as the conventional modulus of $\bar{P} = \{b_k P_k\}$, $P_4 = |\bar{P}|$. The isohelicity therefore results to be again the component of the total hadronic angular momentum along the direction of propagation

$$|\bar{P}\Gamma^{-1} \hat{\delta}^{ij} J_i * P_j *| > = \hat{\lambda}| > . \tag{8.4.47}$$

The point is that this quantity is no longer given by $0, \pm\frac{1}{2}, \pm 1, \dots$, but is instead a continuous function depending on the explicit value of the b's as indicated earlier. The states for the latter representations can therefore be labeled with $\hat{\lambda}$ and b, $| > = |\hat{\lambda}, b_\mu >$.

Again, the continuously varying character of the isohelicity is referred to its projection in our conventional space–time. It is possible to prove that *helicity* and *isohelicity coincide when computed in their respective spaces and isospaces.*

By recalling that the isoinversions $\hat{\pi} * \hat{\tau} * x = -x$ can be easily reduced to conventional inversions $\pi\tau x = -x$ via the factorizations $\hat{\pi} = \pi\hat{1}$, $\hat{\tau}$, $\tau\hat{1}$, their isorepresentations are given by a simple extension of the conventional ones [24,25] and, as such, they will be assumed as known.[110] Finally, the construction of the isodual isoreps of $\hat{P}^d(3.1)$ are an instructive exercise for the interested reader.

8.5: ISOSPECIAL RELATIVITY AND ITS ISODUAL

8.5.A: Conceptual foundations. As it is well known, the operator version of the special relativity was conceived and constructed to represent exterior relativistic systems such as an electron in the hydrogen atom.

As it is equally well known, *the special relativity resulted to be exactly valid for electromagnetic and weak interactions at large,* a validity which has

---

[110] The sole care needed is in regard to the phase factors which must now have an isoexponential character, e.g., $\hat{U}(\hat{\pi}, 0)^2 = \hat{U}(\hat{\pi}, 0) * \hat{U}(\hat{\pi}, 0) = \hat{\omega} = \text{Diag} (\hat{e}^{i\alpha}, \hat{e}^{i\beta}) = \{\text{Diag.} (e^{iT\alpha}, e^{iT\beta})\}$, α, β, real.

been assumed at the foundations of hadronic mechanics.

Recall that all hadrons have approximately the same charge distribution whose radius coincides with the range of the strong interactions. The objective of the covering isospecial relativity is to achieve a form–invariant description of the *strong interactions* under the condition of mutual penetration of the charge distribution of hadrons which are necessary for their activation, resulting in a structurally more general setting which also exists in the interior of nuclei, hadrons and stars.

In figurative terms, we can say that the special relativity provides a Poincaré–invariant description of the atomic structure without any mutual overlapping of the constituents. The covering isospecial relativity has been built instead to provide an isopoincaré–invariant description of the structure of nuclei, hadrons and stars under conditions of mutual overlapping of their constituents.

By its very conception, such objective implies the search of *numerical, experimentally verifiable departures* from the predictions of the special relativity caused by the interior conditions and related nonlinear–nonlocal–nonhamiltonian internal effects.

The reader should keep in mind that these departures are due to nonlinear and nonlocal effects which are *internal* by central assumption. As such, they cannot be detected in the center–of–mass study of a closed–isolated system from the outside via conventional conservation laws, and require instead special procedures for their identification.

This fundamental point is inherent in the very structure of bound states characterized by the Lie–isotopic theory, the closed nonhamiltonian systems. In fact, it has been proved at both the classical level [7] and its operator counterpart (Ch. 11.7) that the global stability of a system and related total conventional conservation laws are admitted, not only by a system of particles at large mutual distances without collisions (e.g., planetary or atomic systems), but also by systems of particles in mutual overlapping (e.g., the structure of Jupiter or a star). Global stability is trivially achieved in the former case via the stability of each individual constituent, while in the latter case it is achieved via a collection of orbits each individually *unstable*, yet with internal exchanges of energy and other quantities compensating each other in such a way to result in total conservation laws.

Since the current *experimental* knowledge on strong interactions is entirely restricted to center–of–mass treatments, it is inapplicable to ascertain the expected existence of internal nonlinear–nonlocal–nonhamiltonian effects, and fundamentally novel experiments must be done (Vol. III).

The current *theoretical* knowledge on strong interactions is also inapplicable for the deviations under consideration because it is entirely of action–at–a–distance *potential* type. On the contrary, the internal effects here considered are *nonpotential* by conception and, as such, they are not mediated by the usual particle exchanges. It is then evident that any given theoretical

prediction, e.g., for the maximal causal speed, solely based on potential interactions has no bearing on those resulting from additional nonpotential effects.

Yet another difference with conventional patterns is due to the fact that contemporary physics admits only *one* relativity for flat space–time. As a result, both particles and antiparticles are treated via the same relativity. Hadronic mechanics admits instead *disjoint* relativities for the treatment of particles and antiparticles. We therefore have the *special and isospecial relativities for particles* as well as the *isodual special and isodual isospecial relativities for antiparticles*.[111]

Even though isoduality and charge conjugation turn out to be equivalent (Ch. II.10), the study of the isodual representation of antiparticles permits fundamentally novel predictions simply beyond the technical capacity of charge conjugation, such as antigravity, the space–time machine, and others.

The main structural methods of the isospecial relativity have been studied earlier in these volumes, including: the generalized notion of isonumbers; the isominkowskian space; the isopoincaré symmetry; and others.

The isotopies have been selected over other possible approaches (e.g., the q–deformations) because they do imply a structural generalization the original theory while possessing an axiomatic, form–invariant structure which coincides with the original structure at the abstract level, as established by the local isomorphisms $\hat{R}(\hat{n},+,*) \approx R(n,+,\times)$, $\hat{M}(x,\hat{\eta},\hat{R}) \approx M(x,\eta,R)$ and $\hat{P}(3.1) \approx P(3.1)$ (which are not possible under q–deformation).

Thus, *the isospecial and the conventional relativities coincide by conception and construction at the abstract, realization–free level*, as known since the original proposal [8] [112].

The primary task in this section is the the identification of the basic postulates of the isospecial relativity and related *deviations* from those of the special relativity, as presented in the recent paper [15]. Their experimental verification will be studied in Vol. III.

8.5.B: Basic postulates and their isodual. As it is well known, the *basic postulates of the special relativity* are:[113]

---

[111] By no means these four relativities exhaust all possible relativities because they are constructed for *reversible* physical conditions. The axiomatic, form–invariant characterization of *irreversibility* requires yet more general relativities of Lie–admissible type studied in the next section.

[112] This basic occurrence suggests caution in the formulation of criticisms on the covering isorelativity because they may result to be criticisms of the Einsteinian axioms.

[113] The references on the special relativity accumulated during this century are so numerous to prohibit any partial outline and, in any case, they are well known. We here limit ourselves to quote the historical account by Pauli in ref.s [5].

POSTULATE 1: The "maximal causal speed" is the speed of light in vacuum

$$V_{Max} = c_0 = \text{speed of light} = \text{constant};\qquad (8.5.1)$$

POSTULATE 2: The addition of speeds u and v follows the relativistic law

$$V_{tot} = (u + v) / (uv / c_0^2);\qquad (8.5.2)$$

POSTULATE 3: The dilation of time and the contraction of space are characterized by the laws

$$\Delta t = \gamma \, \Delta t_0, \quad \Delta L_0 = \gamma \, \Delta L, \quad \gamma = (1 - \beta^2)^{-1/2}, \quad \beta^2 = v_k v_k / c_0 c_0;\qquad (8.5.3)$$

POSTULATE 4: The behaviour of the frequency of electromagnetic waves follows the Doppler shift law for speed v, angle of aberration α and unit vector e = (cos α, sin α) along the direction of light

$$\omega' = \omega(1 - v \cdot e / c_0)\gamma = \omega(1 - v \cos\alpha / c_0)\gamma,\qquad (8.5.4a)$$

$$\cos\alpha' = (\cos\alpha - \beta)/(1 - \beta\cos\alpha),\qquad (8.5.4b)$$

$$\sin\alpha' = \gamma\sin\alpha(1 - \beta\cos\alpha), \quad \beta = v/c_0.\qquad (8.5.4c)$$

POSTULATE 5: Mass verifies the energy equivalence law

$$E = m_0 c_0^2.\qquad (8.5.5)$$

The *basic postulates of the isodual special relativity* are the image of the preceding ones under the change of the sign of the basic unit, $I \to_\backslash 1^d = -1$, thus being defined on the isodual field $R^d(n^d+,\times d)$ of isodual numbers $n^d = n1^d = -n$ with isodual multiplication $n^d \times^d m^d = -n^d m^d$, isodual norm $|n^d|^d = -|n|$, etc. All physical quantities which are positive-definite for the special relativity become negative-definite for the isodual relativity. Particles therefore have negative-definite energy, angular momentum, etc., and evolve backward in time with the understanding, again, that these negative-definite quantities are referred to a negative-definite unit.

The isospecial relativity is based on the isotopy of the basic postulates of the special relativity. They were first proposed in ref. [8], elaborated in ref. [12], studied in detail at the classical level in ref. [11], Chapter IV, and at the operator level in ref. [13].

The postulates for the nonrelativistic-Euclidean case have been studied in preceding chapters. In this section we can therefore assume spherical symmetry and study the case $b_1 = b_2 = b_3 = b_s \neq b_4$ or $n_1 = n_2 = n_3 = n_s \neq n_4$. We shall liberally pass from the characteristic quantities b_μ to their equivalent form $n_\mu = b_\mu^{-1}$ because they are both used in applications depending on the desired emphasis.

Since the structural elements of relativistic hadronic mechanics have been derived in the preceding sections for the most general possible functional dependence of the characteristic quantities, we also present below the *basic postulates of the isospecial relativity* for the most general possible nonlinear, nonlocal and nonhamiltonian conditions when projected in our space-time $M(x,\eta,R)$. Their re-interpretation in $\hat{M}(x,\hat{\eta},\hat{R})$ is provided below.

POSTULATE 1': The "maximal causal speed" in our space-time is given by

$$V_{Max,\ M(x,\eta,R)} = \left| \frac{dr}{dt} \right|_{Max,\ M(x,\eta,R)} = c_0 \frac{b_4}{b_s} = c_0 \frac{n_s}{n_4} = c\ n_s \quad (8.5.6)$$

POSTULATE 2': The addition of speeds u and v follows the "isotopic addition law"

$$v' = \frac{u + v}{1 + u_k\ b_k^2\ v_k\ /\ c_0\ b_4^2\ c_0} ; \quad (8.5.7)$$

POSTULATE 3': The dilation of time and the contraction of space are characterized by the following "time isodilation" and "space isocontraction" laws

$$\hat{\Delta}t = \hat{\gamma}\ \Delta t_0, \qquad \hat{\Delta}L_0 = \hat{\gamma}\ \Delta L, \quad (8.5.8a)$$

$$\hat{\gamma} = (\ 1 - \beta^2\ )^{-1/2}, \qquad \beta^2 = v_k\ b_k^2\ v_k\ /\ c_0\ b_4^2\ c_0 ; \quad (8.5.8b)$$

POSTULATE 4': The behaviour of the frequency of electromagnetic waves follows the "isodoppler shift law" for speed v, isoangle of aberration $\hat{a} = ab_1b_2$ in the (1-2)-plane with characteristic quantities $b_1 = b_2 = b_s$ and unit isovector $\hat{e} = \{isocos\hat{a},\ isosin\hat{a}\} = \{b_s cos\hat{a},\ b_s sin\hat{a}\}$ in the direction of light

$$\hat{\omega} = \omega(\ 1 - v * \hat{e}\ /\ c\ )\ \hat{\gamma} = \omega \frac{1 - v\ b_s\ cos\ \alpha\ /\ c_0\ b_4}{(\ 1 - v^2\ b_s^2\ /\ c_0^2\ b_4^2)^{1/2}}, \quad (8.5.9a)$$

$$cos\ \hat{a}' = (\ cos\ \hat{a}\ -\ \beta\ )\ /\ (1 - \beta\ cos\ \hat{a}\ ), \quad \hat{a} = \alpha\ b_1\ b_2; \quad (8.5.9b)$$

$$sin\ \hat{a}' = \hat{\gamma}\ sin\ \hat{a}(\ 1\ -\ \beta\ cos\ \hat{a}\ ), \quad \beta = \beta\ b_s/b_4, \quad \beta = v/c_0. \quad (8.5.9c)$$

POSTULATE 5': Mass verifies the "energy isoequivalence law"

$$E = m_0\ c^2 = m_0\ c_0^2\ b_4^2 = m_0\ c_0^2\ /\ n_4^2. \quad (8.5.10)$$

The above generalized postulates are implicit in the preceding formulations, e.g., isoplanewaves (8.2.14) or isocasimir invariants (8.4.24); they

recover identically the conventional postulates in vacuum for which $b_\mu = n_\mu = I$; and they coincide with the conventional postulates at the abstract, realization–free level, where we loose all distinctions between $\hat{I}$ and I, x^2 and x^2, $\hat{\beta}^2$ and β^2, $\hat{\Delta}t$ and Δt, $\hat{\omega}$ and ω, etc.

The *basic postulates of the isodual isospecial relativity* are given by the preceding ones under change of sign of the isounit $\hat{I} \rightarrow \hat{I}^d = -\hat{I}$. As a result, all physical quantities which are positive–definite for the isospecial relativity become negative–definite.

In the following sections we shall identify the most salient departures of the isospecial from the special relativity.

8.5.C: Maximal causal speeds higher than the speed of light in vacuum.

The central physical evidence at the foundation of the isospecial relativity is that the speed of electromagnetic waves is a local quantity in our space–time $M(x,\eta,R)$ because dependent on the local density, chemical composition, etc.

In ordinary physical media, such as our atmosphere or water, the speed of light *decreases*, $c = c_0/n_4 < c_0$, as well known. The isospecial relativity predicts the existence of physical conditions, such as those in hyperdense media in the interior of stars with $b_4/b_3 = n_3/n_4 > I$, where the maximal causal speed can be higher than c_0.

The hypothesis that strong interactions could accelerate massive physical particles at speed higher than c_0 was submitted by this author in ref. [27] under the assumption that they have a contact–nonpotential component which is responsible for speeds beyond c_0. In different terms, when dealing with a massive particle in an accelerator (exterior relativistic problem), the only possible interactions are at–a–distance, in which case c_0 is indeed the limiting causal speed because it takes an infinite amount of energy to accelerate a massive particle to the speed c_0.

However, when dealing with interior relativistic conditions, e.g., the same particle in the interior of an exploding star, we have the additional *contact* interactions which do indeed accelerate particles although without any energy consideration. The "breaking of the barrier of the speed of light" c_0 is then consequential.

As an illustration, theoretical studies based on conventional gauge theory conducted by de Sabbata and Gasperini [28] have indicated that the maximal causal speed through a body with the density of a hadron can be of the order of $75c_0$. The current, direct and indirect experimental knowledge on this prediction is studied in Vol. III.

The above considerations refer to the maximal causal speed in our space–time, that is, to the projection of the isospecial relativity in $M(x,\eta,R)$. It should be indicated that *in the isominkowski space–time the maximal causal speed remains c_0* This can be easily proved via functional isoanalysis in $\hat{M}(x,\hat{\eta},\hat{R})$. Recall from Sect. II.8.2.1 that the isodifferentials are given by $\hat{d}x^\mu = T^\mu_{\ \nu}dx^\nu =$

$b_\mu{}^2 dx^\mu$ (no sum). The correct form of the isoline element in $\hat{M}(x,\hat{\eta},\hat{R})$ is therefore given by

$$ds^2 = [\, \partial x^\mu \, \hat{\eta}_{\mu\nu} \, \partial x^\nu \,]\hat{1} = [\, \partial r^k \, b_k{}^2 \, \partial r^k \, - \, \partial t \, c_o{}^2 \, b_4{}^2 \, \partial t \,]\hat{1} \,, \quad (8.5.11)$$

from which one obtains the maximal causal speed in $\hat{M}(x,\hat{\eta},\hat{R})$ for $ds^2 = 0$

$$V_{\text{Max, } \hat{M}(x,\hat{\eta},\hat{R})} = \left| \frac{\partial r}{\partial t} \right|_{\text{Max, } \hat{M}(x,\hat{\eta},\hat{R})} = \frac{\left| \partial r^k \, b_k{}^2 \, \partial r^k \right|^{1/2}}{\left| \partial t \, b_4{}^2 \, \partial t \right|^{1/2}} = c_o \,. \quad (8.5.12)$$

Note that, since the maximal causal speed is $V_{\text{Max}} = c_o b_4 / b_S$, the square root $(1 - v^2 b_S{}^2 / c_o{}^2 b_4{}^2)^{1/2}$ in the definition of the isotopic $\hat{\gamma}$ cannot become imaginary. This author has stressed this point in the preceding ref.s (see, e.g., [9,10]). At times, an absolute value has been added in the preceding literature, i.e., $\hat{\gamma} = |\, 1 - \beta^2\,|^{-1/2}$, but only to exclude the *negative* value of the square root. Note also that the correct definition of *tachyons* is that of *particles travel;ing at speeds higher than the local maximal causal speed*, whioch are called for clarity *isotachyons* hereon. In fact, the isospecial relativity identifies particles traveling at speeds higher than c_o which are ordinary massive particles and not tachyons.

8.5.D: Differentiation of speed of light and maximal causal speed. Another visible departure of the isospecial from the special relativity when both studied in our space–time is the abandonment of the speed of light as the invariant speed in favor of quantity (8.5.6).

This is an intrinsic property of all deformed Minkowski spaces. It represents the maximal causal speed as characterized by an effect following a cause. We are here referring to a physical cause and subsequent effect not necessarily mediated by a photon, such as the impact of a particle against a heavy nucleus of diameter D at one given time t_1 and the consequential emission of particles in the opposite side at a time $t_2 > 0$. The isospecial relativity predicts that the quantity $D/(t_2 - t_1)$ can be *bigger* than c_o.

The conceptual departure of hadronic from quantum mechanics should be focused here. Quantum mechanics studies particles interacting at–a–distance without appreciable overlapping. Cause and effect must then be necessarily mediated by a massless particle such as a photon propagating in vacuum. The maximal causal speed c_o then follows.

Hadronic mechanics studies fundamentally different physical conditions in which particles are in conditions of deep mutual overlapping. Since there is no action–at–a–distance, cause and effect can also be realized without any exchange of particles. The speed of propagation of the latter is then merely dependent on the density and, for sufficiently high densities, it can exceed the speed of propagation of massless particles.

It is easy to verify that *the maximal causal speed in physical media, when computed in our space–time* $M(x,\eta,R)$, *must necessarily be different than the speed of light,* assuming that light can propagate in the media considered.

In fact, the insistence in keeping the speed of light as the maximal causal speed for propagation of light in water leads to a number of inconsistencies, such as the fact that *electrons can propagate in water at speeds higher than the assumed maximal causal speed*, as established by the known *Cherenkov light*.

If one returns to the assumption of c_0 as the maximal causal speed to resolve this inconsistency, one would then have additional problematic aspects, such as the violation of both the conventional and isotopic laws of addition of speeds, none of which yields the speed of light as the sum of two light speeds $u = v = c = c_0/n_4$. In fact, both the conventional and isotopic laws of addition of speeds yield for $u = v = c_0/n_4$ the value $2c_0n_4/(1 + n_4^2)$ which is neither c_0 nor the speed of light in water.

The resolution of the above inconsistencies by the isospecial relativity is immediate. Since water is a *homogeneous and isotropic* medium it must verify the condition $b_s = b_4 = b$ $(n_s = n_4 = = n)$. The maximal causal speed in water for the isospecial relativity therefore remains c_0,

$$V_{Max, M(x,\eta,R)} = c_0 b_4 / b_s = c_0 . \qquad (8.5.13)$$

This permits the interpretation of c_0 as the causal speed of massive particles in water, as experimentally established. The point is that such interpretation necessarily requires the differentiation of the maximal causal speed with the physical speed of light in the medium considered. Also, the addition of two maximal causal speeds in water yields again the maximal causal speed according to law (8.5.7).

All these occurrences are geometrically represented via the simplest possible isotopy of the Minkowskian invariant, the *scalar isotopy*

$$x^2 = (x^t \hat{\eta} x)\hat{1} = [b^2 (x^t \eta x)] b^{-2} \equiv x^2, \quad b_\mu = b, \, T = b^2 1, \hat{1} = b^{-2} 1 , \qquad (8.5.14)$$

which, as one can see, *coincides* with the conventional separation. Nevertheless, the isotopy is not trivial because, e.g., the isolorentz transforms are noncanonical.

The representation of the above case in isospace $\hat{M}(x,\hat{\eta},\hat{R})$ is straightforward. In fact, the isominkowskian geometrization can be visually interpreted as implying the "disappearance of water". Therefore, light propagation in water, when represented in $\hat{M}(x,\hat{\eta},\hat{R})$, is equivalent to light propagation in vacuum.

Even greater inconsistencies emerge if one insists in keeping the speed of light as the invariant speed in our space–time for all media more complex than water, such as transparent media that are inhomogeneous and anisotropic. As we shall see, a necessary condition to resolve these inconsistencies is the general separation in our space–time of the invariant speed from the speed of light, and the use of their identity only in vacuum or in their isominkowskian representation.

8.5.E: Isotopic variation of the meanlife and size of particles. According to the

special relativity the meanlife τ_0 of an unstable particle and the size D_0 of its charge distribution can only vary according to laws (8.5.3) and are independent from the local physical conditions of density, temperature, etc.

According to the isospecial relativity the meanlife and size of particles are subjected to the isotopic laws (8.5.8) and do depend on the interior physical characteristics, i.e.,

$$\hat{\tau} = \tau_0 | 1 - v^k b_k{}^2 v^k / c_0{}^2 b_4{}^2 |^{-1/2}, \quad \hat{D} = D_0 | 1 - v^k b_k{}^2 v^k / c_0{}^2 b_4{}^2 |^{1/2}. \quad (8.5.15)$$

As a first example, consider a neutron in vacuum with its meanlife of 918''. The prediction of the isospecial relativity is that, independently of any nuclear interaction, the neutron has a *different* meanlife when a member of a nuclear structure because nuclei have characteristic constants $b^\circ_\mu \neq 1$. In particular, such constants can be such to imply an *increase* of the meanlife of the particle. This property will be used in Vol. III to study the origin of the stability of the neutron when a member of a nuclear structure.

Therefore, the isotime dilation has fundamental relevance for the experimental verification of deviations from the Minkowskian space expected in the interior of nuclei, hadrons and stars.

This issue will be studied in detail in Vol. III. We here briefly mention that deviations from the Minkowskian behaviour of the meanlife with speed were predicted a long time ago, e.g., by Blochintsev and his school at the JINR in Russia (see, e.g., ref. [29] and quoted papers), Redei and his associates in Italy (see ref. [30]), Kim in Canada (see ref. [31]), by Nielsen and Picek [32] via conventional gauge theory, and others.

By keeping in mind that the center–of–mass trajectory of a hadron in a particle accelerator must obey the special relativity, these authors argued that the only way according to which internal nonlinear–nonlocal–nonhamiltonian effects can manifest themselves in the outside is via a *nonminkowskian behaviour of the meanlife of unstable hadrons with speed.*

The first clear experimental verification of these predictions was achieved at FERMILAB by Aronson et al. [33] for the K°–system in the range 30 to 100 GeV. A subsequent experiment conducted by Grossman et al [34] confirmed the Minkowskian behaviour for the same particles in the *different* energy range of 100 to 350 GeV.

Finally, Cardone et al. [35] showed that the isominkowskian characterization of the medium inside the K°–system and related isotopic time–dilation law proposed by this author [8] permits an exact numerical representation of *both* seemingly contrasting experimental data [33,34].[114]

[114] To minimize intentional misrepresentations this author has encountered several times, we must stress that the verification of the Minkowskian geometry by Grossman et al [34] can only be claimed in the energy-range from 100 to 350 GeV which is DIFFERENT than the range 30 to 100 GeV of Aronson et al. [32]. Moreover, the experiment by

If confirmed, these results will provide an experimental verification of a fundamental assumption of the isospecial relativity, that the hyperdense medium inside hadrons composed by the deep overlapping of wavepackets, wavelengths and charge–distributions of all constituents imply geometric departures from the vacuum (Fig. 11.8.2.2).

The isominkowskian geometrization and related isospecial relativity then follows uniquely and unambiguously because of their "direct universality" (see also later on in this section).

A similar situation occurs for the *dimension* of particles when immersed within hyperdense media. In fact, as it is now familiar, the isominkowskian geometrization necessarily implies an isotopic notion of distance and of its behaviour with speed.

It should be recalled however that the space contraction is still unsettled at the pure level of the special relativity (see, e.g., the critical analysis by Strel'tsov [36] and quoted references). Since the isospecial relativity is an isotopy of the special one, the topics that are still unsettled at the former level evidently carry over at the latter one. We shall therefore abstain from considering the problem of the space contraction until first resolved within the context of the special relativity.

8.5.F: Isodoppler shift of light within physical media.

Consider the familiar Doppler shift law for light propagating in vacuum with null angle of aberration

$$\omega = \omega_0 \left(1 - v/c_0 \right) \left(1 - v^2/c_0^2 \right)^{-1/2} , \tag{8.5.16}$$

Since $v \ll c_0$, $\beta \ll 1$ and $\beta \gg \beta^2$, light is redshifted (i.e., $\omega < \omega_0$) for $v \neq 0$ and $v > 0$ as one can see from the expansion

$$\omega \approx \omega_0 \left(1 - \beta + \tfrac{1}{2}\beta^2 + \dots \right), \quad \beta = v/c_0 . \tag{8.5.17}$$

Consider now the same configuration but for light propagating within a physical medium of low density such as planetary atmospheres or quasar chromospheres. Experimental evidence establishes that the speed of light

Grossman et al. [loc. cit.] is plagues by rather serious flaws in the *theoretical assumptions* of the data elaboration. For instance, the experimenters assumes for such data elaboration a reference frame in which there is no CP violation. This is ultimately equivalent to the assumption of a frame in which the deviation to be tested is absent because both CP violation and anomalous behaviour of the meanlife are dependent on the nonlocality of the structure of the kaons (see in this latter respect, e.g., Kim [31]). Claims of experimental evidence on the exact validity of the special relativity in the interior of kaons are therefore of nonscientific character. The only scientific statement possible at this writing is that the issue is fundamentally unsettled on experimental grounds and in need of comprehensive verifications at *all* energy ranges, beginning precisely with a re–run of the measures by Grossman et al [34] *without* manifestly questionable theoretical assumptions in the data elaboration.

decreases within such physical media, according to the familiar law $c = c/n_4 < c_0$. The use of this lower value of the speed in Eq. (8.5.16) then implies a natural, additional shift toward the red due to the medium,

$$\omega' = \omega_0 (1 - v/c)(1 - v^2/c^2)^{-1/2} \approx$$

$$\approx \omega_0 [1 - v/(c_0/n_4) + \tfrac{1}{2} v^2/(c_0/n_4)^2 + ...] <$$

$$< \omega_0 (1 - v/c_0 + \tfrac{1}{2} v^2/c^2 + ...) \text{ for } c < c_0. \tag{8.5.18}$$

In fact, we still remain with the properties $v \ll c_0/n_4$, $v/(c_0/n_4) \ll 1$, $v/(c_0/n_4) \gg v^2/(c_0/n_4)^2$ and $v/(c_0/n_4) > v/c_0$. The term $v/(c_0/n_4)$ is therefore dominant in the shift resulting in the value $1-v/(c_0/n_4) < 1-v/c_0$ and consequential additional redshift $\omega' < \omega < \omega_0$.

With reference to Fig. 8.2.1, this establishes that, if the original light was redshifted according to the Doppler law (8.5.16) prior to reaching Earth's atmosphere, the same light exits Earth's atmosphere with an *additional* redshift due to the decrease of its speed called *isoredshift*.

But the Doppler law (8.5.18) is not Lorentz invariant because dependent on the noninvariant line element $ds^2 = dr^k dr^k - dt (c_0/n_4)^2 dt$. This leads in a unique and unambiguous way to the isolorentz invariant line element $ds^2 = [dr^k (1/n_k^2) dr^k - dt(c_0/n_4) dt]$ and the isodoppler law of Postulate 3′

$$\hat{\omega} = \omega_0 \frac{1 - (v/c_0)(n_4/n_s)\cos(\alpha\, b_s^2)}{(1 - v^k (1/n_k)^2 v^k / c_0 (1/n_4)^2 c_0)^{1/2}} \tag{8.5.19}$$

which can be written via a series expansion for the case of null angle of aberration and space–isotropy

$$\hat{\omega} \approx \omega_0 [1 - \beta(n_4/n_s) + \tfrac{1}{2}\beta^2(n_4/n_s)^2 + ...] \tag{8.5.20}$$

In summary, besides the modification of the speed of light $c_0 \rightarrow c_0/n_4$, the correct, form–invariant expression requires the additional modification for the speed $v^k v^k = v^2 \rightarrow v^k (1/n_k)^2 v^k = v^2/n_s^2$. The knowledge of the space and time characteristic quantities n_μ are therefore necessary for the experimental verification of the above predictions. Note that the measure of an isoredshift would only yield the ratio n°_4/n_s for the medium considered. The value n°_4 can be computed from the average speed of light in that medium and the value n_s would be known accordingly.

As we shall see in Vol. III, astrophysical measures in *cosmological redshifts of galaxies and quasars* provide a confirmation of the isospecial relativity. In essence, planetary atmospheres or astrophysical chromospheres are such that $n_4/n_s > 1$ thus implying a natural isoredshift of light. In particular, this

confirms the hypothesis first submitted in ref. [12] that *light exits the quasar chromosphere already redshifted.* In fact, light is emitted in the interior of the quasars and propagates in their huge chromospheres estimated in the order of millions of Km before reaching empty space. The hypothesis of ref. [12] is therefore that light exits the quasars chromospheres already redshifted simply because of the decrease of its speed in its interior. In turn, such isoredshift, which is absent for the associated galaxy, permits an exact numerical representation of the measured difference in cosmological redshift between certain quasars and their associated galaxies.

Arp [37] was one of the firsts to provide astrophysical measures according to which certain quasars are at rest with respect to the associated galaxies (or expelled at very low speeds which are insignificant for the redshift). Sulentic [38] then provided experimental evidence of physical connections between certain quasars and their associated galaxies via the superposition of gamma spectroscopic plates. Such evidence evidently rules out the credibility of the Minkowskian interpretation of the difference of redshift between the considered quasars and the associated galaxies because the latter imply such a large difference in speed that the quasar and associated galaxy would have completely separated billions of years ago contrary to gamma spectroscopic evidence.

The isominkowskian interpretation of the difference in redshift was submitted by Santilli [12] as simply due to the decrease of the speed of light in the quasars chromospheres. Mignani [39] subsequently provided an exact numerical interpretation of Arp's data via the isospecial relativity (see Vol. III for all details).

The isoredshift also provides a natural interpretation of the frequency shift measured for light originating from the sun chromosphere since quite some time (see the ref. [40] and literature quoted therein). These measures are particularly relevant for an experimental verification of the isodoppler law because they essentially hold for an angle of aberration $\alpha = 90°$ between the radial motion of Earth with respect to the Sun light, in which case the term in β is null in law (8.5.19).

In additional to the cosmological redshift, quasars have also shown to possess an *internal redshift and blueshift,* that is, a basic frequency usually assumed to be 4680 A° is redshifted. The redshift then increases for values below such frequency, while decreases for value above such frequency but with the understanding that the "internal blueshift" remains cosmologically redshifted (see, e.g., Sulentic [41] for details).

In order to interpret this additional astrophysical information, recall that the refraction of light depends on the wavelength. This implies that the *isodoppler shift possesses a dependency on the frequency itself.* To reach an expression usable for astrophysical verifications, we factorize such a dependence in the ratio

$$\frac{n_4}{n_3} = \frac{n°_4}{n°_3} f(\omega_0), \quad f(\omega_0) > 0, \qquad (8.5.21)$$

where the n°'s represent the remaining contribution after the factorization averaged to constants. In this case the isodoppler law becomes

$$\hat{\omega} = \omega_0 \frac{(1 - v/c_0)(n^{\circ}_4/n^{\circ}_s) f(\omega_0) \cos(\alpha\, b_s^2)}{|1 - \dfrac{v^k n^{\circ}_k{}^{-2} v^k}{c_0\, n^{\circ}_4{}^{-2} c_0} f(\omega_0)^2|^{1/2}} \qquad (8.5.22)$$

and can be written in the approximate form for space symmetry $n_k = n_s$ and null angle of aberration

$$\hat{\omega} \approx \omega_0 \, (1 - \beta\,(n^{\circ}_4/n_s)\,f(\omega_0) + \tfrac{1}{2}\beta^2\,(n_4/n_s)^2\,f(\omega_0)^2 + \ldots. \qquad (8.5.23)$$

As presented by Santilli at the Olympia Conference of 1993 [42][115] (see also the review by Sulentic [41]), the isospecial relativity permits an exact–numerical representation of the available data [40] on the internal quasar redshift and blueshift via a functional dependence of the type $f(\omega) = K_1 \exp K_2(\omega - \omega_0)$, with K_1 and K_2 constants, thus providing an additional remarkable verification of the isospecial relativity.

It should be noted that, while numerous interpretations exist for the cosmological redshift (see, e.g., ref. [43,44]), the interpretation of the internal redshift–blueshift via the isospecial relativity [42] is the only one on record at this time (Summer 1994) to our best knowledge.

The above lines are however only the beginning of the rather complex phenomenology of astrophysical light shift. In fact, the isospecial relativity predicts that *light experiences a blueshift when propagating in media of sufficiently high density*. This is evidently the case for physical media with $n_4/n_s < 1$ which, as we shall see in Vol. III, are precisely the hyperdense media in the *interior* of quasars.

The isospecial relativity therefore predicts that the currently measured cosmological redshift of quasars may in actuality be a difference between the isoblueshift in the interior of the quasars structure and the isoredshift in their chromosphere. The fact that redshift is measured in the outside is merely an indication of the dominance of the latter over the former (see Vol. III for details).

In summary, the isospecial relativity has the following predictions for the behaviour of the frequency shift of electromagnetic waves propagating within physical media:

1) It recovers the experimental evidence that *light does not experience any shift when propagating within homogeneous and isotropic media* such as water for which $n_4 = n_s$, $\hat{\beta} = \beta$ and $\hat{\gamma} = \gamma$. In this case *light does not lose energy* $E = \hbar\omega$

[115] It should be noted that the isoredshift of ref. [42] was primarily referred to the case of angle of aberration $\alpha = 90°$.

to the medium itself and only the wavelength changes.

2) It predicts that *light experiences* a *frequency shift toward the red when propagating within inhomogeneous and anisotropic media of low density (isoredshift) with null angle of aberration* and $n_4/n_s > 1$. In the latter case *light loses energy to the medium.*

3) It predicts that *light experiences a frequency shift toward the blue when propagating within inhomogeneous and anisotropic physical media of sufficiently high density (isoblueshift) with null angle of aberration* and $n_4/n_s < 1$. In the latter case *light acquires energy from the medium.*

4) It predicts that *the isoredshift and isoblueshift are frequency dependent* resulting in an internal dependence respect to a characteristic frequency.

5) It predicts that *the isotopic shift of frequency exists also for angle of aberration* $\alpha = 90°$. In this case the shift is the opposite than that for $\alpha = 0$ because $v*\hat{e} \equiv 0$ and the term in β is absent in law (8.5.19), resulting in the isotopic law $\hat{\omega} = \omega_0[1 + \beta^2(n^4/n_s)^2 + ...]$, with $\hat{\omega} > \omega = \omega_0(1 + \beta^2 + ...)$ for $n_4/n_s > 1$ and $\hat{\omega} < \omega$ for $n_4/n_s < 1$..

The above studies have lead to the prediction that a component of the tendency toward the red of Sun light at sunset is precisely of isotopic origin [9,11], that is, due to the established decrease of speed of light in our atmosphere. The experimental resolution of this prediction is advocated in Vol. III because it would imply a resolution here on Earth of the problem of quasars red/blue/shifts.

We should finally mention that deviations from the special relativity caused by light dissipating energy within physical media were studied at the turn of this century (see, e.g., Lorentz [45] and the comments by Pauli in ref.s [5], Sect.s 6 and 36), but they were regrettably abandoned following the establishing of the special relativity. This is unfortunate because the imposition of an invariant description would have lead to a necessary generalization of the structure of the minkowski space. The isotopic structure uniquely follows, again, from its direct universality.

We finally mention for completeness that the aberration still presents unsettled aspects at the pure Minkowskian level (see, e.g, [46]).

8.5.G: Mutation of the intrinsic characteristics of particles and the notion of isoparticle. According to the special relativity, elementary particles have perennial and immutable intrinsic characteristics of rest energy, magnetic moments, charge, spin, etc., irrespective of their physical conditions. For instance, an electron has rest energy of 0.511 MeV, spin $\frac{1}{2}$, magnetic moment $\mu_e = 1$, charge e, etc., irrespectively of whether it is in the hydrogen atom or in the core of a collapsing star.

The isospecial relativity predicts instead that elementary particles in interior relativistic conditions experience an alteration of their *intrinsic* characteristics called *mutation,*[116] as first conjectured by this author back in

---

[116] The origin of the term "mutation" is mathematical. It was submitted by this author

1978 [47] via a generalization of the Dirac equation with nonpotential interactions.

The dependence of the meanlife of particles on the local conditions has been discussed earlier. Postulate 5' clearly shows the local variation of the rest energy which assumes the value in our space–time $\tilde{E} = mc^2 = mc_0^2/n_4^2$. Note that the mutation of the rest energy is a direct consequence of the isopoincaré symmetry and, appears in the mutation of the structure of the second–order isocasimir $P^2 = -(m_0 c_0)^2 \rightarrow \hat{P}^2 = -(m_0 c_0/n_4)^2$. This implies that an electron, when immersed in the hyperdense medium inside a star geometrized by the characteristic quantity b_4, acquires a value of its rest energy which is *different* than the value 0.511 MeV.

The mutation of the intrinsic and orbital angular momenta have been studied in Sect. II.8.4.E They also originate from the isopoincaré symmetry, and are characterized, this time, by the mutation of the Pauli–Lubanski isocasimir invariant $W^2 \rightarrow \hat{W}^2$.

The mutations of the intrinsic magnetic moment, charge and other characteristics are then a necessary consequence, as we shall see in Ch. II.10.

Because of these occurrences, ref. [11] introduced the term of *isoparticle* to denote a particle characterized by the isospecial relativity, that is, by an isounitary irreducible isorep of the isopoincaré symmetry. Its primary difference with the ordinary particles as characterized by the special relativity are precisely the mutations of the *intrinsic* characteristics due to the transition from the conventional Casimir invariants to their isotopic generalizations.

The understanding of the isospecial relativity requires the knowledge that *particles and isoparticles coincide at the abstract level when represented in their respective space–times.* In different terms, the mutations exist only when the intrinsic characteristics of particles are projected in our space–time, while in isospace–time the intrinsic characteristic experience no mutation at all.

This occurrence is similar to the preservation of the maximal causal speed c_0 in isospace–time, while a mutation of the same speed occurs when projected in our space–time.

The notion of isoparticle is at the foundation of the primary predictions of the isospecial relativity, i.e., novel structure models of nuclei, hadrons and stars which, in turn, imply the prediction for a novel subnuclear energy originating at the level of individual hadrons called *hadronic energy* [53]. As such, the mutations of the intrinsic characteristic of particles will be a primary subject of study in Vol. III.

At this point it is sufficient to illustrate the experimental foundations of the mutations in their simplest possible form. Recall that perfectly rigid bodies are a philosophical abstraction and do not exist in nature. A hadron such as a neutron cannot therefore be perfectly rigid and must experience a deformation

---

[47] to characterize alterations of physical characteristics under "isotopies" (in which the new symmetry is isomorphic to the original one), and to differentiate them from alterations of physical characteristics due to "deformations" (in which case the new symmetry is no longer isomorphic to the original one)

of its charge distribution under sufficiently intense external fields or collisions.

Note that the amount of the deformation for given external conditions is open to scientific debate, but the existence of the deformation itself is beyond scientific doubts.

According to classical electrodynamics, the deformation of a charged and spinning sphere necessarily implies the alteration of its intrinsic magnetic moment. The deformation of the shape of the neutron therefore necessarily implies an alteration of its intrinsic magnetic moment which is an example of the mutations under consideration here.

This establishes the existence of physical conditions under which the special relativity is inapplicable or at best ineffective for the representation of hadrons as extended objects undergoing deformations of their charge distributions. In fact, the conventional representation requires the computation of form factors in second quantization with known problematic aspects whenever the actual shape is nonspherical and undergoes deformations because of the evident breaking of the rotational symmetry.

By comparison, the covering isospecial relativity represents actual nonspherical shapes and all their infinitely possible deformations beginning at the purely classical level (see Fig. 11.8.4.2 and ref. [12]). After all, the special relativity was conceived for point-like particles while its isotopic covering has been constructed for extended-deformable particles.

At a closer scrutiny, the inapplicability of the special relativity originates from its true foundations, the rotational symmetry SO(3), which is notoriously applicable only to rigid particles. By the same token, the applicability of our covering relativity originates from its own foundations, the isorotational symmetry SÔ(3), which has been constructed for deformable particles (Ch. 11.6).

In summary, the simplest possible illustration of an isoparticle and related mutation is given by a neutron in interactions with sufficiently intense external fields such to cause a deformation of its charge distribution (without necessarily altering the other characteristics of spin, charge, etc.). The alteration of the intrinsic magnetic moment is then consequential.

This simplest possible mutation is, by no means, new because it was conjectured since the early stages of nuclear physics. For instance one can read in ref. [48], p.31, that the intrinsic magnetic moment of a nucleon may be different when in close proximity to another nucleon, e.g., when a member of a nuclear structure.

This old hypothesis is very naturally represented in an exact-numerical way by the isospecial relativity. In fact, as we shall see in detail in Vol. III, the hypothesis is preliminarily confirmed by Rauch's experiment [49] on the 4π-spinorial symmetry of a thermal beam of neutrons in interaction with the intense electric and magnetic field of Mu-metal nuclei. These measures show a median angle systematically smaller than 72o° although the measures are not yet final. Rauch's data are interpreted in an exact-numerical way by the isospecial

relativity via the isodirac equation with *conventional* spin merely representing a small deformation of the neutron and of the related intrinsic magnetic moment [50].

Finally, the mutation of the magnetic moment of nucleons has been used by this author for the first achievement on record of the exact–numerical representation of the total magnetic moment of few–body nuclei, as first presented at the meeting "Deuteron 1993" at the JINR in Dubna [50].

Stated in different terms, after over half a century of attempts via the use of all possible corrections, the special relativity has not permitted an *exact–numerical* representation of the total magnetic moment of the deuteron and of the *few–body* nuclei[117] based on *conventional* values of the magnetic moments of the nucleon constituents. On the contrary, the exact–numerical representation of the same quantity via the covering isospecial relativity is direct and immediate because a mutation of the intrinsic magnetic moment is now permitted.

Once the mutation of the intrinsic magnetic moment is theoretically and experimentally established, the mutations of the remaining characteristics for physical conditions more complex than the nuclear ones is then inevitable, and so are their rather deep consequences.

One can equivalently arrive at the mutation of the intrinsic characteristics of particles via the renormalization theory. As it is well known, all interactions imply renormalizations. The mutation of the *intrinsic* characteristics then emerges from the non*lagrangian* character of the interactions considered, as established since their original proposal [47].

8.5.H: Isotopic variation of the units of space and time. According to the special relativity, space and time are "universal" in the sense that they are the same for all possible observers throughout the Universe with the same speed relative to an inertial system.

While preserving the conventional variation with speed, *the isospecial relativity predicts that space and time have an additional variation due to the change of their isounits with local physical conditions.*

Recall that the isospecial relativity is based on the following *isounits of space and time in isominkowski space*

$$1_s = \{ b_1^{-2}, b_2^{-2}, b_3^{-2} \}, \quad 1_t = b_4^{-2}, \quad b_\mu = b_\mu(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, ...) > 0 . (8.5.24)$$

Recall also that, when the isospace $\hat{M}(x,\hat{\eta},\hat{R})$ is projected in our space-time, it yields the generalized separation $x^\mu \hat{\eta}_{\mu\nu} x^\nu$ which can be interpreted as a conventional separation $\bar{x}^\mu \eta_{\mu\nu} x^\nu$ with local coordinates $\bar{x} = (b_\mu x^\mu)$ (no sum). Thus, the projection of $\hat{M}(x,\hat{\eta},\hat{R})$ in $M(x,\eta,R)$ can be represented via conventional coordinates $x = \{x^\mu\}$ and the following *isounits of space and time in our*

---

[117] For many-body nuclei the situation is different because of the large number of parameters at hand.

Minkowski space

$$\hat{1}_s = \{\, b_1^{-1}, b_2^{-1}, b_3^{-1} \,\}, \qquad\qquad \hat{1}_t = b_4^{-1} . \qquad\qquad (8.5.25)$$

Recall the central property of the isotopic methods, that no symmetry or geometry can possibly identify its own unit because it is and remains external. This basic property is the mathematical representation of the fact that, here on Earth, we can do all possible *comparative* measures of time, but we cannot possibly measure its isounit $\hat{1}_t$ and the same occurs for space. If such an *external* unit is indeed a local quantity, the variation of time with that of the local conditions under the same speed is then consequential.

A central problem of the isospecial relativity is the identification of the dependence of the isounit which may alter the behaviour of our time. The answer submitted in the next chapter is that *the primary dependence of the isounit affecting the time evolution is that on the local gravitational field.*

In different terms, the representation of the contact—resistive forces due to physical media via the isounit has clearly a mathematical character which, as such, may not affect our time evolution. However, the representation of gravitation via the isounit studied in the next chapter may well have a direct physical meaning, in which case the alteration of the units of space and time would be consequential.

An important prediction of the isospecial relativity is that, under the assumption of the same speed relative to an inertial system, time on Jupiter's atmosphere evolves in a way different than time on Earth's atmosphere because of the significant differences in the two gravitational fields.

A fundamental experimental problem raised by the isospecial relativity is whether we live in a conventional space and time with units $1_\mu = 1$ or we live in a space and time with isounits $\hat{1}_\mu$ given by Eq.s (8.5.25).

It is evident that a local character of the unit of space and time would have far reaching implications at all epistemological, theoretical and experimental levels. For instance, the terms "age of the Universe" would lose their conventional meaning under a locally varying time.

The identification of the primary predictions for antiparticles can be easily derived from the preceding ones via isoduality and they will not be considered for brevity.

8.5.i: Isominkowskian classification of physical media. As one can see from the preceding section, the isospecial relativity predicts fundamentally different physical events, such as isoredshift or isoblueshift, depending on the values of the characteristic quantities $b_\mu = 1/n_\mu$.

A classification of all possible physical media into nine physically different types was introduced by this author in ref.s [9,11] via the use of the isominkowskian geometry. This classification has emerged to be of fundamental

character for the practical applications of hadronic mechanics because the knowledge of the *isominkowskian type* of a given medium, even without a numerical knowledge of the characteristic quantities, identifies the main lines of the dynamics.

In order to study the isominkowskian classification, let us recall that the characteristic b-functions or the average $b^°$-constants must necessarily be different for different media. For instance, one must expect that the $b^°$-constants, say, for Jupiter's atmosphere (ignoring gravitation) are different than those of Earth's atmosphere because of the differences in size, density, pressure, temperature, etc. Similarly, one must expect that the characteristic $b^°$-constants of the $K^°$ particle are different than those for the proton because the two particles have different densities.

Recall that the special relativity predicts the general laws which must be verified by a Lagrangian or a Hamiltonian, but cannot possibly predict their numerical value unless sufficient experimental data on the case at hand are known.

Along exactly the same lines, the isospecial relativity predicts the general laws to be obeyed by a Lagrangian or Hamiltonian and the characteristic quantities, but it cannot possibly predict their numerical value which must be derived from experimental measures for each case considered.

As we shall see in Vol. III, when sufficient data are available, the isospecial relativity does indeed provide numerical predictions of all characteristic $b^°$-quantities. This is the case for numerous physical occurrences, such as the structure of mesons, or the electron pairing in superconductivity, or the experimental data on the behaviour of the mean life with speed, etc

We now review the physical interpretation of the isometric. In regard to the fourth component $\hat{\eta}_{44} = - b_4^2 = -1/n_4^2$, the isominkowski spaces generalize the conventional index of refraction $n = n^°_4 = b^°_4{}^{-1}$ to all possible physical media, whether transparent or opaque to light .

If the medium is not transparent, $\hat{\eta}_{44}$ represents a purely geometric characterization of the density of the medium considered, similar to (but physically different than) the geometrization of gravitation provided by the component g_{44} of a Riemannian metric.

The knowledge of the value $n^°_4 = b^°_4{}^{-1}$ is therefore fundamental for all applications of the isospecial relativity because it provides a first basic characterization of the type of medium considered.

Note that n_4 can be smaller or bigger than 1. If the medium is transparent then $n_4 > 1$ and we have light propagating within ordinary media. If, however, $n_4 < 1$, it does not mean that light is propagating at a speed $c = c_0/n_4 > c_0$, because those media are generally hyperdense and therefore opaque to light. In this latter case the quantity n_4 represents a purely geometric property.

A first interpretation of the space components of the isometric when averaged to constants $\hat{\eta}_{kk} = b^°_k{}^2$ is the direct representation of the actual

nonspherical shape of the particle considered, such as spherical ellipsoids with semiaxes characterized by b^{-2}_k , k = 1, 2, 3 (Fig. 11.8.4.2). However, other interpretations are possible depending on the case at hand. For instance, the phenomenological data of ref. [35] show that, in the behaviour of the meanlives of unstable hadrons with speed, the space–components of the isometric represent a geometrization of the *internal effects which are nonlinear in the velocities.*

Finally, the difference between the time and space components is a direct representation of the anisotropy of the medium considered. This difference is characterized by the value

$$\hat{\beta}^2 = \frac{v_k\, b_k^2\, v_k}{c_0\, b_4^2\, c_0} \;\neq\; \beta^2 = \frac{v_k\, v_k}{c_0^2}\,, \tag{8.5.26}$$

or, equivalently, $\hat{\gamma} \neq \gamma$.

Inspection of the isolorentz transformations (8.3.19) and values (8.5.26) then permits the classification of all possible isominkowskian geometries first into *three groups* depending on whether $\hat{\beta} = \beta$, $\hat{\beta} > \beta$, or $\hat{\beta} < \beta$, and then into three particular cases per each group depending on whether $n_4 = b^{-1}$ = 1, < 1 or > 1. This results into the *nine types* of different physical media of Fig. 8.5.1.

ISOMINKOWSKIAN CLASSIFICATION OF PHYSICAL MEDIA

GROUP I: $\hat{\beta} \equiv \beta$, $\hat{\gamma} = \gamma$;

  $\left\{\begin{array}{l}\end{array}\right.$
  TYPE 1: $n_s = n_4$, n_4 = 1;
  TYPE 2: $n_s = n_4$, $n_4 > 1$;
  TYPE 3: $n_s = n_4$, $n_4 < 1$;

GROUP II: $\hat{\beta} > \beta$, $\hat{\gamma} \leqq \gamma$;

  $\left\{\begin{array}{l}\end{array}\right.$
  TYPE 4: $n_s < n_4$, n_4 = 1;
  TYPE 5: $n_s < n_4$, $n_4 > 1$;
  TYPE 6: $n_s < n_4$, $n_4 < 1$;

GROUP III: $\hat{\beta} < \beta$, $\hat{\gamma} > \gamma$;

  $\left\{\begin{array}{l}\end{array}\right.$
  TYPE 7: $n_s > n_4$, n_4 = 1;
  TYPE 8: $n_s > n_4$, $n_4 > 1$;
  TYPE 9: $n_s > n_4$, $n_4 < 1$;

FIGURE 8.5.1: A schematic view of the isominkowskian classification of physical media made–up of matter into three groups and nine different types which is at the foundation of the applications of the isospecial relativity in diversified fields. The corresponding classification for antimatter merely requires the negative values of the characteristic quantities. The reader should be aware that the

classification of this figure is different than that of the preceding literature because we have use for symmetry the same sequence $n_4 = 0, > 1$ and < 1 for all groups.

Media of Group I are homogeneous and isotropic. Light propagating in them experiences no shift (i.e., it loses or gains no energy). The medium of Type 1 is evidently the conventional Minkowski space. Media of Type 2 are water and other transparent substances in which the speed of light is $c = c_0/n_4 < c_0$. Media of Type 3 are not transparent to light in which case the quantity $c = c_0/n_4 > c_0$ does not represent a physical speed but merely constitutes a geometric characteristic of the medium itself. As an example, Animalu [54] indicates that superconductors appear to be media of Type 3.

Media of Group II are inhomogeneous and anisotropic and therefore share a natural isoredshift (i.e., they loses energy to the medium itself) according to the rules

$$\hat{\omega} \approx \omega_0(1 - \beta\,(n_4/n_s) + \tfrac{1}{2}\beta^2\,(n_4/n_s)^2 + ...) < \omega' \approx \omega_0(1 - \beta + \tfrac{1}{2}\beta^2 + ...\,), \quad (8.5.28)$$

because $\hat{\beta} > \beta$ but $\hat{\beta}$ remains much smaller than one and thus $\hat{\beta} > \hat{\beta}^2$. Media of Type 4 are under study for the representation of ordinary conductors where the maximal causal speed remains $c = c_0 b_4 = c_0$ yet the resistance to the propagation of charges is represented by the isotopy. Media of Type 5 have been systematically used for the representation of planetary atmospheres and astrophysical chromospheres [9,11,39,42]. As an example, an average 2% reduction of the speed of light in our atmosphere of 2% yields the numerical value $n^\circ_4 \approx 1.004$ while the prediction of a contribution of isoredshift at the horizon yields the value $n^\circ_3 \approx 003$ Media of Type 6 are used for the study of hyperdense quasars chromospheres [11].

Media of Group III are also inhomogeneous and anisotropic but of a topology different than that of Group II. In fact, they share a natural isoblueshift (i.e., light acquires energy from the medium) in the following sense

$$\hat{\omega} \approx \omega_0(1 - \beta\,(n_4/n_s) + \tfrac{1}{2}\beta^2\,(n_4/n_s)^2 + ...) > \omega' \approx \omega_0(1 - \beta + \tfrac{1}{2}\beta^2 + ...\,), \quad (8.5.29)$$

because $\hat{\beta} < \beta, \hat{\beta} \ll 1, \hat{\beta} \ll \hat{\beta}^2$. Media of Type 7 are unknown at this writing. Media of Type 8 have been used to geometrize nuclei. Finally, media of Type 9 are the densest media identified by the isospecial relativity and they occur the interior of hadrons or of collapsing stars.

As a simple but effective illustration of the isominkowskian geometrization of the interior of hadrons, consider the phenomenological calculations by Nielsen and Picek [32] for the interior of pions and kaons conducted via the use of the conventional gauge theories in the Higgs sector, which confirm the apparent deviation from the Minkowskian metric of the general type

$$\eta_{vacuum} = \text{diag.} (1, 1, 1, -1) \rightarrow \eta' ((1 - \alpha/3), (1 - \alpha/3), (1 - \alpha/3), -(1 + \alpha)), \quad (8.5.29a)$$

$$\alpha = (-3.79 \pm 1.37) \times 10^{-3} \text{ for } \pi^{\pm}, \; \alpha = (+0.61 \pm 0.17) \times 10^{-3} \text{ for } K^{\pm}. \quad (8.5.29b)$$

But the isominkowskian geometrization is universal. It is therefore simple to see that generalized metric (8.5.29a) is one of the simplest possible isominkowskian metrics with realization

$$\eta' \equiv \hat{\eta} = \text{diag.} (b_1^2, b_1^2, b_3^2, - b_4^2), \quad b_1^2 = b_2^2 = b_3^2 = b_s^2 = 1 - \alpha/3, \; b_4^2 = 1 + \alpha, \quad (8.5.30)$$

with the numerical values

$$\pi^{\pm}: \quad b_s^3 \cong 1 + 1.2 \times 10^{-3} \, (n_s^2 \cong 0.9988), \; b_4^2 \cong 1 - 3.79 \times 10^{-3} \, (n_4^2 \cong 1.0038), \quad (8.5.31a)$$

$$K^{\pm}: \quad b_s^2 \cong 1 - 2 \times 10^{-4} \, (n_s^2 \cong 1.0002), \; b_4^2 \cong 1 + 6.1 \times 10^{-4} \, (n_4^2 \cong 0.9994), \quad (8.5.31b)$$

Consequently, according to the results of ref. [32], *the hadronic medium inside pions is of isominkowskian Type 6,* while *the hadronic medium inside kaons is of isominkowskian Type 9.* Since all remaining hadrons have a density bigger than that of kaons, we can predict that *all hadrons heavier than kaons are media of isominkowskian Type 9.* As we shall see in Vol. III, these results are confirmed by all experimental values of the characteristics functions currently available.

Far from being a mere mathematical curiosity, the above isominkowskian geometrization carries the following implications:

1) It reveals a structural difference between pions and kaons which is beyond any possibility of conventional quark theories. In fact, pions have a maximal causal speed in $M(x,\eta,R)$ of $0.997c_0 < c_0$, while kaons have the value $1.004c_0 > c_0$ with consequential, rather subtle structural differences studied in Vol. III.

2) Ref. [32] uses conventional relativistic methods in the elaboration of data showing deviations from Minkowskian behaviour. The isominkowskian geometrization identifies instead rigorous methods for the data elaboration which are compatible with the deviation themselves, such as the isoscattering theory of Ch. II.12,

3) Despite deformation (8.5.39), the fundamental Lorentz symmetry remains exact, contrary to the statement of ref. [32] where α is called the "Lorentz asymmetry parameter". In fact, the general invariance of isometric (8.5.30) is the isolorentz symmetry $\hat{L}$ computed for the isounit

$$\hat{1} = \text{diag.} \{ (1 - \alpha/3)^{-1}, (1 - \alpha/3)^{-1}, (1 - \alpha/3)^{-1}, (1 + \alpha)^{-1} \}. \quad (8.5.32)$$

The isomorphism $\hat{L} \approx L$ then follows from $\hat{1} > 0$. The explicit form of the isolorentz transforms is given by merely *plotting* the b-values (8.5.30) in Eq.s

(8.3.19) without any calculation. Note that the Lorentz *symmetry* is preserved, but the Lorentz *transforms* are necessarily lost. For all details we refer the reader to Vol. III.

8.5.J: Direct universality of the isospecial relativity. As final comments we study in more details the reader's attention the "direct universality" of the isospecial relativity, that is, its capability to provide a form-invariant description in isominkowski space of all nonlinear-nonlocal-nonhamiltonian systems admitting signature-preserving generalizations $\hat{\eta}$ of the Minkowski metric η (universality), directly in the x-frame of the observer (direct universality).

This property was studied in detail by Aringazin [55] who showed that all the numerous generalizations of the Einsteinian behaviour of the meanlife of unstable particles with speed existing in the literature (see, e.g., ref.s [29–33]) are particular cases of the isotopic law

$$\hat{\tau} = \tau_0 (1 - v^k b_k^2 \hat{e}^k / c_0)(1 - v_k b_k^2 v_k / c_0 b_4^2 c_0)^{-1/2}, \qquad (8.5.33)$$

because they can be all obtained from the preceding law via different expansions in terms of different coefficients and with different truncations.

In fact, Aringazin [loc. cit.] correctly considers the primary dependence of the space functions $b_1 = b_2 = b_3 = b_s$ as being that in the velocity, $b_s = b_s(v)$. This allows expansions of the type

$$b_s(v) = 1 + \lambda_0 + \lambda_1 \gamma + \lambda_2 \gamma^2 + \lambda_3 \gamma^2 + ... , \qquad (8.5.34a)$$

$$\gamma = (1 - \beta^2)^{-1/2} < 1, \quad \beta = v/c_0 < 1, \quad \lambda_k \ll 1 . \qquad (8.5.34b)$$

The isotopic time contraction can then be expanded in the form

$$\hat{\tau} = \tau_0 \hat{\gamma} = \tau_0 \gamma \{ 1 + \lambda_0 \gamma^2 + \lambda_1 (1 + \lambda_0) \gamma^3 + [\tfrac{1}{2}\lambda_1^2 + \lambda_2 (1 + \lambda_0] \gamma^4 + ...\} \qquad (8.5.36)$$

which yields as a particular case:

A) The generalized law by Blochintsev [29], Redei [30] and others

$$\tau = \tau_0 \gamma (1 + \lambda_0 \gamma^2), \quad \lambda_0 = 10^{25} a_0, \qquad (8.5.37)$$

where a_0 is a certain universal length;

B) The generalized law by Nielsen and Picek [32]

$$\tau = \tau_0 \gamma (1 + \lambda_0 \gamma^2), \quad \lambda_0 = 4 \alpha / 3 ; \qquad (8.5.38)$$

C) The generalized law by Aronson et al. [33]

$$\tau = \tau_0 \gamma (1 + b_\chi^{(N)} \alpha^{(N)}), \alpha^{(N)} = E_N / m_{oN}, \quad N = L, S, \qquad (8.5.39)$$

where E_N is the kinetic energy, m_{oN} is the mass of the particle K_L or K_S and the b_χ's are certain slope parameters.

Again, the geometric unification of existing laws into the unique isotopic law (8.5.33) is not a mere mathematical curiosity because it carries important *experimental* implications. In fact, in the absence of the isospecial relativity, the experimenters face a variety of seemingly different generalized law, with consequential inability to select which of them should be tested. Under the isospecial relativity the law is unique, although admitting different specialization for different physical conditions, thus avoiding ambiguities on which law to test.

Moreover, Aronson et al. [loc. cit.] identified a velocity dependence of the mass difference for the K°-system $\Delta m = m_L - m_S$ which can also be interpreted as a natural consequence of the isotopic principle of equivalence $E_k = m_k [c_o b_4(v)]^2, k = L, S$. Moreover, Aronson et al. [loc. cit.] achieved their results via the redefinition of the <Minkowskian coordinates

$$x \in M(x,\eta,R) \;\to\; \bar{x} = x (+ b_\chi^{(N)} a^{(N)}) \in M(\bar{x},\eta,R),$$ (8.5.40

which is precisely along the basic assumptions of the isospecial relativity, the generalization of the units of space and time according to rules (8.5.25), thus yielding the isotopic interpretation

$$b^{(N)} = 1 + b_\chi^{(N)} a^{(N)}.$$ (8.5. 41)

The applicability of the isospecial relativity for studies [33] is then evident. But perhaps the most important conclusion of ref. [33] is that the measured deviations cannot be interpreted via conventional interactions, thus confirming their novel *isogeometric* character.

A virtually endless number of connections exist between the isospecial relativity and various generalizations existing in the literature. As a mere indication, we here mention that another generalization of type (8.5.25) restricted to time was proposed by Milne [56] according to the law

$$\tau = t_0 + t_0 \,\mathrm{Ln}\,(\,t/t_0\,),$$ (8.5.42)

where t is the so-called a*tomic time,* τ is the *gravitational time* and t_0 is the time in which $t = t_0 = \tau$. The above notion of time can be easily reformulated via the isounit of time. A number of other isotopic reformulation of existing generalizations will be studied in Vol. III.

The first point conveyed in this section is that a*ll the rather numerous generalizations of conventional Minkowskian formulations existing in the literature are a particular case of the isospecial relativity.*

The second point conveyed in this section is that *departures from the Minkowskian geometries cannot be elaborated with Minkowskian methods because of evident problems of intrinsic consistency, and must therefore be elaborated with isominkowskian methods, i.e., isofields, isospaces, isospecial functions, etc.*

The final and perhaps most important point conveyed in this section from which all others follow is that *the rather general belief that the Lorentz symmetry and the basic postulates of the special relativity are violated for a signature–preserving departure from the Minkowskian geometry is erroneous, because said symmetry and postulates remain exactly valid when treated with the appropriate isotopic methods.*

8.6: GENOSPECIAL RELATIVITY AND ITS ISODUAL

8.6.A: Conceptual foundations. By no means the isospecial relativity and its isodual of the preceding section exhaust all possible generalizations of the special relativity, because physics is a discipline that will never admit final theories (see, e.g., Fig. 6.1, pp. 250–251 of ref. [7] on the open chain of generalized relativities which are conceivable with current knowledge).

In quantum mechanics there is only *one* relativity for relativistic conditions on flat space–time, the special relativity, which therefore characterizes all physical events, including particles and antiparticles.

In hadronic mechanics we have the following *hierarchy* of relativities with increasing complexity and methodological needs to represent a progressively increasing complexity of systems:

  1) Conventional special relativity with basic interval

$$x^2 = [(x - y)^t \eta (x - y)] I, \quad \eta = \text{diag. (I, I, I, -1),} \quad 1 = \text{diag. (1, 1, 1, 1),} \quad (8.6.1)$$

and conventional Lie structure for the P(3.1)–invariant characterization of particles in exterior relativistic conditions on the conventional space $M(x,\eta,R)$;

  2) Isodual special relativity with basic interval

$$x^{2d} = [(x - y)^t \eta^d (x - y)] I^d \equiv x^2, \quad \eta^d = -\eta, \quad I^d = -I, \quad (8.6.2)$$

and isodual Lie structure for the $P^d(3.1)$–invariant characterization of antiparticles in exterior relativistic conditions in isodual space $M^d(x,\eta^d,R^d)$;

  3) isospecial relativity with basic interval of topological Class I

$$x^2 = [(x - y)^t \hat{\eta}(x, \dot{x}, \ddot{x}, \phi, \partial\phi, \partial\partial\phi, \mu, \tau, ...)(x - y)] \hat{1}, \quad (8.6.3a)$$

$$\hat{\eta} = T\eta = \hat{\eta}^\dagger, \quad T = T^\dagger > 0, \quad \hat{1} = \hat{1}^\dagger = T^{-1} > 0, \quad (8.6.3b)$$

and Lie–isotopic structure for the $\hat{P}(3.1)$–invariant characterization of particles in interior relativistic conditions in isospace $\hat{M}(x,\hat{\eta},\hat{R})$;

  4) Isodual isospecial relativity with basic interval of Class II

$$x^{2d} = [(x - y)^t \, \eta^d(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \mu, \tau, ...) (x - y)] \hat{1}^d \equiv \hat{x}^2, \quad \hat{\eta}^d = -\hat{\eta}, \, \hat{1}^d = -\hat{1},$$

$$(8.6.4)$$

and isodual Lie–isotopic structure for the $\hat{P}^d(3.1)$–characterization of antiparticles in isodual isospace $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$;

5) genospecial relativity with basic interval of Class 1

$$x^2 = [(x - y)^t \, \tilde{\eta}(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \mu, \tau, ...) (x - y)] \hat{1}, \quad \tilde{\eta} = \hat{T}\eta \neq \tilde{\eta}^\dagger, \, \hat{1} = \hat{T}^{-1},$$

$$(8.6.5)$$

with a Lie–admissible structure for the $\hat{P}(3.1)$–admissible characterization of particles in a suitable space (see below); and

6) Isodual genospecial relativity with basic invariant of Class II

$$x^{2d} = [(x - y)^t \, \tilde{\eta}^d(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, \mu, \tau, ...) (x - y)] \hat{1}^d, \quad \tilde{\eta}^d = -\tilde{\eta}, \, \hat{1}^d = -\hat{1},$$

$$(8.6.6)$$

with isodual Lie–admissible structure for the $\hat{P}(3.1)^d$–admissible characterization on antiparticles on a suitable isodual space.

As one can see, the basic methods for particles are given by the chain

LIE METHODS $\subset$ LIE-ISOTOPIC METHODS $\subset$ LIE-ADMISSIBLE METHODS

with isodual images for antiparticles.

The above chain of methods has been conceived for the quantitative treatment of a corresponding increase in the complexity of physical conditions. Lie methods describe closed–isolated–reversible systems with local–differential–potential interactions. The Lie–isotopic methods describe closed–isolated systems with reversible center–of–mass trajectory and internal interactions of both local–differential–potential as well as nonlocal–integral–nonpotential types. Finally, the Lie–admissible methods have been constructed for the quantitative treatment of open systems in irreversible conditions under the most general possible nonlinear–nonlocal and nonpotential external interactions.

The main physical significance of the genospecial relativity is therefore the axiomatization of irreversibility and the identification of its origin in the ultimate level of the structure of the Universe, that of elementary particles in open–nonconservative conditions such as a proton in the core of a star when considering the rest of the system as external.

To avoid un–necessary repetitions, the understanding of this section requires a knowledge of the preceding studies of Lie–admissible methods.

8.6.B: Genominkowskian geometrization of irreversibility. The most effective way to achieve an axiomatic, form–invariant characterization of the four possible directions of time for irreversible systems is by constructing the theory on a generalized notion of field which does admit four inequivalent classes. The

answer is now known and it is given by the genoreal fields of Figure 7.6.1, i.e.,

$$^<\!\hat{R}(^<\!\hat{t},+,<) \qquad\qquad \hat{R}^>\!(\hat{t}^>,+,>)$$

$$^{d<}\!\hat{R}(^{d<}\!\hat{t},+,d<) \qquad 0 \qquad \hat{R}^{>d}(\hat{t}^{>d},+,>d) \qquad \text{time}$$
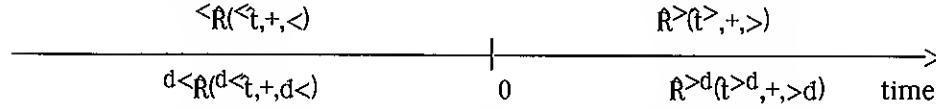
The simplest and effective realization is given by *Jannussis complex time* [58] with the following four realizations $\hat{t}^> = \hat{t}1^> = t(n + im)$, $^<\!\hat{t} = ^<\!1_t = (n - im)t$, $\hat{t}^{>d} = t(-n - im)$, $^{d<}\!\hat{t} = ^{d<}\!1_t = (-n - im)t$, where t is the ordinary real time.

The *genospecial relativity* is the generalization of the special relativity admitting at all its levels the isounit $1^>$ or its Hermitean version $^<\!1 = (1^>)^\dagger$. The *isodual genospecial relativity* is then the image of the preceding one under isoduality. These definitions require that the totality of the backgrounds methods must be constructed in such a way to admit the assumed genounit as the correct unit of the theory.

The first step is therefore the construction of the *genominkowski spaces* (Sect. 1.7.5) $^<\!\hat{M}^>(x,^<\!\hat{\eta}^>,^<\!\hat{R}^>)$ (where only direction at the time can be used) with general interval hereon assumed for simplicity in the diagonal form

$$x^{<\hat{2}>} = [\,(x-y)^t \, ^<\!\hat{\eta}^>(\dot{x},\ddot{x},\Phi,\partial\Phi,\partial\partial\Phi,\mu,\tau,...)\,(x-y)\,]\,^<\!1^> \epsilon \,^<\!\hat{R}^>(^<\!\hat{n}^>,+,<>)\,,\text{(8.6.7a)}$$

$$^<\!\hat{\eta}^> = \,^<\!\hat{T}^>\eta\,,\quad \eta = \text{diag.}\,(1,1,1,-1),\quad ^<\!1^> = (^<\!\hat{T}^>)^{-1} \neq (^<\!1^>)^\dagger\,,\quad \text{(8.6.7b)}$$

$$^<\!\hat{T}^> = \text{diag.}\,(^<\!\hat{b}^>_1{}^2,^<\!\hat{b}^>_2{}^2,^<\!\hat{b}^>_3{}^2,^<\!\hat{b}^>_4{}^2)\,,\quad ^<\!\hat{b}^>_\mu \neq \,^<\!\hat{b}^>_\mu \neq 0,\qquad \text{(8.6. 7c)}$$

where the upper bar denotes complex conjugation, the characteristic quantities $^<\!\hat{b}^>_\mu$ are complex-valued functions and the real part of the genotopic element $^<\!\hat{T}^>$ is of Kadeisvili Class I.

Rercall that the genominkowskian spaces are isomorphic to the conventional spaces because, as in the isominkowskian case, jointly with the deformation of the metric $\eta \to \,^<\!\hat{\eta}^> = \,^<\!\hat{T}^>\eta$, there is a lifting of the unit of an amount inverse of such deformation, $1 \to \,^<\!1^> = (^<\!\hat{T}^>)^{-1}$.

Note that *the genominkowskian interval coincide with the conventional ones for complex genounits*, i.e., for $\hat{1}^> = n+im$, $x^{<\hat{2}>} \equiv x^2$. Thus Jannussis complex time is admitted by an *ordinary* Minkowskian separation, only reinterpreted in a way more general then the conventional, one.

The *isodual genominkowskian spaces* $^<\!\hat{M}^{>d}(x,^<\!\hat{\eta}^>,^<\!\hat{R}^>)$ are characterized by the isodual generic interval on the isodual genofield, also assumed in the diagonal form

$$x^{<\hat{2}>d} = [\,(x-y)^t \, ^<\!\hat{\eta}^{>d}(\dot{x},\ddot{x},\Phi,\partial\Phi,\partial\partial\Phi,\mu,\tau,...)\,(x-y)\,]\,^<\!1^{>d} \, \epsilon \,^<\!\hat{R}^{>d}(^<\!\hat{n}^{>d},+,<>d),$$
$$\text{(8.6.8a}$$

$$^<\!\hat{\eta}^{>d} = \,^<\!\hat{T}^{>d}\eta = -^<\!\hat{\eta}^>\,,\qquad ^<\!1^{>d} = -^<\!1^>\,,\qquad \text{(8.6.8b)}$$

$$^<\!\hat{T}^{>d} = \text{diag.}\,(^<\!\hat{b}^{>d}_1{}^2,^<\!\hat{b}^{>d}_2{}^2,^<\!\hat{b}^{>d}_3{}^2,^<\!\hat{b}^{>d}_4{}^{d2})\,,\qquad ^<\!\hat{b}^{>d}_\mu = \,^<\!\hat{b}^>_\mu\,,\quad \text{(8.6.8c)}$$

The genominkowskian geometry is essentially the same as the isominkowskian one, except that the isounit is no longer Hermitean, thus becoming a genounit. This permits a direct geometrization of irreversibility, that is, its representation via the metric itself.

As an illustration, one of the central problems of irreversibility is the construction of the generalized light cone which is applicable under such irreversibility, that is, for *speeds of light which* are *locally variable in an irreversible way.* This problem is readily solved by the *genolight cone* which is essentially the isolight cone referred to a nonhermitean isounit. The important point is that the genolight cone also coincides with the conventional cone at the abstract level.

The construction of the isodual genominkowski spaces then follows familiar lines here omitted for brevity.

8.6.C: Lie-admissible genopoincare' symmetry and its isodual. We now introduce the mathematical structure of the Lie-admissible branch of hadronic mechanics which consists of:

1) The *forward and backward enveloping genoassociative algebras* $\hat{\xi}^>$ with the same elements A, B, ... of the quantum mechanical envelope ξ but now equipped with the forward and backward products $A>B = AT^>B$ and $A<B = A^<TB$, respectively, and related genounits $1^> = (T^>)^{-1}$, $^<1 = (^<T)^{-1}$. The isodual images $\hat{\xi}^{>d}$ then have the products $A>^dB = -A>B$ and $A^d<B = -A<B$ with related isodual genounits $1^{>d} = -1^>$ and $^{d}{}^<1 = - {}^<1$.

2) The *forward and backward complex genofields* $^<\hat{C}^>(^<\hat{c}^>,+,<>)$ with elements $^<\hat{c}^> = c \, ^<1^>$, related methodology of genoproducts, genonorms, etc. (Ch. 1.2), and isodual images $^<\hat{C}^{>d}(^<\hat{c}^{>d},+,<>d)$.

3) The *forward and backward genohilbert space* $s^<\mathcal{H}^>$ with genostates $| >$ and composition $< \, ^<1^> \, > \, = \, < \, | \, ^< T^> \, | \, >^<1^>$ on the complex genofield $^<\hat{C}^>(^<\hat{c}^>,+,<>)$. The *isodual genohilbert spaces* $^<\mathcal{H}^{>d}$ has the isodual genostates $< \, |^d = - \, (\, | \, > \,)^\dagger$ (Sect. 1.6.3) and isodual compositions $< \, ^<1^> \, >^d \, = \, < \, | \, ^<T^{>d} \, | \, >^<1^{>d}$ on the isodual genofields $^<\hat{C}^{>d}(^<\hat{c}^{>d},+,<>d)$.

We assume the reader is familiar with the basic properties of the above mathematical structures, such as the fact that the action of an operator $A \in \hat{\xi}^>$ on a state is still modular and associative,[118] but of the four different genotopes $A>| >$, $< |<A$, $< |^d <^dA$ and $A>^d| >^d$ each one solely definable over the corresponding genofield.

These features permit the occurrence, which seems to be unique for hadronic mechanics, in which the Hamiltonian $H = K + V$ is Hermitean but *nonconserved* according to the basic *Lie-admissible equations* [47]

$$i \, \frac{^<\partial^>A}{^<\partial^>t} \, = \, i \, ^<b_4^{>-2} \, \frac{dA}{dt} \, = \, (A , H) \, = \, A < H - H > A , \quad H = H^\dagger , \quad (8.6.9)$$

[118] This is a necessary condition to avoid *Okubo's No Quantization Theorem* (App. II.3.B).

for which we have in our space–time the *time–rate–of–variation of the energy*

$$i \frac{<\hat{\partial}^> H}{<\hat{\partial}^> t} = \ = H < H - H > H = H(^< T - T^>) H \neq 0 , \qquad (8.6.10)$$

of which the conservation law is a simple particular case. The integrated form is given by the *Lie–admissible group* [47]

$$A(t) = \hat{e}^{>iH>t^>} A(0) <{}^<\hat{e}^{-i\ {}^< t < H} = e^{iHT^> t} A(0) e^{-it\ ^< T H} , \qquad (8.6.11)$$

where the two actions to the right and to the left are naturally ordered resulting in an isobimodule (Sect. 1.7.6). Note that each action is characterized by the genounitary transform, e.g., $\hat{0}^> = \hat{\exp}^>(i H > t^>)$, $\hat{0}^> > \hat{0}^{>\dagger} = \hat{0}^{>\dagger} > \hat{0}^> = \hat{1}^>$.

The operator form of the four-momentum now requires four different realizations depending on the assumed time arrow

$$p_\mu > | > = - i \hat{\partial}^>_\mu | > = - i \hat{b}^>_\mu{}^{-2} \partial_\mu | > , \quad < | < p_\mu = i < |{}^<\hat{\partial}_\mu = i < | \partial_\mu {}^<\hat{b}_\mu{}^{-2},$$

$$< |{}^{d}<^{d} p_\mu = - i < |{}^<\hat{\partial}_\mu = - i < | \partial_\mu {}^{d<}\hat{b}_\mu{}^{-2}, \ p_\mu >^d | >^d = i \hat{\partial}^>_\mu | > = i \hat{b}^{>d}_\mu{}^{-2}\partial_\mu | >$$

$$(8.6.12)$$

To compute the Lie–admissible relativistic genocommutation rules, note that products such as $A > B = A T^> B$ and $A < B = A^< T B$ are different when computed in the original quantum mechanical space, that is, with respect to the *same unit* I, evidently because the two operators $T^>$ and $^< T$ are different, and we shall write $A > B_I \neq A < B_I$. However, when computed in their respective genoenvelopes over their respective genofields, the two products coincide because $A T^> B$ is computed with respect to the genounit $\hat{1}^> = (T^>)^{-1}$ while the product $A^< T B$ is computed with respect to the genounit $^<\hat{1} = (^< T)^{-1}$, and we shall write

$$A > B_{\hat{1}^>} = A < B_{^<\hat{1}} = AB_I \qquad (8.6.13)$$

The above rules imply that *the Hamiltonian* H *is indeed conserved in genospace and it is nonconserved only when the equation of motion are projected in our space–time* as in Eq.s (8.6.

$$i \frac{<\hat{\partial}^> H}{<\hat{\partial}^> t} = \ = H < H_{^<\hat{1}} - H > H_{\hat{1}^>} = 0 , \qquad (8.6.14)$$

This is a fully expected occurrence because of the abstract identity between conservative and nonconservative systems when treated in their respective spaces and genospaces.

Next, we note that the fundamental property (II.8.4.5) persists under genotopies, and we shall write

$$\partial^>_\mu x_\nu \quad = \quad x_\nu {}^<\partial_\mu \quad = \quad \eta_{\mu\nu} \, . \tag{8.6.15}$$

where $\eta_{\mu\nu}$ is the *conventional* Minkowski metric.

The above properties permit the calculation of the *fundamental relativistic genocommutation rules on genospaces*

$$(x_\mu , p_\nu) > | > \; = (x_\mu < p_\nu - p_\nu > x_\mu) > | > = (x_\mu {}^<T \, p_{\mu \triangleleft} - p_\nu \, T^> x_{\mu \uparrow >}) \, T^> | > \; =$$

$$= \; (x_\mu {}^<T \, p_{\mu \triangleleft} - x_\mu \, T^> p_{\nu >} + i \eta_{\mu\nu}) T^> | > \; = \; i \eta_{\mu\nu} > | > \, . \tag{8.6.16a}$$

$$(x_\mu , x_\nu) > | > \; = (p_\mu , p_\nu) > | > \; = \; 0 \, . \tag{8.6.16b}$$

We reach in this way, apparently for the first time, the important result that the algebraic structure of the quantum brackets is lifted into a Lie–admissible form, but the eigenvalues of the brackets remain the original ones. This confirms the abstract unity of the Lie–admissible branch of hadronic mechanics and quantum mechanics, this time, at the relativistic level.

We should stress again that expressions (8.6.14) are valid on genospaces under property (.8.6.12). To see the corresponding properties when the Lie–admissible brackets are computed in ordinary spaces it is best to introduce the unified notation $a = \{x^\mu, p_\nu\}$. The *Lie-admissible commutation rules in ordinary spaces* can the be written

$$(a^\mu , a^\nu) = a^\mu {}^<T \, a^\nu - a^\nu \, T^> a^\mu = i \, S^{\mu\nu}(a, p) \, , \tag{8.6.17}$$

where $S^{\mu\nu}(r, p)$ is Lie–admissible, that is, such that $\Omega^{\mu\nu} = S^{\mu\nu} - \bar{S}^{\nu\mu}$ is Lie (see the corresponding classical case in Sect. II.1.5).

The differences between rules (8.6.16) and (8.6.17) are evident. In particular, the coordinates and momenta are commutative in the former but not in the latter.

The construction of the Lie–admissible generalization of the Poincaré symmetry in genospaces is then a straightforward extension of the nonrelativistic analysis of Sect. II.7.6. It is easy to see that the *two* isosymmetries $\hat{P}(3.1)$ and $\hat{P}^d(3.1)$ are turned into *four* genosymmetries, the *forward genopoincaré symmetry* $\hat{P}^>(3.1)$, the *backward genopoincaré symmetry* ${}^<\hat{P}(3.1)$, the *isodual forward genopoincaré symmetry* $\hat{P}^{>d}(3.1)$, and the *isodual backward genopoincaré symmetry* ${}^{d<}\hat{P}(3.1)$, for the form–invariant description of systems in the corresponding four directions of time.

The parameters of the above genosymmetries are the conventional ones w $= \{w_k\} = \{\theta,v,a\}$, $k = 1, 2, ..., 10$, although they are reformulated as genonumbers ${}^<\hat{w}>$ or their isodual ${}^<\hat{w}{}^{>d}$. The generators are also the conventional relativistic ones $X = \{X_k\} = \{M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, p_\alpha\}$. The transformations are expressed via genomodular actions on the genominkowskian space, e.g., $x' = \hat{g}^> > x$, and their connected components ${}^<\hat{P}>_+{}^\uparrow(3.1)$ are representable via the genounitary

transforms

$$0> = \hat{e}>^{iX_k > \hat{w}>_k}, \quad <0 = <\hat{e}^{-i <\hat{w}_k < X_k}. \qquad (8.6.18)$$

The explicit form of the genopoincaré transformations is the same as that in Eq.s (11.8.3.34) with the sole proviso that the real–valued b's are replaced by the complex–valued $<b>$'s.

The *Lie–admissible genopoincaré group* $<P>_+^{v\psi}(3.1)$ for the transformation of a physical quantity is given by the isobimodular structure

$$A(w) = \hat{e}>^{iX_k > \hat{w}_k>} > A(0) < <\hat{e}^{-i <\hat{w}_k < X_k} = e^{iX_kT>w_k} A(0) e^{-iw_k<TX_k}. \qquad (8.6.19)$$

The *Lie–admissible genopoincaré algebra* $<P>_+^{\dagger}(3.1)$ can then be derived from the preceding Lie–admissible group via known procedures (Ch. 11.7) and can be written in genospaces

$$(M_{\mu\nu}, M_{\alpha\beta})>|> = i(\eta_{\nu\alpha}M_{\beta\mu} - \eta_{\mu\alpha}M_{\beta\nu} - \eta_{\nu\beta}M_{\alpha\mu} + \eta_{\mu\beta}M_{\alpha\nu})>|> \quad (8.6.$$

$$(M_{\mu\nu}, P_\alpha)>|> = i(\eta_{\mu\alpha}P_\nu - \eta_{\nu\alpha}P_\mu)>|>, \quad (P_\mu, P_\nu)>|> = 0, \qquad (8.6.20)$$

where the brackets (...,...) in the l.h.s. are Lie–admissible while η in the r.h.s. is the conventional Minkowski metric.

Thus, the structure constants of $<\hat{p}>_+^{\dagger}(3.1)$ formally *coincide* with those of the conventional Poincaré algebra $p_+^{\dagger}(3.1)$, by confirming not only the local isomorphism $<\hat{p}>_+^{\dagger}(3.1) \approx \hat{p}_+^{\dagger}(3.1)$, but also the identity at the abstract level of the conventional and isotopic symmetries. This implies *the abstract identity between relativistic quantum and hadronic mechanics in its most general possible, Lie–admissible form.*

The *genocasimir invariants* are structurally the same as the abstract ones (II.8.3.38) although now realized on $<\mathcal{H}>$ and can be written

$$\hat{C}^{(0)}>|> = 1>>|> = |>, \qquad (8.6.21a)$$

$$C^{(2)}>|> = P^2>|> = \hat{\eta}>^{\mu\nu}P_\mu > P_\nu >|> = P_\mu > P^\mu >|>, \qquad (8.6.21b)$$

$$C^{(4)}*|\hat{\Phi}> = \hat{W}>^2 >|> = \hat{\eta}>^{\mu\nu}\hat{W}>_\mu > \hat{W}_\nu >|>, \quad \hat{W}_\mu = \tfrac{1}{2}\epsilon_{\mu\alpha\beta\rho}M^{\alpha\beta} > P^\rho, \qquad (8.6.21c)$$

The isodual genosymmetry $<P>^d_+{}^{\dagger}(3.1)$ can be constructed via isoduality. The study of the remaining aspects is left to the interested reader. For instance, the image of rules (8.6.20) in our space–time is unknown at this writing.

8.6.D: Genospecial relativity and its isodual. The *genospecial relativity* is a covering of the isospecial relativity for the invariant description of open–irreversible systems under the genopoincaré symmetry on genospaces for each direction of time.

The basic postulates are characterized by a genotopy of the conventional

Postulates 1–5 of the special relativity (Sect. II.8.5.B) which, when represented in their appropriate genospace, coincide with the latter at the abstract level. As an example, the maximal causal speed in genospace can be proved to remain c_0 even though the actual speed of light can be locally varying in an irreversible way. A similar situation holds for the other postulates.

The primary physical relevance of the genospecial relativities is to provide a description of irreversible processes at the ultimate, elementary level of matter in all four possible directions of time which: is derivable from first axioms; invariant under the genopoincaré symmetry; is a relativistic extension of the genogalinean relativity of Sect. II.7.5; and it is such to coincide with the conventional quantum description of reversible processes at the abstract, realization–free level.

These conditions permit the resolution of a long standing, fundamental problem of contemporary physics, the origin of irreversibility, this time, at the relativistic level. As recalled in Sect. II.7.5, such a problem has been created by the fact that, on one, side, physical reality is manifestly irreversible while, on the other side, quantum mechanics is structurally reversible. This fundamental dichotomy has then stimulated numerous attempts to reconcile the limitations of quantum mechanics with physical reality.[119]

Hadronic mechanics eliminates the very existence of the problem. In fact, the new mechanics is structurally irreversible in its most general possible, Lie-admissible form, and admits a reversible branch only as a particular case. Irreversibility then originates in the ultimate elementary structure of matter, such as in one individual electron in the interior of a star. Macroscopic irreversibility is only a consequence due to a large collection of individually irreversible elementary events.

The above resolution of the origin of irreversibility was proposed by this author as the very essence of hadronic mechanics [47]. A collection of the original papers along these lines can be found in ref. [60]. These first works are formulated on conventional Hilbert spaces over conventional fields and, as such, lack their axiomatic and form–invariant characters. The formulation of Lie-admissible structures on genohilbert space over genofields has been articulated for the first time in this presentation.

Besides the above application, the genospecial relativity also has additional applications, such as the possibility of rendering physically admissible various complex extensions of conventional real quantities, spaces, etc. which are generally believed to be outside our physical reality. In fact, the latter become admissible when referred to a suitably selected complex unit (see, e.g., the complex extensions of the Minkowski space in Sect. II.9.7).

---

[119] This is another field with a rather vast literature all essentially based on conventional quantum mechanics and, therefore, not directly connected with these studies, with very few exceptions, such as the studies by Prigogine and his group (see, e.g., ref. [59] and quoted literature).

The best illustration is via *Jannussis' complex time* [58] which, if written in the conventional form $t_1 + it_2$ *appears* to be outside the physical reality. However, the same complex time can be written $t_1 + it_2 = t(n_1 + in_2)$ and becomes fully admitted by the abstract Einsteinian axioms under the assumption of the time isounit $\hat{1}^>_{44} = (n_1 + in_2)^{-1}$. In the final analysis, as noted earlier, the space-time intervals for the conventional time and for Jannussis complex time *coincide*.

As we shall see in Vol. III, the above main features permit the Lie-admissible reformulation of a large body of theories based on the complex extension of real structures, with intriguing epistemological, theoretical and experimental implications.

8.7: ANTIGRAVITY

In this section we outline one of the most intriguing predictions of our isodual special relativity, the prediction of *antigravity* for *elementary* particles (such as positrons) in the field of matter, while isoselfdual bound states of particles and antiparticles (Sect.7.8.c) are *attracted*. This prediction was first studied by the author in ref. [51]. The corresponding treatment via the isogeneral relativity on a curved space will be done in the next chapter.

The reader should be aware that the search for antigravity dates back to the early stages of physics (see the review [61]), and includes a large variety of attempts, some of which have even been patented (see, e.g., ref.s [62]). However, *all* these attempts are based on mechanical and other means, and *none* of them is based on the actual reversal of the attractive character of gravity.

This scenario is due to the fact that, as is well known, the Minkowskian and Riemannian geometries offer no possibility of any nature to reverse the sign of the gravitational field, and can at best predict a *decrease* of the gravitational force of antiparticles in the field of matter which, as such, remains always attractive.

The advent of our isominkowskian and isoriemannian geometries and their isoduals evidently alters this scientific scene. In fact, antigravity is predicted by a simple interplay between the isogeometries and their isodual for both flat and curved spaces.

In this section we initiate our studies of antigravity at the simplest possible level, that of the *exterior relativistic problem of particles and antiparticles*. The fundamental relativity of this section is therefore the *conventional* special relativity for the *exterior* dynamical problem of *particles* complemented with our *isodual special relativity* for the *exterior* dynamical problem of *antiparticles*. The interior counterpart will be studied in the next chapter.

We can actually begin with the representation of particles via the conventional Euclidean space

$$E(r,\delta,R), \qquad \delta = \text{diag.}\,(\,1,\,1,\,1\,), \qquad r^2 = r^i\,\delta_{ij}\,r^j \;\in\; R(n,+,\times)\,, \quad I = \delta\,, \qquad (8.7.1)$$

and the representation of antiparticles in the isodual Euclidean space

$$E^d(r,\delta^d,R^d),\; \delta^d = \text{diag.}\,(\,-1,\,-1,\,-1),\;\; r^{2d} = (\,r^i\,\delta^d_{ij}\,r^j\,)\,I^d \;\in\; R(n,+,\times)\,, \; I^d = -1. \quad (8.7.2)$$

The relativistic extension is then straightforward.

In order to understand the antigravity predicted by hadronic mechanics, it is recommendable to understand first the representation of the behaviour of particles and antiparticles under electromagnetic interactions.

Consider a conventional massive *particle* and its *antiparticle* in exterior conditions in vacuum with charge −e and +e, respectively (say, an electron and a positron). Suppose that these particles enter into the gap of a magnet with constant magnetic field **B** as in Fig. 8.7.1.

As is well known, visual experimental observation establishes that the two particles have spiral trajectories of *opposite orientation*. But this behaviour occurs for *the representation of both particles in the same Euclidean space*. The situation in hadronic mechanics is different, as described by the following:

Proposition 8.7.1 [51]: *The trajectory of a charged particle in Euclidean space under a magnetic field and the trajectory of the corresponding antiparticle in isodual Euclidean space <coincide>.*

Proof: Let the particle has negative–definite charge −e in Euclidean space, that is, the value −e is defined with respect to the positive–definite unit +1 of the underlying field of real numbers R(n,+,×), and is under the influence of the magnetic field **B**. The characterization of the corresponding antiparticle via isoduality implies the reversal of the sign of all physical quantities, thus yielding the charge +e in the isodual Euclidean space, as well as the reversal of the magnetic field −**B**, although now defined with respect to the negative–definite unit −1. It is then evident that the trajectory of a particle with charge −e in the field **B** defined with respect to the unit +1 in Euclidean space and that for the antiparticle of charge +e in the field −**B** defined with respect to the unit −1 in isodual Euclidean space coincide. q.e.d.

An important property can be expressed via the following

Corollary 8.7.1 [loc. cit.]: *The antiparticle reverses its trajectory when projected from its isodual space into the conventional space.*

The consistency of the isodual characterization of the electromagnetic behaviour of antiparticles will be studied in detail in Ch. II.10. At this moment it is sufficient to verify its consistency in Euclidean space and its isodual. Consider the *repulsive* Coulomb force among two *particles* of *negative* charges $-q_1$

and $^{-}q_2$

$$F = K \times (-q_1) \times (-q_2) / |r|^2 > 0 . \qquad (8.7.3)$$

where the operations of multiplication $\times$ and division $/$ are the conventional ones of the underlying field $R(n,+,\times)$. Under isoduality we have

$$F^d = K^d \times^d (-q_1)^d \times^d (-q_2)^d /^d |r|^{2d} = -F < 0 , \qquad (8.7.4)$$

where $\times^d = -\times$ and $/^d = -/$ are the isodual multiplication and isodual division of the underlying field $R^d(n^d,+,\times^d)$.

But the isodual force $F^d = -F$ occurs in the isodual Euclidean space and it is therefore defined with respect to the unit -1. As a result, isoduality correctly represent the *repulsive* character of the Coulomb force for two *antiparticles* with *positive* charges.

The Coulomb force between a *particle* and an *antiparticle* can only be done in hadronic mechanics by projecting the antiparticle in the conventional space of the particle or viceversa. In the former case we have

$$F = K \times (-q_1) \times (-q_2)^d / |r|^2 < 0 , \qquad (8.7.5)$$

thus yielding an *attractive* force, as experimentally established. In the projection of the particle in the isodual space of the antiparticle we have

$$F = K^d \times^d (-q_1) \times^d (-q_2)^d /^d |r|^{2d} > 0 . \qquad (8.7.6)$$

But this force is now referred to the unit -1, thus resulting to be again attractive.

We shall have ample opportunity to verify in the remaining parts of this volume and in Vol. III that isoduality does indeed provide a characterization of antiparticle in a way completely consistent with experimental evidence for electromagnetic interactions.

Once these aspects are understood, the prediction of antigravity becomes so simple to appear trivial, and can be expressed via the following

Hypothesis 8.7.1 [51]: *Antigravity is provided by the projection of the gravitational field of antiparticles (or antimatter) in the gravitational field of particles (or matter) and viceversa.*

The above hypothesis will be studied at the full gravitational level on curved spaces in Ch. II.9. At this preliminary stage, we merely study the hypothesis in Euclidean space. Consider the Newtonian gravitational force of two *particles* of masses m_1 and m_2

$$F = -G \times m_1 \times m_2 / |r|^2 < 0 , \qquad (8.7.7)$$

where the minus sign has been added for consistency with law (8.7.3).

Within the context of quantum mechanics, the masses m_1 and m_2 remain positive, as well known. This yields *gravitational attraction* among any pair of particle–particle, antiparticle–antiparticle or particle–antiparticle.
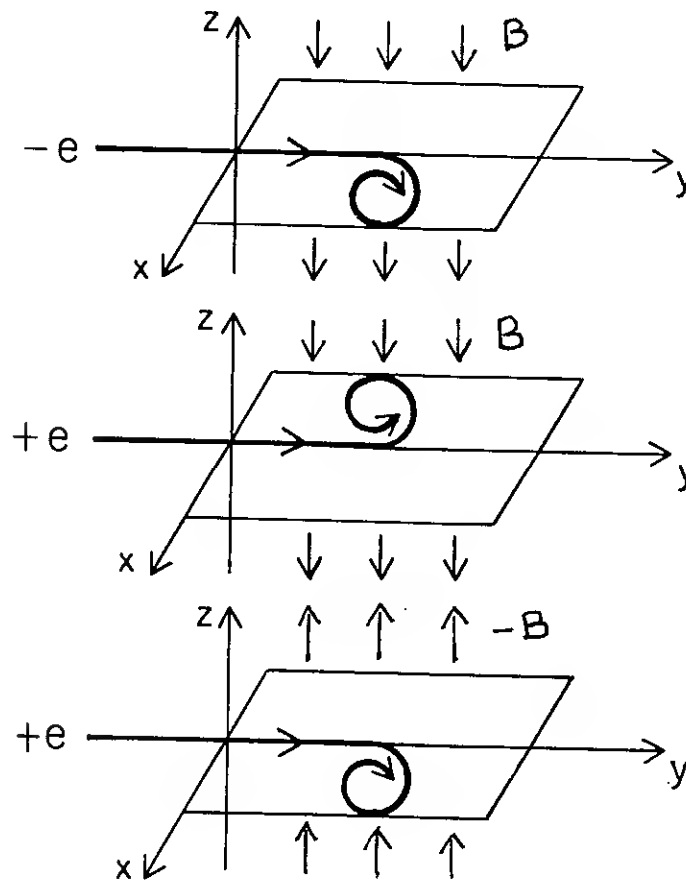
TRAJECTORIES OF ANTIPARTICLES IN ISODUAL SPACES

FIGURE 8.7.1: A schematic view of the identity of the trajectories of particles in Euclidean space and of the corresponding antiparticles in isodual Euclidean space.

Within the context of hadronic mechanics the situation is different. In fact,

antiparticles are now represented via isodualities into a different space. The case of antiparticle–antiparticle under isoduality yields the law

$$F^d = -G^d \times^d m_1{}^d \times^d m_2{}^d /^d |r|^{2d} > 0 . \tag{8.7.8}$$

But this force is again defined with respect to the negative unit −1. The isoduality therefore correctly represent the *attractive* character of the *gravitational* force among two *antiparticles*.

EXPERIMENT ON THE GRAVITY OF ANTIPARTICLES

PHOTONS

NEUTRONS

ANTINEUTRONS
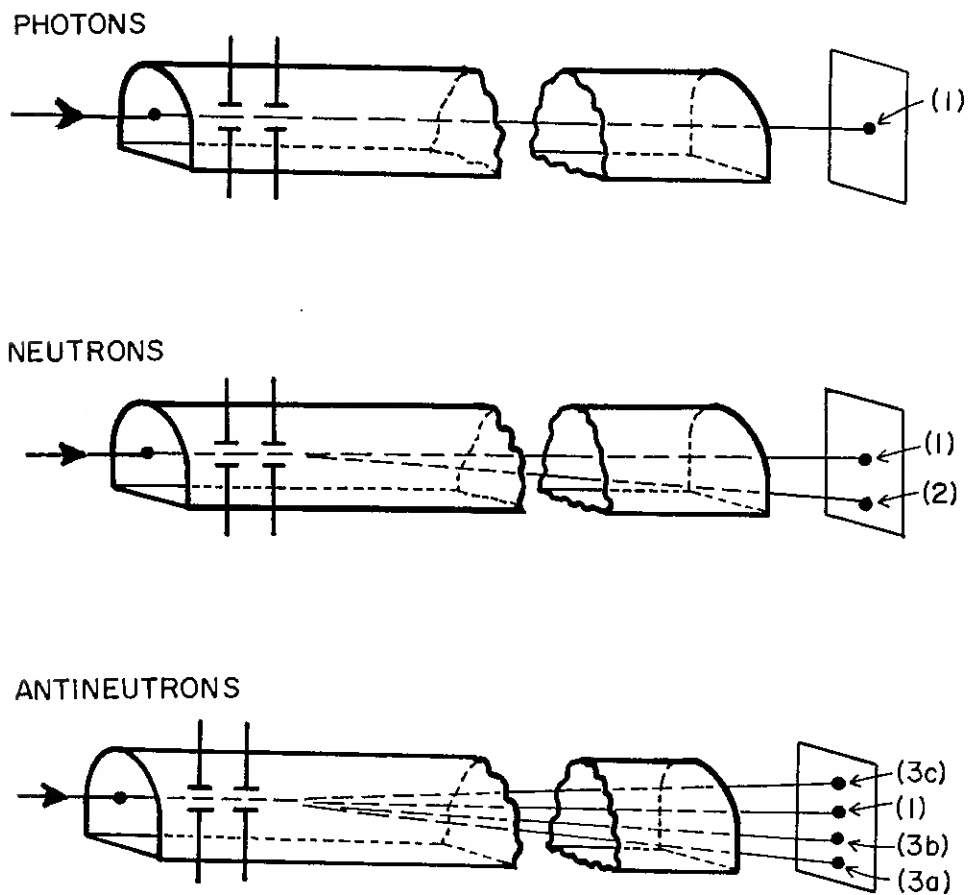
FIGURE 8.7.2: The experiment on the *gravity of antiparticles* proposed by this

author during recent visits (summer 1994) to the two-miles long linear tunnel at SLAC, Stanford–USA via the use of electrons and positrons and to the one-kilometer long tunnel at the JINR, Dubna–Russia, via the use of neutrons and antineutrons. As illustrated in the figure, the experiment essentially consists of the following three measures [51]:

1) Measure the "point of no gravity" at the end of the tunnel via a collimated optical beam;

2) Measure the downward displacement at the end of the tunnel experienced by a collimated beam of low energy particles due to the gravitational attraction of Earth which, for sufficiently low energy (e.g., of the order of KeV, is indeed measurable after 1 Km with current technology, such as scintillators, interferometric apparata or other means;

3) Measure the displacement at the end of the tunnel experienced by a collimated beam of antiparticles with the same low energy of the particles, and see whether: a) it is the same as that of particles as generally expected, b) it is attractive but less than that of particles, as predicted in ref. [61], or 3) it is a complete reversal of that of particles as predicted in ref. [51].

The case of particle-antiparticle under isoduality requires the projection of the antiparticle in the space of the particle,

$$F = -G \times m_1 \times m_2^d / |r|^2 > 0, \qquad (8.7.9)$$

which is now *repulsive*, thus illustrating the antigravity as per Hypothesis 8.7.1 Similarly, if we project the particle in the space of the antiparticles we have

$$F^d = -G^d \times^d m_1 \times^d m_2^d /^d |r|^{2d} < 0, \qquad (8.7.10)$$

which is also repulsive because referred to the unit -1.

Needless to say, model (8.7.9) is just a primitive illustration of Hypothesis 8.7.1 in the simplest possible Euclidean/isodual Euclidean case. Its extension to the Minkowskian/isodual Minkowskian case is straightforward and therefore omitted for brevity. The most general possible treatment of gravity will be done at the interior gravitational level in the next chapter in which case, as we shall see, the isoriemannian geometry and its isodual permit for the first time the complete reversal of the sign of the curvature for antiparticles in the gravitational field of particles (or viceversa).

The experimental verification of antigravity is fully feasible with current technology and actually rather simple (see figure 8.7.1).

The results of this section can be summarized via the following:

Proposition 8.7.1 [loc. cit.]: *The representation of antiparticles via isoduality renders the gravitational interactions equivalent to the electromagnetic ones, in the sense that the Newtonian gravitational law*

becomes equivalent to the Coulomb law.

As we shall see in the next chapter, the implications of antigravity are rather deep indeed. As an indication, it implies the *identification* (rather than "unification") of the electromagnetic and gravitational interactions at its ultimate level, the origin of mass (Sect. 11.9.3). In fact, such an identification renders antigravity consequential because it implies that gravitation can be attractive and repulsive exactly as it is the case for the electromagnetic interactions.

Moreover, as we shall see in Sect. 11.9.7, antigravity permits the theoretical prediction of an experimentally verifiable "space-time machine".

APPENDIX 8.A: ISOTOPIC UNIFICATION OF SIMPLE LIE-ALGEBRAS

In Ch. 11.4 we presented the following possibility first formulated in the original proposal for the Lie-isotopic theory [19] as the ultimate meaning of the covering notion of universal enveloping isoassociative algebra:

Conjecture: *All simple Lie algebras of Cartan's classification of dimension n over a field of characteristic zero can be unified by a Lie-isotopic algebra of the same dimension.*

In Ch. 11.6 we proved the above conjecture for the case of dimension three, by showing that the the the infinite family of isotopies of the rotational algebra $so(3)$, when assumed of Kadeisvili Class III, includes all three-dimensional simple Lie algebras, the compact $so(3)$ and and noncompact $so(2.1)$, as well as algebras which are simply beyond the technical capabilities of Cartan's classification, such as the isodual algebras $so^d(3)$, $so^d(2.1)$ and the four infinite families of isotopies $\hat{so}(3)$, $\hat{so}^d(3)$, $\hat{so}(2.1)$ and $\hat{so}^d(2.1)$.

The validity of the conjecture for the case of simple Lie-algebras of dimension six was proved in ref. [8]. In essence, the reformulation of the isotopies of the Lorentz algebra of Sect. 11.8.3.A for the case of Class III readily implies as particular cases:

1) $so(4)$ for T = diag. (1, 1, 1, –1), $so(3.1)$ for T = diag. (1, 1, 1, 1) and $so(2.2)$ for T = (1, 1, –1, 1), thus unifying all simple Lie algebra in Cartan's classification of dimension six (up to local isomorphisms), plus

2) the isodual algebras $so^d(4)$ for T = diag. (–1, –1, –1, 1), $so^d(3.1)$ for T = diag. (–1, –1, –1, –1) and $so^d(2.2)$ for T = (–1, –1, 1, –1) which are outside the technical possibilities of Cartan's classification, plus

3) the following six infinite classes of isotopies also outside the representational possibilities of Cartan's classification: $\hat{so}(4) \sim so(4)$, $\hat{so}^d(4) \sim so^d(4)$,

$s\hat{o}(3.1) \sim so(3.1)$, $s\hat{o}^d(3.1) \sim so^d(3.1)$, $s\hat{o}(2.2) \sim so(2.2)$, and $s\hat{o}^d(2.1) \sim so^d(2.2)$.

In the main text of this chapter we have strictly imposed the condition that all isotopies of the Lorentz algebra are of Class 1, in which case they only lead to the isotopes $s\hat{o}(3.1)$. This is due to the fact indicated earlier that no known physical event can deform the signature of the Minkowski space, thus implying a positive–definite isotopic element T in the deformation $\eta \rightarrow \hat{\eta} = T\eta$, $T > 0$.

However, the reader should be aware that, from a mathematical viewpoint, isotopic methods have much broader possibilities because they imply the loss of distinction between compactness and noncompactness both in the algebras as well as in the explicit form of the transformations, with the ensuing possibility of unification that are evidently precluded by the conventional Lie theory.

For these unifying reasons, one should note that the isotopies of the Lorentz symmetry in their original derivation [8] were constructed as isotopes of **so(4)**, rather than of **so(3.1)** and they unified in a natural way the Lorentz **so(3.1)** and de sitter **so(2.2)** algebras.

We finally mention that the Conjecture considered in this appendix has not been proved for dimensions other than 3 and 6. We merely indicate in this respect that the additional of other individual dimensions is straightforward, but the general proof is not trivial. The theorem of isotopic unification of conventional fields of Sect. 1.2.9 has been worked out for the most difficult part of the general unification, the inclusion of the exceptional algebras.

APPENDIX 8.B: ISOTOPIES AND QUANTUM DEFORMATIONS OF THE POINCARE´ SYMMETRY

In the main text of this chapter we have studied the *isotopies* $\hat{P}(3.1)$ *of the Poincaré "symmetry"* $P(3.1)$. An alternative line of research has been studied by several authors (see, e.g., ref.s [63-65] and literature quoted therein), and can be divided into two groups:

A) *The q–deformations of the Poincaré algebras* which are constructed along the *Drinfeld–Jimbo deformation* or its variants (e.g., braided and multiparameters), and have noncommuting four momentum operators; and

B) *The k–deformations of the Poincaré algebra* which are constructed along the *Hopf algebras* or its variants, and have commuting four momentum operators.

In this appendix we shall study the k–deformations of ref.s [64,65] also known as *quantum deformations* $p_k(3.1)$ *of the Poincaré algebra* $p(3.1)$, where k is a mass parameter. We shall then indicate the connection between the quantum deformation and the isotopies.

The *quantum deformed Poincaré algebras* of ref.s [64.65] can be written in terms of the conventional generators of $P(3.1)$ and *conventional commutation rules* (see Eq.s (1.1) of ref. [65])

$$[M_i , M_j] = i \, \epsilon_{ijk} \, M_k \, , \, [P_{\mu, \, p_\nu}] = 0 \, , [N_i , M_j] = i \, \epsilon_{ijk} \, N_k \, , \qquad (8.B.1a)$$

$$[M_i , P_j] = i \, \epsilon_{ijk} \, P_k \, , [M_i , P_4] = 0 \, , \quad [N_i , P_4] = i \, P_i \, , \qquad (8.B.1b)$$

$$[M_i , P_j] = i \, k \, \delta_{ij} \sinh (P_4 / k) \, , \, [N_i , N_j] = - \, i \, \epsilon_{ijk} \, [M_k \cosh (P_4/k) - (2k)^{-2} P_k \, (\, P.M) \,]$$
$$(8.B.1c)$$

where, as one can see, only the last commutators are deformed. The deformed Casimir invariants are given by ([loc. cit.], eq.s (1.4) and (1.5))

$$C^{(2)} = \delta^{ij} \, P_i \, P_j - [\, 2 \, k \sinh (P_4/k) \,]^2 \, , \qquad (8.B.2a)$$

$$C(4) = - \delta^{ij} \, W_i \, W_j + [\cosh (P_4/k) - \delta^{ij} \, P_i \, P_j] \, W_4^2 \, , \qquad (8.B.2b)$$

where W represents the *deformed* Pauli–Lubianski operator. The explicit form of the quantum deformed boosts leaving invariant (8.B.2a) are given by rather complex expressions in terms of Jacobi elliptic functions which are omitted here for brevity (see [loc. cit.], Eq.s (2.9)–(2.12)).

The connections of the above quantum deformation and the isotopies of the Poincaré algebra are intriguing indeed. In fact, both approaches are based on a deformation of the basic Minkowski space which, in momentum formalism, is evidently given by deformed Casimir (8.B.2) which we write in the form

$$" P^2 \, " = P_1 P_1 + P_2 P_2 + P_3 P_3 - P_4 [\, 2 \, k \sinh (P_4 / k) / P_4 \,]^2 P_4 \, , \quad (8.B.3)$$

both approaches characterize a generalized notion of particle because of the structural generalization of the Casimir invariants (Sect. II.8.5.G) and have other common features.

The two approaches are however applied to different physical conditions. In fact, quantum deformations (8.B.1) are applied to conventional exterior relativistic conditions , such as to an intriguing m_{elm}/k correction of the energy level of the hydrogen atom [64], while the isotopies are applied to relativistic particles in interior conditions with conventional as well as nonlinear–nonlocal–nonhamiltonian interactions.

The mathematical differences between the two approaches are visible. In fact, the isolorentz symmetry of Sect. II.8.3.B immediately yields the invariance of separation (8.B.3) via merely plotting in expressions (II.8.3.19) in momentum form the values $b_1 = b_2 = b_3 = 1$, $b_4 = 2k\sinh(P_4/k)/P_4$ without any need for the complicated Jacobi elliptic functions.

Also, the exponentiation of quantum deformations into *groups* is still open at this writing (see however the recent studies [65]) because of the technical complexities of the approach (such as lack of uniqueness of the exponential function itself), while the exponentiation of the corresponding isotopic approach is elementary. As a result of this occurrence, the isotopies of the Poincaré

algebras have a simple and unambiguous exponentiation into the corresponding *isosymmetry*, while this is not the case for ≅q- and) k-deformations.

We should finally mention that the transition from the conventional Casimir invariant $\eta^{\mu\nu}P_\mu P_\mu$ to the deformed invariant (8.B.3) can be easily proved to be *noncanonical*. As a result, the underling time evolution is *nonunitary*. The formulation of quantum deformations on conventional Hilbert spaces over ordinary fields therefore implies the now familiar problematic aspects which are common to all deformations [66] (lack of form-invariance of the theory under its own time evolution; lack of preservation of Hermiticity-observability at all times due to *Lopez's lemma* 11.3.C.1 (p. II.122); lack of a measurement theory because of the lack of a basic unit which is preserved at all times; etc., see App. I.7.9.A for details). On the contrary, the isopoincaré symmetry on isohilbert spaces over isofields avoids all these problematic aspects, as shown in this chapter.

The comparison of the simplicity of the isotopic Poincaré symmetry and the rather complex character of the corresponding quantum deformation of the Poincaré algebra is instructive. In fact, the comparison focuses the attention on the fact that *the complexity of quantum deformations is due to the preservation of the conventional Lie algebra product* AB – BA *(which, at any rate, is not preserved in time)*, while *the comparative simplicity and effectiveness of the isotopic approach is due to the use of the structurally generalized Lie product* ATB – BTA *where* T *is the 4×4 matrix representing the deformation of the underlying Minkowski space (which is indeed preserved at all times).*[120]

In the final analysis, one should keep in mind that quantum deformations can study only one *single deformation at the time*, e.g., deformations (8.B.1c). By comparison, the isotopies of Poincaré are directly universal, that is, they provide a unified invariance for all infinitely possible deformations $\eta^{\mu\nu}P_\mu P_\nu \rightarrow \hat{\eta}^{\mu\nu}(P, \dot{P}, \ddot{P}, \psi(P), \partial\psi(P), \partial\partial\psi(P), \mu, \tau, ...)P_\mu P_\nu$, $\hat{\eta} = T\eta$, $T > 0$.

APPENDIX 8.C: HERTZ-MOCANU NONINERTIAL RELATIVITY

As recalled in App. II.7.A, ultimate physical problems generally remain open and

---

[120] The transformation of commutation rules (8.A.1) under nonunitary time evolutions $UU^\dagger = 1 \neq O$ can be easily proved to yield the *isotopic* commutators, $U[X_iX_j - X_jX_i]U^\dagger = X_i'TX_j' - X_j'TX_i'$, $X = (M, N, P)$, $X' = UXU^\dagger$, where the isotopic element is $T = (UU^\dagger)^{-1}$ and the invariant isounit is $\hat{1} = T^{-1}$. In turn, as now familiar, the latter isocommutators remain form-invariant under the most general possible nonunitary transforms, provided that they are treated via the isotopic formalism of hadronic mechanics. These occurrences illustrate that *the isopoincaré algebra is simply unavoidable even when non desired*. Note that the isopoincaré algebra is universal even for all possible isotopically mapped quantum deformations.

are at time debated for centuries. Besides the ether, this is also the case of the historical alternative between Einstein's relativity [3] proposed in 1905 and *Hertz's relativity* [67] proposed some fifteen earlier. This alternative was considerably debated during the first part of this century, the issue was believed to be settled in favor of Einstein's relativity thereafter, although its study has been resumed of lately and subjected to detailed scrutiny by C. 1. Mocanu [68-70] and others. In this appendix we shall briefly review the main lines.

Einstein's and Hertz's relativities are different in objectives, technical realization and arena of physical applicability. In fact, Einstein's relativity is conceived for *inertial frames*, it is based on conventional mathematics and applies for relativistic conditions of particles in uniform motion. By comparison, Hertz's relativity was conceived for *noninertial frames* as occurring in the physical reality of our environment, it is based on a generalized formalism called *Helmholtz's calculus*, and applies for *nonrelativistic* conditions of particles in *nonuniform* motion.

Despite these visible differences, numerous criticisms were moved by Lorentz, Poincaré, Weyl and others against Hertz's relativity in favor of Einstein's version. The main contentions were that Hertz's relativity inevitably leads to the *ether theory* with the related privileged reference system while its extrapolation to relativistic speed leads to results contrary to experimental evidence. By comparison, these problematic aspects are absent in Einstein's theory. Despite all that, by no means the issue is settled, and studies in the topic are here encouraged, because of the evident possibility of a *relativity for nonuniform motion in flat spaces without any need of gravitation.*

Mocanu's first provided a comprehensive review of Hertz's relativity and underlying Helmholtz calculus in monographs [68]. He then constructed in a subsequent series of papers [69] a relativistic extension of Hertz's original formulation which apparently avoids the original problematic aspects. The latter theory is here referred to as the *Hertz-Mocanu relativity* (see ref. [70] for the latest presentation).

A central tool is *Helmholtz's total derivative* of a vector F in a three-dimensional Euclidean space $E(r,\delta,R)$

$$\frac{d_H F}{dt} = \frac{\partial F}{\partial t} + v \nabla \cdot F + \nabla \times (F \times v), \qquad (8.C.1)$$

where v is a generally *nonuniform* speed. *Hertz-Maxwell's equations* then read

$$\nabla \cdot B = 0, \quad \nabla \cdot E = \rho/\epsilon_0, \quad \nabla \times B = (d_H E/dt) c_0^{-2}, \quad \nabla \times E = - d_H B/dt \tag{8.C.2}$$

It is evident that for $v = 0$ the above equations coincide with Maxwell's equations. However for v small but not null, Eq.s (8.C.2) are different than Maxwell's form. Iteration of Hertz's contributions for increasing values of the speed then leads to deviations which have been the subject of the criticisms

recalled earlier.

However, Mocanu has introduced a different form of iterative procedure which is omitted here for brevity (see ref. [70]) which essentially yields Maxwell's electrodynamics of relativistic particles with the sole difference that the *Lorentz gamma*

$$\gamma = (1 - v^2 / c_o^2)^{-1/2}, \tag{8.C.3}$$

is replaced by *Helmholtz–Mocanu gamma*

$$\tilde{\gamma} = (1 - v^2 / c_o^2)^{-1} = \gamma^2. \tag{8.C.4}$$

The *Hertz–Mocanu boosts* in (3–4)-coordinates are the given by

$$x'^3 = \gamma_H (x^3 - \beta t), \quad x'^4 = \gamma_H (x^4 + \beta x^3), \quad \beta = v / c_o, \tag{8.C.5}$$

where the reader should keep in mind that *the speed v is not uniform*. For the reconstruction of the law of relativistic composition of speed and of other laws of the special relativity in the above nonuniform setting we refer the interested reader to ref. [70].

The lack of final character of the criticisms on Hertz relativity is now evident. In fact, all criticisms are based on the results of the *special* relativity, that is, for *uniform* motion, while no final experimental result exists at this writing for *nonuniform motion in flat spaces*.

A few comments on the Hertz–Mocanu relativity and the isospecial relativity by this author are in order. Recall that the latter too has been conceived for *noninertial frames* as a condition of applicability to the frames of the physical reality. In fact, the emerging general isolorentz transforms are *nonlinear* (Sec. II.8.3). As a result, the speeds v of the isospecial relativity are not restricted to be uniform.

It should be however indicated that *our isospecial relativity does not include as a particular case the Hertz–Mocanu relativity*. This can be seen, e.g., from the fact that the identity of transforms (8.C.5) and the isolorentz boosts of Sect. II.8.3

$$\hat{\gamma} (x^3 - \beta x^4) = \tilde{\gamma} (x^3 - \beta x^4), \quad \hat{\gamma} (x^4 - \beta B^2 x^3) = \tilde{\gamma} (x^4 + \beta x^3), \tag{8.C.6a}$$

$$\hat{\gamma} = |1 - \beta^2 B^2|^{-1/2}, \quad \tilde{\gamma} = (1 - \beta^2)^{-1}, \quad \beta = v / c_o, \quad B = b_s / b_4, \tag{8.C.6b}$$

does not exist for $v \neq c_o$, i.e., there exists no solution in the $B = b_3/b_4$ term yielding the above identities.[121]

This result should be expected from the fact that the isospecial relativity is an isotopy of the special relativity and, therefore, it preserves the topological

---

[121] Note that the value $B^2 = [(1 + \beta^2)^2 - 1] / \beta^2$ readily yields the identity $\gamma_s \equiv \gamma_H$, but this is insufficient to yield the identity of transforms (8.C.6) because of the B-term in the isotransformation of time.

characteristics of the latter, including the crucial preservation of the signature of space-time (+, +, +, –). On the contrary, the Hertz–Mocanu relativity is not an isotopy of the special relativity and it is based on a separation between space and time much along the *Galilean* case of App. 11.7.A, thus resulting to be more aligned with the *relativistic Galilean symmetry* [71] than with the isotopies of Poincaré.

For additional studies on generalizations of the special relativity we regret to be unable to review for brevity, we refer the reader to Animalu [72].

REFERENCES

1. H. A. LORENTZ, Amst. Proc. **6**, 609 (1905)
2. H. POINCARÉ, Comptes Rendues, Paris **140**, 1504 (1905)
3. A. EINSTEIN, Ann. Math. **17**, 891 (1905)
4. H. MINKOWSKI, Math. Ann. **68**, 472 (1910)
5. W. PAULI, *Theory of Relativity*, Pergamon Press, London (1958); H. A. LORENTZ, A. EINSTEIN, H. MINKOWSKI and H. WEYL, *The Principles of Relativity: A collection of Original Memoirs*, Methuen (1923); *Collected Papers of Albert Einstein*, J. Stachel, Editor, Princeton Univ. Press, Vol.s I–V (1988 on); A. TYAPKIN, *Poincaré*, Molodaya Gvardi, Moscow, Russia (1982)
6. R. M. SANTILLI, *Foundations of Theoretical Mechanics*, Vol. 1: *The Inverse Problem in Newtonian Mechanics*, Springer–Verlag, Heidelberg/New York (1978)
7. R. M. SANTILLI, *Foundations of Theoretical Mechanics*, Vol. II: *Birkhoffian Generalization of Hamiltonian Mechanics*, Springer–Verlag, Heidelberg/New York (1983)
8. R. M. SANTILLI, Lett. Nuovo Cimento **37**, 545 (1983)
9. R. M. SANTILLI Hadronic J. Suppl. **4A**, issue no. 3 (1988)
10 R. M. SANTILLI, *Isotopic Generalizations of Galilei's and Einstein's Relativities*, Vol. I: *Mathematical Foundations*, Hadronic Press, Palm Harbor, FL (1991); second edition, Academy of Sciences of Ukraine, Kiev, in press
11. R. M. SANTILLI, *Isotopic Generalizations of Galilei's and Einstein's Relativities*, Vol. II: *Classical Isotopies*, Hadronic Press, Palm Harbor, FL (1991); second edition, Academy of Sciences of Ukraine, Kiev, in press
12. R. M. SANTILLI, Hadronic J. Suppl. **4B**, issue no. 3, (1989)
13. R. M. SANTILLI, Hadronic J. **15**, 1 (1991)
14. R. M. SANTILLI, in *The Mathematical Legacy of Hanno Rund*, J. V. Kadeisvili, Editor, Hadronic Press, Palm Harbor, FL (1993)
15. R. M. SANTILLI, J. Moscow Phys. Soc. **3**, 255 (1993)
16. A. K. ARINGAZIN, A. JANNUSSIS, M. NISHIOKA, D. F. LOPEZ and B. VELJANOSKI, *Santilli's Lie–Isotopic Generalizations of Galilei's and Einstein's Relativities*, Kostarakis Publisher, Athens, Greece (1990)

17. J. V. KADEISVILI, *Santilli's Isotopies of Contemporary Algebras, Geometries and Relativities,* Hadronic Press, Palm Harbor, FL (1992); second edition, Academy of Sciences of Ukraine, Kiev, in press

18. J. LÔHMUS, E. PAAL and L. SORGSEPP, *Nonassociative Algebras in Physics,* Hadronic Press, Palm Harbor, FL (1994)

19. R. M. SANTILLI, Hadronic J. 1, 223 and 1279 (1978)

20. R. M. SANTILLI, *Lie-Admissible Approach to the Hadronic Structure,* Vol. I: *Nonapplicability of Galilei's and Einstein's Relativities ?,* Hadronic Press, Palm Harbor, FL (1978)

21. R. M. SANTILLI, *Lie-Admissible Approach to the Hadronic Structure,* Vol. II: *Coverings of the Galilei and Einstein Relativities ?* Hadronic Press, Palm Harbor, FL (1982)

22. R. M. SANTILLI, seminar at the International Congress of Mathematicians, Zurich, August 1994, and in *ICM-94 Short Communications,* p. 82

23. W. RÜHL, *The Lorentz Group and Harmonic Analysis,* Benjamin, New York (1970)

24. N. I. VILENKIN and U. A. KLIMYK, *Representations of Lie Groups and Special Functions,* Vol.s I–III, Kluwer, Dordrecht, England (1993)

25. I. M. GEL'FAND, R. A. MINLOS and Z. Ya. SHAPIRO, *Representations of the Rotation and Lorentz Groups and their Applications,* MacMillan, New York (1963)

26. J. V. KADEISVILI, in *The Mathematical Legacy of Hanno Rund,* J. V. Kadeisvili, Editor, Hadronic Press (1993)

27. R. M. SANTILLI Lett. Nuovo Cimento 33, 145 (1982)

28. V. DESABBATA and M. GASPERINI, Lett. Nuovo Cimento 34, 337 (1982)

29. D. I. BLOCHINTSEV, Phys. Lett. 12, 272 (1964); Sov. Phys.–Usp. 9, 405 (1966)

30. L. B. REDEI, Phys. Rev. 145, 999 (1966)

31. D. Y. KIM, Hadronic J. 1, 1343 (1978)

32. H. B. NIELSEN and I. PICEK, Nuclear Phys. B 211, 269 (1983)

33. S. H. ARONSON, G. J. BOCK, H-Y. CHANG, and E. FISHBACH, Phys. Rev. Lett. 48, 1306 (1982) and Phys. Rev. D 28, 495 (1983)

34. N. GROSSMAN, H. KELLER, C. JAMES, M. SHUPE, K. THORNE, P. BORDER, M. J. LONGO, A. BERETVAS, A. CARACAPPA, T. DEVLIN, H. T. DIHEL, U. TOSHI, K. KUEGER, P. C. PETERSEN, S. TEIGE and G. B. THOMSON, Phys. Rev. Lett. 59, 18 (1987)

35. F. CARDONE, R. MIGNANI and R. M. SANTILLI, J. Phys. G: Nucl. Part. Phys. 18, L61 and L141 (1992)

36. V. N. STREL'TSOV, Hadronic J. 17, 105 (1994)

37. H. ARP, *Quasars Redshift and Controversies,* Interstellar Media, Berkeley (1987)

38. J. W. SULENTIC, Astrophys. J. 54, 343 (1987)

39. R. MIGNANI, Phys. Essays 5, 531 (1992)

40. P. MARMET, IEEE Trans. Plasma Science 17, 238 (1988)

41. J. W. SULENTIC, in *Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Plenum, London (1994)

42. R. M. SANTILLI, in *Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Plenum, London (1994)

43. H. ARP, G. BURBRIDGE, F. HOYLE, V. J. NARDIKAR and N. C. WICRAMASINGH, Nature **346**, 807 (1990)

44. H. ARP, in *Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Plenum, London (1994)

45. H. A. LORENTZ *Versuch einer Theorie der Elektrishen und Magnetishen Erscheinungen in bewenglen Körpern*, Leyda (1895)

46. T. E. PHIPPS, Apeiron, in press

47. R. M. SANTILLI, Hadronic J. **1**, 574 (1978)

48. J. M. BLATT and V. F. WEISSKOPF *Nuclear Physics*, Wiley, New York (1964)

49. H. RAUCH, Hadronic J. **4**, 1280 (1981); and in *Proceedings of the International Symposium on the Foundations of Quantum Mechanics*, Japanese Phys. Soc., Tokyo (1983)

50. R. M. SANTILLI, in *Proceedings of Deuteron 1993*, JINR, Dubna, Russia (1994)

51. R. M. SANTILLI, Hadronic J. **17**, 257 (1994)

52. R. M. SANTILLI, Hadronic J. **17**, 285 (1994)

53. R. M. SANTILLI, Hadronic J. **17**, 311 (1994)

54. A. O. E. ANIMALU, Hadronic J. **17**, 349 (1994)

55. A. K. ARINGAZIN, Hadronic J. **12**, 71 (1989)

56. E. A. MILNE, *Kinematic Relativity*, Clarendon Press, London (1948)

57. R. M. SANTILLI, Algebras, Groups and Geometries **10**, 273 (1993)

58. A. JANNUSSIS, Hadronic J. 13, 399 and 415 (1990); in *Proceedings on the Third International Workshop on Symmetry Methods in Physics*, G. Pogosyan et al., Editors, JINR, Dubna, Russia (1994); and in *Frontiers of Fundamental Physics*, F. Selleri and M. Barone, Editors, Plenum, London (1994)

59. 1. PRIGOGINE, in *Order and Fluctuations in Equilibrium and Nonequilibrium Statistical Mechanics*, G. Nicolis et al., Editors, Wiley, N. Y. (1981)

60. A. SCHOEBER, Editor, *Irreversibility and Nonpotentiality in Statistical Mechanics*, Hadronic Press, Palm Harbor, FL (1984)

61. M. M. NIETO and T. GOLDMAN, Physics Reports **205**, 221 (1991) and **216**, 345 (1992)

62. D. H. CHILDRESS, *The Antigravity Handbook*, Network Unlimited Press, Stelle, Illinois, 1993; B. C. EBERSHAW, *Antigravity Propulsion Devices*, D. & B. Associates, 1980.

63. V. DOBREV, in *Proceedings of the Second Wigner Symposium*, Clausthal 1991, World Scientific, Singapore; J. LUKIERSKY, A. NOVICKI, H. RUEGG and V. TOLSTOY, Phys. Lett. B **264**, 331 (1991); O. OGIEVETSKI, W.B. SCHMIDKE, J. WESS and B. ZUMINO, Comm. Math. Phys. **50**, 495 (1992); S. GILLER, J. KUNZ, P. KOSINSKY, M. MAJEWSKI and P. MASLANKA, Phys. Lett. B. **286**, 57 (1992)

64. J. LUKIERSDKI, A. NOWICKI and H. RUEGG, Phys. Lett. B **293**, 344 (1992); J.

LUKIERSKI, H. RUEGG and W. RÜHL, Phys. Lett. B **313**, 357 (1993)

65. J. LUKIERSKI and H. RUEGG, Phys. Lett. B **329**, 189 (1994); S. MAJID and H. RUEGG, Phys. Lett. B **334**, 348 (1994)

66. D. F. LOPEZ, Hadronic J. **16**, 429 (1993)

67. H. R. HERTZ, Ann Phys. 40 and 41 (1890

68. C. I. MOCANU, *Hertzian Relativistic Electrodynamics and its Associated Mechanics,* Vol.s I and II, Hadronic Press, Palm Harbor, FL (19931), second edition, in press

69. C. I. MOCANU, Hadronic J. **10**, 61, 153 and 231 (1987); **11**, 35 and 55 (1988); Found. Phys. Lett. **5**, 443 (1992)

70. C. I. MOCANU, in *Proceedings of the International Conference on the Frontiers of Fundamental Physics,* F. Selleri and M. Barone, Editors, Hadronic Press, Palm Harbor, FL (1994)

71. P. ROMAN, J. J. AGHASSI and R. M. SANTILLI, Phys. Rev. D **1**, 2753 (1970); J. Math. Phys. **11**, 2297 (1970); and Nuovo Cimento **5A**, 551 (1971)

72. A. O. E. ANIMALU, Hadronic J. **9**, 61 (1986); **10**, 321 (1987); in *Proceedings of the Fifth Workshop on Hadronic Mechanics,* Nova Science, New York (1993); and Preprint, University of Irvine, CA (1993)

NOTE ADDED TO THIS SECOND EDITION

An alternative formulation of the isodifferential calculus on isominkowski space based on the definitions $\hat{\partial}_\mu = \hat{1}^\mu{}_\nu \, dx^\nu$ and $\partial/\partial x^\mu = \hat{T}_\mu{}^\nu \partial/\partial x^\nu$ (rather than expressions (8.4.2) and (8.4.3)) with consequential reformulation of the isospecial relativity and its isodual, has been worked out in the papers below

R. M. SANTILLI, Nonlocal-integral isotopies of differential calculus, mechanics and geometries, Rendiconti Circolo Matematico di Palermo, in press

R. M. SANTILLI, Isotopic lifting of Newtonian mechanics, Revista Tecnica, **18**, (1995), in press; Isotopic lifting of analytic and quantum mechanics, Revista Tecnica **19**, (1996), in press; and Isotopic lifting of differential geometries, Revista Tecnica, **19**, (1996), in press.

A mathematical study of the Conjecture of App. 8.A (on the possible isotopic unification of all simple Lie algebras of Cartan's classification with the same dimension) has been conducted in

Gr. TSAGAS, A study on the classification of Lie–Santilli algebras, Algebras, Groups and Geometries, **13**, (1996) in press)

9: ISOTOPIC, GENOTOPIC AND ISODUAL FORMULATIONS OF GENERAL RELATIVITY

9.1: STATEMENT OF THE PROBLEM.

In this chapter we consider the field originated by Riemann [1], Einstein [2,3], Schwartzchild [4,5], Freud [6] and several others (see, e.g., ref.s [7-9]), today known under the name of the *general theory of gravitation* or *general relativity* for short.

We shall assume the Riemannian geometry [9] as being *exactly* valid for the exterior gravitational problem in vacuum and introduce its axiom—preserving isotopies and genotopies for interior problems which are nonlinear in the velocities and other variables, nonlocal—integral and non—first—order—Lagrangian when projected in the original space, but isolinear, isolocal and isolagrangian when represented in isospace.

Unfortunately, while we did assume the special relativity at the foundations of our relativistic studies, we shall be unable to assume the general relativity at the foundations of our gravitational studies for a variety of reasons.

The first is that, unlike the special, the general relativity is afflicted by a considerable number of rather serious problematic aspects of *physical* character which have accumulated in the refereed literature of this century but remained essentially ignored. It is evident that no truly scientific appraisal of the general relativity can be done without a serious consideration and eventual resolution of these problematic aspects.

Contrary to popular beliefs, the most controversial aspects of the general relativity are its *experimental verifications.* In fact, verifications centrally dependent on curvature, such as the bending of light, are intrinsic in the *Riemannian geometry* and, as such, they are equally admitted by other non-Einsteinian theories of gravitation. The remaining verifications are centrally dependent on the so-called *post-Newtonian approximation* (see, e.g., ref. [8])

which is far from being unique or established on rigorous scientific grounds owing to the lack of resolution of the dissident literature in refereed journals. In fact, the selection of a *different* post–Newtonian approximation of the *same* field equations leads to *different* numerical values in disagreement with experiments, thus establishing the unsettled character of the current experimental profile.

Moreover, the general relativity is afflicted by problematic aspects of *geometrical* character which also have remained largely ignored. For instance, we have shown in Lemma 1.5.6.2 that Einstein's tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}$ *does not* preserve under isotopies the vanishing character of the covariant divergence (or contracted Bianchi identity). We shall also show in this chapter that Einstein's field equations do not possess the general form requested by *the Freud identity* of the Riemannian geometry [6,7]. It is evident that, lacking a resolution one way or the other of these additional problematic aspects, the geometric axioms of the general relativity cannot be assumed at the foundations of our studies.

As we shall see, the isotopies and genotopies imply an inevitable two–fold revision of gravitation. The first is that, when working in a curved space, only those axioms which are invariant under isotopies and genotopies can be accepted. This condition alone is sufficient to prevent the use of Einstein's field equations because of the above mentioned lack of preservation under isotopies of the null character of the covariant divergence of the basic tensor $G_{\mu\nu}$.

The second revision is due to the fact that *relativistic hadronic mechanics of the preceding chapter already admits* a *full description of gravitation on a "flat" isominkowski space.* In fact, the theory has been constructed for the isotopies of the Minkowski metric $\eta \rightarrow \hat{\eta} = T\hat{\eta}$ with an arbitrary functional dependence, including that in the space–time coordinates, $\hat{\eta}(x, \ldots)$. But the metric $g(x)$ of a (3.1)–dimensional Riemannian space always admit the factorization $g(x) = T(x)\eta$. Therefore, all possible Riemannian metrics $g(x)$ can be *identically* interpreted as being the metrics of an isominkowskian space $g(x) \equiv T(x)\eta = \hat{\eta}(x)$.

It follows that relativistic hadronic mechanics, as presented in the preceding chapter, already incorporates a full description of gravitation as embedded in its isounit. This yields a new formulation of gravitation with far reaching implications owing to its *flat* character, as well as a new quantization of gravity based on the same axioms of the special relativity which therefore ensure its consistency.

In different terms, a fully consistent quantum theory of gravity already exists. It did creep in un–noticed because it is imbedded in the *unit* of conventional relativistic quantum mechanics.

It is hoped that these introductory remarks are sufficient to indicate the novel perspective of the gravitational studies presented in this chapter.

We shall begin with a brief review of the arguments of Ch. 1.1 according to which general relativity is *inapplicable* to *interior* gravitational problems. We shall then review the problematic aspects of the theory for *exterior* gravitational

problems and identify the general class of field equations which resolve the latter problematic aspects from a purely geometric viewpoint, including the Freud identity.

After these introductory parts, we shall present the nonlocal–integral formulation of interior gravitation on a curved space–time as suggested by the isoriemannian geometry of Sect. 1.5.6. We shall then identify the exterior limit of the interior field equations and show that they coincide with the exterior field equations independently derived in the preceding section. This important result confirms the overall geometric consistency of the isotopies, and establishes the rather forceful inevitability of the proposed new class of field equations. An outline of the genoriemannian counterpart is included for the characterization of irreversibility, this time, from a gravitational viewpoint.

We shall then pass to the flat isominkowskian reinterpretation of gravitation and identify some of its intriguing implications, including the isotopic quantization of gravity, the isotopic treatment of gravitational singularities and other aspects. The section also includes the isodual formulation for antimatter, the formulation of antigravity at the gravitational level and the genotopic extensions for irreversible conditions.

The chapter ends with the main lines of a structurally novel cosmology which is implies by the preceding studies.

A technical knowledge of the isotopies and genotopies of the Riemannian geometry (Sect.s 1.5.6 and 1.7.7) as well as of relativistic hadronic mechanics (Ch. II.8) is a pre–requisite for the understanding of this chapter.

9.2: INAPPLICABILITY OF GENERAL RELATIVITY FOR THE INTERIOR GRAVITATIONAL PROBLEM

9.2.A: Exterior and interior gravitational problems. Let us begin by recalling the historical distinction between the *exterior gravitational problem,* consisting of point–like test bodies moving in the homogeneous and isotropic vacuum, and the *interior gravitational problem,* consisting of extended test bodies moving within inhomogeneous and anisotropic physical media.

This distinction was introduced by the founders of analytic dynamics, and kept up to the first part of this century (see, e.g., Schwartzschild's *two* papers, the first famous one on the *exterior* problem [4] and the second little known paper on the *interior* problem [5], or the early well written treatises in gravitation, e.g., [10,11]).

Regrettably, the above distinction was progressively relaxed, up to the current condition of virtual complete silence in the specialized literature, and the treatment of all gravitational problems as occurring solely in vacuum. In particular, the distinction has been eliminated where it is needed most, in the limiting interior conditions of gravitational collapse, black holes, big bang

theories and all that.

9.2.B: Inequivalence of the interior and exterior problems.

The point-like approximation of test bodies implies the local-differential character of the applicable geometry. In the exterior gravitational problem the actual size and shape of the test body has no impact on the trajectory. Test bodies can therefore be effectively approximated as being *massive points*, as first stated by Galilei and subsequently embraced by Einstein [3].

The equations of motion are then characterized by *ordinary or partial differential equations*, such as the familiar forms in Euclidean space

$$m \, d^2r/dt^2 = F(t, r, \dot{r}), \quad F = \text{local-differential}, \quad (9.2.1)$$

The *exact* validity of local-differential geometries for the exterior problem then follows. This illustrates the reason for our assumption of the conventional Riemannian geometry as being *exactly* valid for the exterior problem in vacuum jointly with its tangent Minkowski space.

In the interior problem within physical media trajectories are directly affected by the actual shape of the test body. Interior problems are therefore intrinsically *nonlocal* hereon referred to an essential dependence on surface integrals (at the classical level) or volume integrals (at the particle level). As an example, two space-ships with the same mass and speed but different shapes have different re-entry trajectories in atmosphere whose characterization requires an integral over the surface σ of the space-ship.

The (classical) equations of motion are therefore *ordinary or partial differential equations plus an integral term*, such as the following forms also in Euclidean space

$$m \, d^2r/dt^2 = F(t, r, \dot{r}) + \mathfrak{F}(\sigma, ...), \quad \mathfrak{F}(\sigma) = \int_\sigma d\sigma \, K(\sigma, ...), \quad (9.2.2)$$

where K is a suitable functional, and the additional functional dependence besides σ can be on the velocities $\dot{r}$, the accelerations $\ddot{r}$, the local density μ, temperature τ, and any other needed interior quantity.

The exterior and interior problems are therefore structurally inequivalent on topological grounds. In fact, the former admits known local-differential topologies, while the latter requires an *integro-differential topology* (Sect. 1.1.5) as evidently necessary to accommodate the integro-differential forces $\mathfrak{F}(\sigma, ...)$.

The above geometric inequivalence of the exterior and interior problems is known since a long time and it is at times called *Cartan's legacy* because (E.) Cartan was the first to show that *the Riemannian geometry simply cannot recover "all" Newtonian systems of our physical reality*.

Moreover, exterior problems can only admit action-at-a-distance, potential-Lagrangian interactions, while interior problems have additional nonpotential-non-(first-order)-Lagrangian forces with zero range. For instance,

missiles in atmosphere admit integral resistive forces $\mathcal{F}(\sigma)$ which can be approximated via power series in the velocities, $\mathcal{F} \approx -\sum_n \gamma_n \dot{r}^n$, $\gamma_n > 0$. As known in *engineering* rather than physical circles, these expansions have nowadays surpassed the truncation at the tenth power for the recent very high speeds of missiles in atmosphere or satellites during re–entry. These interior systems are simply beyond any hope of representation via a Lagrangian in the r–frame of the observer (see later on the problems occurring in the use of coordinate transforms).

The necessary and sufficient conditions for the existence of a potential or a Lagrangian were studied in detail by this author in monographs [12,13] and they resulted to be the *conditions of variational selfadjointness* originated by Helmholtz back in 1887 (see the historical note in [loc. cit.]). Exterior problems result to be *essentially selfadjoint* (SA), that is, verifying the conditions of SA in the frame of the observer, and we shall write

$$m\, d^2r/dt^2 = F^{SA}(t, r, \dot{r}), \quad F = \text{local–differential and SA}, \qquad (9.2.3)$$

while interior problems result to be *essentially nonselfadjoint* (NSA), that is, violating the conditions of SA in the frame of the observer, and we shall write

$$m\, d^2r/dt^2 = F^{SA}(t, r, \dot{r}) + \mathcal{F}^{NSA}(\sigma, \ldots), \quad \mathcal{F}^{NSA} = \text{nonlocal–integral and NSA.} (9.2.4)$$

Ref. [13] then introduced the *Birkhoffian generalization of Hamiltonian mechanics* for the analytic representation of the latter systems (in their local–differential approximation), whose Lagrangian counterpart is of *second–order* (e.g., acceleration–dependent).

In summary, *the studies of monographs* [12,13] *established the topological, geometric and analytic inequivalence of the exterior and interior problems and the lack of representation of the latter in the frame of the observer via a first–order Lagrangian.*

9.2.C: Irreducibility of the interior to the exterior problem. Whenever

exposed to the complexity of the interior problem, a rather general attitude is that of reducing it to a collection of exterior systems in vacuum. For instance, when exposed to the equations of motion of a satellite during re–entry in atmosphere with drag force $F^{NSA} = -\sum_{n=1,\ldots,10} \gamma_n \dot{r}^n$, a rather general attitude is that of reducing such system to its point–like elementary constituents in the expectation that such reduction would turn the original interior system into a collection of exterior ones in vacuum.

As recalled in Ch. 1.1, the latter reduction has been proved to be impossible by the so–called *No–Reduction Theorems* [14] which essentially establish that a *classical interior system (such as a satellite during re–entry in atmosphere) with a continuously decaying angular momentum, simply cannot be reduced to a (finite) collection of ideal elementary particles in exterior conditions in vacuum,*

each one with conserved angular momentum. Viceversa, a finite collection of elementary particles all in stable orbits with conserved angular momenta, simply cannot reproduce a classical object whose center of mass has a continuously decaying angular momentum.

A second No-Reduction Theorem was formulated [loc. cit.] for the inconsistency of the reduction of a classical system manifestly noninvariant under the rotational (and therefore, the Lorentz) symmetry because of the continuous decay of the angular momentum, to an ideal collection of point-like particles verifying such symmetry (and viceversa).

A third No-Reduction Theorem [loc. cit.] shows the impossibility of a consistent reduction of a classical object in irreversible conditions to an ideal collection of quantum mechanical particles all in reversible conditions (and viceversa).

At any rate, even ignoring these No-Reduction Theorems, a quantum version of gravity (which is an evident pre-requisite for these reductions), is still far from being achieved in a final form accepted by the scientific community at large for numerous independent reasons such as: the open technical problems in the operator formulation of curvature; the null value of the Hamiltonian of Einstein's gravitation as compared to the necessity of a non-null Hamiltonian in quantum mechanics; the unsettled character of the classical field equations themselves which have to be quantized; etc.

Interior systems with continuously decaying angular momenta are *purely classical.* The achievement of a purely classical representation is therefore a necessary pre-requisite before possible operator formulations may acquire physical credibility.

After achieving the awareness of the impossibility of a consistent reduction of interior to exterior problems, a further rather general attitude is that of transforming interior nonlagrangian systems into new coordinates in which a conventional Lagrangian representation exists.

It was shown in monograph [13] that, under the necessary smoothness and regularity conditions in a star-shape region of the variables, and under a necessary local-differential approximation, *all nonselfadjoint interior systems admit an indirect analytic representation,* that is, they always admit transformations of the local variables under which a Lagrangian or a Hamiltonian exists (this property is also known as the *Lie-Koening theorem*).

However, the needed transformations $\{r, p\} \to \{r', p'\} = \{r'(t, r, p), p'(t, r, p)\}$ are *necessarily noncanonical* (trivially, because the original system is nonhamiltonian by assumption), and *necessarily nonlinear* (trivially, because linear transformations cannot possibly eliminate forces of the type $F = -\gamma r^{-n}$). The transformation of a system which is nonlagrangian in the coordinates of the observer into an equivalent Lagrangian system therefore implies:

1) The local-differential *approximation* of the nonlocal interior forces via power series in the velocities and other means as a necessary condition to apply

the Lie–Koening theorem;

2) The inability to realize the transformed frame in actual experiments (e.g., an experiment feasible in our r–frame is evidently not realizable in the transformed frame r' = α exp(βrṙ), α, β reals) and, last but not least:

3) The inapplicability of Einsteinian theories evidently because of new Lagrangian frames are highly nonlinear and, therefore, highly *noninertial.*

In short, the reduction of nonlagrangian to Lagrangian systems defeats the very objectives for which it was intended for.

By no means the above comments exhaust all reasons for the inapplicability of the general relativity to interior conditions. For instance, as now familiar, one of the objectives of a correct theory for interior problem is a direct geometric representation of the inhomogeneity and anisotropy of interior physical media which is sufficient, alone to require a structural revision of the general relativity.

Another argument is related to the locally varying character of electromagnetic waves in interior conditions studied in the preceding chapter. This implies the lack of interior applicability of general relativity because of its known inability to represent such physical reality. This aspect is also sufficient, alone, to require a structural revision of the general relativity because of the evident need to generalize in a suitable way the fundamental metric of the theory.

Particularly inspiring is the care in which the founders of gravitation presented *interior* solutions as being *merely approximate* (see, e.g., Schwartzschild [5]), a scientific care which has virtually disappeared in the contemporary literature in the field.

In the final analysis, typical interior problems such as gravitational collapse, black holes or big bang theories are not composed of ideal point particles moving in vacuum, but rather of extended wavepackets and charge distributions of hadrons in conditions of total mutual penetration and compression in large numbers into small regions of space. The nonlinearity in the velocity, nonlocal–integral as well as nonpotential–nonlagrangian character of these interior conditions, and the consequential inapplicability of the *Riemannian geometry*, let alone Einstein's gravitation, are then beyond credible doubts (see Weiss [15] for problems of scientific ethics and accountability on this issue).

9.3: PROBLEMATIC ASPECTS OF GENERAL RELATIVITY FOR THE EXTERIOR GRAVITATIONAL PROBLEM

9.3.A: Einstein's conception of gravitation in vacuum. We are now sufficiently equipped to review the problematic aspects of general relativity in

vacuum. Our analysis is restricted to the *exterior gravitation of an astrophysical body with null total charge and null electric and magnetic moments*, in which case *Einstein's field equations* on a conventional (3+1)-dimensional Riemannian space $\Re(x,g,R)$ are given by[122]

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = 0 . \qquad (9.3.1)$$

As we shall see, the above conception of exterior gravitation is afflicted by numerous, rather serious and independent problematic aspects whose only known solution is a structural revision of the theory itself.

In this section we shall outline the direct and indirect, physical and geometrical arguments according to which: 1) Eq.s (9.3.1) miss a term in their l.h.s., the *isotopic scalar* (1.5.6.18); 2) Eq.s (9.3.1) also miss a term in the r.h.s., a nowhere null source *in vacuum* which is first-order in magnitude even for bodies with null total charge and null electric-magnetic moments; and 3) a consistent theory of gravitation must admit equivalent formulations in both the Riemannian *and the Minkowskian space* .

We review below the study of ref. [34] on the above problematic aspects and identify the general class of field equations for their possible resolution.

9.3.B: Incompatibility of general relativity with the electromagnetic origin of mass.

A first problematic aspect of general relativity studied in detail by this author two decades ago [16], the incompatibility of Einstein's field equations (9.3.1) and the electromagnetic origin of mass. According to established knowledge, the mass M of all elementary particles has an electromagnetic origin which is of first-order in magnitude, with second-order corrections due to the short range, weak and strong fields. This implies the necessary presence in the r.h.s. of the field equations in vacuum of a nowhere null, first-order source even for bodies with null total electromagnetic phenomenology.

In fact, ref. [16] established that a neutral particle with mass M possesses in its environment an electromagnetic field $F^{\mu\nu}{}_{elm}$ whose energy-momentum tensor $T^{\mu\nu}{}_{elm}$ is of such first-order in magnitude to essentially account for the entire mass of the particle, the balance being due to weak and strong short range interactions (s.r.i),

$$M \approx \int dv\, T^{00}{}_{Elm}, \qquad M = \int dv\, (T^{00}{}_{Elm} + t^{00}{}_{s.r.i.}) . \qquad (9.3.2)$$

This implies the complete elimination of the mass tensor $M^{\mu\nu}$ in the field equations and its replacement with the fields originating mass itself, according to the identification [53]

---

[122] The extension of the theory to astrophysical bodies with charge and electric-magnetic moments essentially implies the addition in the r.h.s. of a source with a known very small contribution to curvature which, as such, will be ignored.

$$M^{\mu\nu}{}_{matter} \equiv T^{\mu\nu}{}_{elm} + t^{\mu\nu}{}_{s.r.i.} , \quad T^{\mu\nu}{}_{elm\,;\,\nu} \equiv 0 , \qquad (9.3.3)$$

where the last expression represents a known property of the electromagnetic field in Minkowski space.

The basic equations for gravitation in vacuum emerging from studies [16] are therefore given by the following modification of Einstein's equations (9.3.1)

$$G^{\mu\nu} = k\,(T^{\mu\nu}{}_{elm} + t^{\mu\nu}{}_{s.r.i.}) , \quad T^{\mu\nu}{}_{elm\,|\,\nu} \equiv 0 , \qquad (9.3.4)$$

where the tensor $T^{\mu\nu}{}_{elm}$ is a nowhere null, first-order source even for bodies with null total charge and elm moments, the tensor $t^{\mu\nu}{}_{s.r.i.}$ is of second-order in magnitude compared to the preceding one; and the covariant derivative $|\,\nu$ is now in a Riemannian space.

The verification of these results was done in ref. [16] via the classical relativistic calculations of the π° particle as a bound state of one constituent with charge +q and its antiparticle with charge −q in high dynamical conditions at a mutual distance of 1 fm $(10^{-13}$ cm). Even though the π° has null total charge and null elm moments, the calculations showed the existence in its *exterior* of a first-order electromagnetic source $T^{\mu\nu}{}_{elm}$ such to verify general laws (9.3.2) and (9.3.3). The same result holds for charged particles such as the $\pi^\pm$ with exactly the same laws, only implying a third contribution in the energy-momentum tensor which is therefore tacitly implied hereon.

But a macroscopic body is essentially a large collection of particles and their interactions. Summation over all particles constituents and related fields of a given astrophysical body then yields laws (9.3.2) and (9.3.3), with the only difference that, in the latter case, the tensor $T^{\mu\nu}$ is the superposition of the electromagnetic field of a large number of constituents.

Studies [16] also indicate the possibility of the following identifications of the gravitational and inertial masses

$$M_{gr} = \int dv\, T^{00}{}_{Elm} , \qquad M_{in} = \int dv\, (T^{00}{}_{Elm} + t^{00}{}_{s.r.i.}) , \qquad (9.3.5)$$

A primary result of ref. [16] can therefore be expressed via the following:

Lemma 9.3.1: *By ignoring in first approximation weak and strong interactions, the electromagnetic origin of mass requires the "identification" of the gravitational and electromagnetic field in both the interior and exterior problems.*

In different terms, the studies of ref. [16] permitted the elimination of the vexing problem of "unification" of the gravitational and electromagnetic fields via their "identification" in the ultimate structure of matter, that at the level of elementary constituents. Equivalently, ref. [16] suggested the transition from the "description" of the gravitational field as currently done, to the study of its

"origin' in the fields which originate matter itself.

Rather than being a mere mathematical curiosity, the above results have deep physical implications which can be expressed via the following

Corollary 9.3.1.A: *The "identification" of the electromagnetic and gravitational field implies the existence of antigravity for antimatter in the field of matter or viceversa.*

In fact, the identification implies the necessary equivalence of electromagnetic and gravitational interactions for both attractive and repulsive conditions, exactly as outlined in Sect. 11.8.7. These aspects will be additionally studied in detail in the next section via the isodual Riemannian representation of the gravitational field of antimatter. As we shall see, Corollary 9.3.1.A will be confirmed in its entirety.

Ref. [16] concluded with: A) the proof that, under the validity of Maxwell's electrodynamics, $T^{\mu\nu}_{elm}$ is purely classical and can be identically null only for opposite charges at rest with respect to each other and at null mutual distances; B) the identification of methods for the calculation of *approximate expressions* of the $T^{\mu\nu}_{elm}$ tensor for a very large number of constituents; and C) the proposal of specific experiments for the verification via neutron interferometry and other means, that a sufficiently large electromagnetic field (such as that of currently available very large magnets) does indeed create a gravitational field.

In summary, *the electromagnetic origin of the mass of elementary particles renders Einstein's gravitation directly incompatible with Maxwell's electrodynamics*. In fact:

> Either one accepts Einstein's conception of gravitation, In which case Maxwell's theory must be structurally modified into a form syileding a null electromagnetic field outside neutral hadrons, with consequential restructuring of the contemporary theory of elementary particles to eliminate the primary electromagnetic origin of mass;

> Or one accepts Maxwell's electrodynamics with the consequential, primary, electromagnetic origin of the mass, in which case Einstein's gravitation must be structurally modified to admit a first-order electromagnetic source in the exterior problem in vacuum of bodies with null total charge and electromagnetic moments.

No compromise is on record, to our best knowledge, since the identification of the above incompatibility two decades ago.

9.3.C: Incompatibility of general relativity with the weight of relativistic particles. The results of the preceding sections can be reached in a variety of alternative ways. The first that comes to mind is that identified by Lopez [17].

In essence, Einstein's reduction of gravitation to pure geometry without source (or solely to the Riemannian metric $g^{\mu\nu}$) is in disagreement with

experimental evidence that particles at relativistic speeds have weight as it is the case for all masses irrespective of their speed.

In fact, when passing from the Riemannian spaces $\Re(x,g,R)$ to the tangent Minkowskian space $M(x,\eta,R)$, we have the reduction via *normal coordinates* $g^{\mu\nu}(x) \to \eta^{\mu\nu}$ which is in essence the contemporary formulation of the *equivalence principle* (see, e.g., ref. [8]). But Einstein's field equations (9.3.1) have no source. Therefore, for the considered astrophysical bodies will null total elm phenomenology, *all gravitational effects "disappear" at the limit of the general relativity to the local Minkowski space under normal coordinates.* This evidently implies the inability for relativistic particles to have a weight in the gravitational field of Earth which is contrary to experimental evidence, e.g., as available in the contemporary high energy accelerators.

The conventional literature is usually silent in the above occurrence (although without published counter-arguments to our knowledge), and all limits of gravitation to a flat geometry are restricted to the *Euclidean* rather than Minkowskian space (this si the post-Newtonian approximation indicated earlier, see [8]).

At a deeper inspection, Lopez's argument [loc. cit.] has serious conceptual, theoretical and experimental implications which must be addressed. An evidently necessary condition to resolve Lopez's argument is that a correct gravitational theory must have a first-order gravitational source $T^{\mu\nu}$ which is nowhere null and which, as such, persists in the tangent Minkowskian space. In this case, the gravitational field is represented by the curved metric $g^{\mu\nu}(x)$ *as well as by the source* $T^{\mu\nu}$. Under normal coordinates $g^{\mu\nu}(x)$ would always be reduced to the Minkowski metric $\eta^{\mu\nu}$, but the source $T^{\mu\nu}$ would persist, thus having the pre-requisite for the preservation of weight at the Minkowskian limit.

It follows that Lopez' argument [loc. cit.] is intimately connected with the problematic aspect of the preceding subsection. In fact, both arguments independently lead to the same conclusion, the need for a nowhere-null first-order source in the r.h.s. of the field equations in vacuum. Its identification with the electromagnetic field follows from the elm origin of mass and so does the prediction of antigravity.

Moreover, the most direct resolution of the problem is to require a *gravitational theory not only with a nowhere-null source, but which can be equivalently written in both the Riemannian and the Minkowskian spaces.* In this case we would have a dual treatment of gravitation, first with flat relativistic spaces and then with curvature.

By no mean Lopez' argument is new. In fact, criticisms on the equivalence principle can be traced back to its inception. Among them we quote the studies by Logunov and his school [(see ref.s 18-20] and quoted literature) which are based precisely on the condition of admitting equivalent gravitational theories in the Minkowski and Riemannian spaces. In fact, *Logunov's relativistic field equations* can be written (see, e.g., ref. [18], p. 423)

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = (8\pi / \sqrt{-g})(T^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} T + t^{\mu\nu}), \qquad (9.3.6a)$$

$$T^{\mu\nu}{}_{|\mu} = 0, \qquad t^{\mu\nu} = (\mu^2 / k)(1 + \delta^{\mu}{}_{\alpha}\, \delta^{\nu}{}_{\beta}\, \eta^{\alpha\beta}), \qquad (9.3.6b)$$

where $T^{\mu\nu}$ is the nowhere–null source discussed above with divergenceless scalar T and η is the conventional Minkowski metric.

As one can see, Logunov's theory belongs to the class of *two metrics theories* with numerous examples (see, e.g., [8]). The most important difference between Logunov's theory and the other is that one of the two metrics is Minkowskian in the former, rather than being both Riemannian as in the latter, thus permitting the dual formulation in Minkowski and Riemannian spaces. Another basic difference is given by the rather unique structure of the field equations which will be discussed later on in this section.

9.3.D: Incompatibility of general relativity with antimatter. As is well known, antimatter was discovered in the *negative–energy solutions of field equations.* It follows that a *necessary condition to achieve consistency between gravitation and relativistic field theory, the gravitational equations in vacuum must admit negative–energy solutions.*

It is evident that Einstein's gravitation fails to verify this additional, rather fundamental requirement. An effective way to verify the above compatibility condition is, again, the joint formulation of gravitation in both Riemannian and Minkowskian spaces. The emergence of negative–energy solutions is then ensured because they are inherent in *all* relativistic field equations.

By no means this additional problematic aspect is of marginal relevance. In fact, *the inability to include Einstein's gravitation in the unified gauge theories of weak and electromagnetic interactions may originate precisely in the structural disparities on the description of antiparticles.* In fact, unified gauge theories are *bona fide* relativistic field theories, thus possessing negative–energy solutions, while Einstein gravitation is not.

But there is more. *The current representation of the entire Universe with Einstein's gravitation prohibits astrophysical studies on the possible existence of stars, galaxies and quasars made up of antimatter.* After all, light is identically emitted from both matter and antimatter. Current Einsteinian theories are structurally unable to confront this problem in the needed effective and quantitative way.

As we shall see, one of the challenges posed by isotopic geometries to astrophysics is precisely that of ascertaining whether some of the stars, galaxies and quasars we see in the Universe are indeed made up of antimatter.

But there is still more. *The description of matter and antimatter with Einstein's gravitation prohibits quantitative studies on antigravity.* In fact, as shown earlier in this volume, the latter studies are fundamentally dependent on a consistent representation of antimatter.

At a deeper analysis, it emerges that the condition for a consistent gravitation theory of admitting negative-energy solutions is sufficient, alone, to require a structural revision of Einstein's gravitation.

In the final analysis, we should not forget that antimatter did not exist at the time of the inception of gravitation [3].

9.3.E: Additional problematic aspects of Einstein's gravitation studied by Yilmaz and others. By no means the problematic aspects of general relativity outlined above exhaust all those existing in the literature.

Numerous additional problematic aspects have been identified by Yilmaz (see ref.s [21–25] and large number of references quoted therein). To avoid a prohibitive length we here indicate some of them without a detailed treatment:

1) Einstein's gravitation is incompatible with the Newtonian description of planetary motion [loc. cit.]. The post Newtonian approximation of Einstein's gravitation is not Newtonian mechanics, but rather the so-called *Hooke's mechanics.* The main difference is that the former admits a conventional Keplerian center for a planetary system, while the latter admits a non-Keplerian center with infinite inertia, thus violating the principle of action and reaction. This study establishes the lack of uniqueness of the post Newtonian approximation.

2) Einstein's gravitation is incompatible with the special relativity [loc. cit.]. The relativistic limit of the gravitational conservation laws of the energy-momentum yields only the conservation of the *rest* energy and not that of the *total* relativistic energy. Also, the relativistic limit of electromagnetic radiation in Einstein's theory is incompatible with experimental evidence and its description by the special relativity. These results evidently complement the lack of weight for relativistic particle indicated earlier.

3) Einstein's gravitation is incompatible with the experimental tests in gravitation [loc. cit.]. According to the studies here considered, *Einstein's gravitation does represent the 43" correction in the advancement of the perihelion of Mercury, but not the basic Newtonian term 532".* This is due to the inability of gravitation to recover Newtonian mechanics (lack of uniqueness of the post-Newtonian approximation). Moreover, *Einstein's gravitation is unable to provide a consistent explanation of the measured bending of light in a gravitational field.* This is due to the inability of Einstein's theory to achieve the identity of the active and passive mass, thus implying infinite mass for the source, and the consequential inability to represent the bending of light in a fully consistent way.

4) Einstein's gravitation is unable to represent many-body systems [loc. cit.]. As clearly indicated in his limpid writings [3], Einstein conceived his gravitation for the description of *one* test body in the gravitational field of an external mass. As such, *Einstein gravitation is a one-body theory by conception.* Despite this well known historical point, Einstein's theory has been often used for many-body systems as requested by the physical reality, say, of a test body in a

binary system. Investigations done by Yilmaz [loc. cit.] have however shown that the "many–body solutions" of Einstein's equations are not technically sound and they are certainly unsettled on scientific grounds.

5) Einstein gravitation is incompatible with quantum mechanics [loc. cit.]. The notorious problematic aspect in the quantization of Einstein's gravitation is the null value of its Hamiltonian which is at variance with the quantum mechanical need of a non–null Hamiltonian. However, what is less known is that the operator formulation of a gravitational theory which is incompatible with Newtonian and relativistic theories is bound to have a host of hidden problematic aspects of *structural* character.

For all the above and otheraspects we refer the interested reader to the original literature [21–25] and quoted papers. The point important for this analysis is that *the general origin of all the problematic aspects identified by Yilmaz is the lack of source in vacuum, i.e., the assumption* $G_{\mu\nu} = 0$.

Yilmaz [loc. cit.] has finally shown that *all* problematic aspects 1), 2), 3), 4), 5) above are resolved by the following generalization of Einstein's equations in vacuum

$$G^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = k (\tau^{\mu\nu} + t^{\mu\nu}), \tag{9.3.7}$$

where $t^{\mu\nu}$ is *Yilmaz' stress–energy tensor* and $\tau^{\mu\nu}$ is a source tensor. In fact, Yilmaz has shown that the addition of the above source implies a *non–null Hamiltonian* with consequential regaining of the compatibility with Galilean, relativistic and quantum description.

Even though expressed with the emphasis on the stress–energy tensor, all arguments by Yilmaz [loc. cit.] agree with the preceding physical arguments on the need of a nowhere–null source in vacuum.

It should be finally indicated that Yilmaz arguments have been published and propagated for decades, although they have remained essentially ignored in the specialized literature (the comments by Weiss [30] on issues of scientific ethics raised by this occurrence should be inspected).

To avoid a prohibitive length, we are unable to report additional problematic aspects of Einstein's gravitation studied by several other authors.

9.3.F: Problematic aspects of general relativity from the Freud identity. We now pass to a study of the problematic aspects of Einstein's gravitation of *geometric character.*

P. Freud [6] proposed an identity of the Riemannian geometry back in 1939 which was studied in detail by Pauli [7], but thereafter ignored for several decades by virtually the entire literature in gravitation.

The Freud identity was rediscovered by Yilmaz [25] who brought it to the attention of this author. In turn, this author brought the Freud identity to the attention of the late H. Rund who studied it in detail in one of his last papers [26].

This author also constructed the axiom-preserving isotopies of the Freud identity in ref. [27], studied them in details in ref.s [28,29] and reported the results in Sect. 1.5.6.

These studies have essentially established that *the Freud identity requires the presence in general of a nowhere-null source in vacuum which is contrary to Einstein's conception of gravitation, Eq.s (9.3.1), and fully aligned with all physical arguments of Sect.s 9.3.B–9.3.E.*

The *Freud identity* in its original form [6] (see also Pauli [7] and Rund [26]) can be written

$$\tilde{G}_\mu{}^\nu = \tfrac{1}{2}\sqrt{-g}\,\frac{\partial\,\tilde{G}}{\partial\,g^{\alpha\beta}{}_{,\nu}}\,g^{\alpha\beta}{}_{,\mu} - \tfrac{1}{2}\sqrt{-g}\,\tilde{G}\,\delta_\mu{}^\nu - \frac{\partial\,\tilde{V}^{\nu\rho}{}_\mu}{\partial x^\rho}, \qquad (9.3.8)$$

where

$$\tilde{G}_\mu{}^\nu = \sqrt{-g}\,G_\mu{}^\nu, \qquad\qquad \tilde{V}^{\nu\rho}{}_\mu = \sqrt{-g}\,V^{\nu\rho}{}_\mu, \qquad (9.3.9a)$$

$$\tilde{G} = g^{\mu\nu}(\Gamma_\mu{}^\rho{}_\sigma\,\Gamma_\nu{}^\sigma{}_\rho - \Gamma_\mu{}^\rho{}_\nu\,\Gamma_\rho{}^\tau{}_\tau), \qquad (9.3.9b)$$

$$2\,V_\mu{}^{\rho\sigma} = g^{\alpha\beta}(\,\delta_\mu{}^\rho\,\Gamma_\alpha{}^\sigma{}_\beta - \delta_\mu{}^\sigma\,\Gamma_\alpha{}^\rho{}_\beta\,) + (\,\delta_\mu{}^\sigma\,g^{\rho\tau} - \delta_\mu{}^\rho\,g^{\sigma\tau}\,)\,\Gamma_\tau{}^\gamma{}_\gamma +$$

$$+ g^{\sigma\alpha}\,\Gamma_\mu{}^\rho{}_\alpha - g^{\mu\alpha}\,\Gamma_\mu{}^\sigma{}_\alpha, \qquad (9.3.9c)$$

$G_\mu{}^\nu$ is Einstein's tensor and the Γ's are the familiar Christoffel's symbols.

Lemma 9.3.1 (Rund [loc. cit.]): *The Freud identity (9.3.8) is valid for all possible Riemannian spaces with nonsingular and symmetric metric, irrespective of dimension and signature.*

In different terms, Eq.s (9.3.8) are an intrinsic property of the Riemannian geometry, as it is the case for all other identities. The identity must therefore be verified by all consistent theories.

Note also the full agreement of the Freud identity with the need for a source in vacuum studied in the preceding subsections. Note also that the Freud identity holds for all possible Riemannian spaces. As such, it must be verified also for bodies with null total charge and null total electric-magnetic moments.

The lack of general verification of identity (9.3.8) by Einstein's equations (9.3.1) is evident because of the non-null value of the r.h.s. of the former for the exterior problem in vacuum and the null value of the r.h.s. in the latter.

In essence, Einstein's equations are the particular case in which the r.h.s. of the Freud identity is null. As such, no geometric incompatibility can be claimed and the solutions of Eq.s (9.3.1), such as the Schwartzchild's exterior solution [4] do indeed satisfy the Freud identity, because this is the case for all possible Riemannian metrics [26]. The point is that this is a highly restricted class, and definitely not the general class of theories of gravitation admitted by the Riemannian geometry.

Yilmaz [25] recalled that the quantities

$$U_\mu{}^\nu = \tfrac{1}{2} \left(\frac{\partial \tilde{G}}{\partial g^{\alpha\beta}{}_{,\nu}} \ g^{\alpha\beta}{}_{,\mu} - \tilde{G}\,\delta_\mu{}^\nu \right), \quad u_\mu{}^\nu = (1/\sqrt{-g})\, \frac{\partial \tilde{V}_\mu{}^{\nu\rho}}{\partial x^\rho}, \quad (9.3.10)$$

are *pseudotensors*, as known since Pauli [7]. Each of them contains a pseudo part $z_\mu{}^\nu$, as a result of which we have the reformulation in terms of true tensors $W_\mu{}^\nu$ and $w_\mu{}^\nu$,

$$U_\mu{}^\nu - u_\mu{}^\nu = = (U_\mu{}^\nu - z_\mu{}^\nu) + (- u_\mu{}^\nu + z_\mu{}^\nu) = \tau_\mu{}^\nu + t_\mu{}^\nu, \quad (9.3.11)$$

Yilmaz's important result [loc. ci.] is therefore that a theory of exterior gravitation in vacuum, to be compatible with the Freud identity, must possess a nowhere null source in vacuum of the type

$$G^{\mu\nu} = (R^{\mu\nu} - \tfrac{1}{2} R\, g^{\mu\nu}) = \tau_\mu{}^\nu - t_\mu{}^\nu, \quad (9.3.12)$$

where $\tau^{\mu\nu}$ and $t^{\mu\nu}$ are now *bona fide tensors* characterized by Eq.s (9.3.11).

Equations (9.3.12) derived from the Freud identity must be compared with Eq.s (9.3.4) derived from the electromagnetic origin of the gravitational field. It is evident that the tensor $\tau^{\mu\nu}$, as derived above, does not verify in general the electromagnetic property $\tau^{\mu\nu}{}_{|\nu} = 0$. However, one can add and subtract in the r.h.s. of Eq.s (9.3.12) a tensor $\tilde{\tau}^{\mu\nu}$ such that $(\tau^{\mu\nu} + \hat{\tau}^{\mu\nu})_{|\nu} = 0$, and we can write

$$\tau^{\mu\nu} + t^{\mu\nu} \equiv (\tau^{\mu\nu} + \hat{\tau}^{\mu\nu}) + (t^{\mu\nu} - \hat{\tau}^{\mu\nu}) \equiv g^{\mu\nu} + \mathcal{G}^{\mu\nu}, \quad (9.3.13a)$$

$$g^{\mu\nu} = \tau^{\mu\nu} + \hat{\tau}^{\mu\nu}, \quad g^{\mu\nu}{}_{|\nu} \equiv 0, \quad \mathcal{G}^{\mu\nu} = t^{\mu\nu} - \hat{\tau}^{\mu\nu}. \quad (9.3.13b)$$

We see in this way that *the Freud identity is fully compatible with the primary electromagnetic origin of mass.* We also see the *necessity for a non-null Yilmaz's stress-energy tensor* $t^{\mu\nu}$ *in vacuum.* In fact, Yilmaz's tensor necessarily emerges to turn the tensor $W^{\mu\nu}$ with $W^{\mu\nu}{}_{|\nu} \neq 0$ of the Freud identity into the form $(W^{\mu\nu} + \tilde{W}^{\mu\nu})_{|\nu} \equiv 0$ needed for compatibility with the electromagnetic origin of mass.

9.3.G: Geometric incompleteness of Einstein's tensor. In inspecting the preceding sections, the reader may have the impression that Eq.s (9.3.8) or (9.3.12) resolve all the problematic aspects of Einstein's exterior gravitation. This is not the case because of the *geometric incompleteness* of Einstein's tensor $G^{\mu\nu}$ identified by this author in 1988 [27], studied in details in ref.s [28,29] and outlined in Sect. 1.5.6. It is important for clarity to review the main argument here.

Throughout the studies of this and of the preceding volume we have illustrated the axiom-preserving character of the isotopies. Such character is so strong that a given property is not a true geometric axiom unless it is preserved under isotopies. The occurrence is confirmed by the null value of the covariant

divergence of the metric tensor, $g^{\alpha\beta}(x)|_\beta \equiv 0$, which is preserved under isotopies despite the arbitrary functional dependence of the isometric, $\hat{g}^{\alpha\beta}(x, \dot{x}, \ddot{x}, \mu, \tau, ...)|_\beta \equiv 0$ (see Sect. 1.5.6). The same axiom–preserving character is verified by all other properties of the Riemannian geometry, except *Einstein's tensor* $G^{\mu\nu}$.

In fact, we have established in Lemmas I.5.6.2 and I.5.6.3 that *Einstein's tensor* $G^{\mu\nu} = R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$ *is geometrically incomplete because it does not preserve under isotopy the vanishing character of its covariant divergence (contracted Bianchi identity).* Specifically, the following property in $\Re(x,g,R)$

$$G^{\mu\nu}|_\mu = (R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu} R)|_\nu \equiv 0, \tag{9.3.14}$$

is not preserved by its isotopic image $\hat{G}^{\mu\nu}$ in isoriemannian spaces $\Re(x,\hat{g},\hat{R})$,

$$\hat{G}^{\mu\nu}{}_{|\mu} = (\hat{R}^{\mu\nu} - \tfrac{1}{2}\hat{g}^{\mu\nu} \hat{R})_{|\nu} \neq 0. \tag{9.3.15}$$

where $|\nu$ is the isocovariant derivative (1.5.5.8).
However, if Einstein's tensor $G^{\mu\nu}$ is "completed" in the form

$$S^{\mu\nu} = G^{\mu\nu} + \tfrac{1}{2} g^{\mu\nu}\Theta, \tag{9.3.16a}$$

$$\Theta = g^{\mu\nu} (\Gamma_\mu{}^\rho{}_\sigma \Gamma_\nu{}^\sigma{}_\rho - \Gamma_\mu{}^\rho{}_\nu \Gamma_\rho{}^\tau{}_\tau) \equiv \Gamma_{\alpha\beta\gamma} \Gamma_\mu{}^\beta{}_\nu (g^{\beta\nu} g^{\mu\gamma} - g^{\beta\gamma} g^{\mu\nu}), \tag{9.3.16b}$$

then it preserves under isotopies the null value of the covariant divergence,

$$S^{\mu\nu}|_\mu = (G^{\mu\nu} +\tfrac{1}{2}g^{\mu\nu}\Theta)|_\nu \equiv 0 \quad \to \quad \hat{S}^{\mu\nu}{}_{|\mu} = (\hat{G}^{\mu\nu} + \tfrac{1}{2}\hat{g}^{\mu\nu}\hat{\Theta})_{|\nu} \equiv 0, \tag{9.3.17}$$

What is rather remarkable is that the expression Θ emerging from the exterior limit of the isoriemannian quantity $\hat{\Theta}$, Eq.s (1.5.6.18), *coincides* with the quantity $\hat{G}$ emerging in the Freud identity, Eq.s (9.3.9b).

9.3.H: Proposed general class of gravitational equations in vacuum from the Freud identity. By combining all the preceding studies, we can now proposed the following *general class of field equations in vacuum* as requested by "all" Riemannian identities, including the Freud identity, and the electromagnetic origin of mass:

$$S^{\mu\nu} = R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu} R + \tfrac{1}{2}g^{\mu\nu}\Theta = k (S^{\mu\nu}{}_{elm} + \mathcal{F}^{\mu\nu}), \tag{9.3.18a}$$

$$S^{\mu\nu}{}_{elm}{}_{|\mu} = 0, \tag{9.3.18b}$$

where

$$\varepsilon_\mu{}^\nu{}_{elm} + \mathcal{F}_\mu{}^\nu = \tfrac{1}{2}(1 / k) (\frac{\partial \Theta}{\partial g^{\alpha\beta}{}_{,\nu}}g^{\alpha\beta}{}_{,\mu}) - (1 / k\sqrt{-g}) \frac{\partial V^{\nu\rho}{}_\mu}{\partial x^\rho}, \tag{9.3.19a}$$

$$M = \int dv\,(\,g^{00}{}_{elm} + \mathfrak{F}^{00}\,), \qquad \Theta = g^{\mu\nu}(\,\Gamma_{\mu}{}^{\rho}{}_{\sigma}\,\Gamma_{\nu}{}^{\sigma}{}_{\rho} - \Gamma_{\mu}{}^{\rho}{}_{\nu}\,\Gamma_{\rho}{}^{\tau}{}_{\tau}\,), \qquad (9.3.19b)$$

$$2\,V_{\mu}{}^{\rho\sigma} = g^{\alpha\beta}(\,\delta_{\mu}{}^{\rho}\,\Gamma_{\alpha}{}^{\sigma}{}_{\beta} - \delta_{\mu}{}^{\sigma}\,\Gamma_{\alpha}{}^{\rho}{}_{\beta}\,) + (\,\delta_{\mu}{}^{\sigma}\,g^{\rho\tau} - \delta_{\mu}{}^{\rho}\,g^{\sigma\tau}\,)\,\Gamma_{\tau}{}^{\gamma}{}_{\gamma} +$$

$$+ g^{\sigma\alpha}\,\Gamma_{\mu}{}^{\rho}{}_{\alpha} - g^{\rho\alpha}\,\Gamma_{\mu}{}^{\sigma}{}_{\alpha}, \qquad \Theta \equiv \tilde{G}. \qquad (9.3.19c)$$

The additional condition of admitting an equivalent formulation in Minkowski space should be kept in mind (see Sect. II.9.5).

As one can see, Einstein's gravitation is structurally incompatible with the above equations because of they lack of the isoscalar term in the l.h.s. and the two tensors in the r.h.s.

On the contrary, Logunov's relativistic gravitation [18–20] does indeed belong to the above class. In fact, the basic equations can be written, e.g., in form (9.3.6) which have structure (9.3.18) under: 1) the identification $T \equiv \Theta$, 2) a suitable realization of the $V\mu\nu_{\rho}$ tensor to yield the tensor $t^{\mu\nu}$ of Eq.s (9.3.6b), and c) the electromagnetic realization of the tensor $T^{\mu\nu}$.

Note that Eq.s (9.3.18) are reducible to Yilmaz' form (9.3.12). Therefore, all Yilmaz' arguments on the resolution of the problematic aspects of Einstein's equations also apply to Eq.s (9.3.18). The latter are however different than Yilmaz' equations (9.3.12) because of the electromagnetic origin of the tensor $g^{\mu\nu}{}_{elm}$ as well as the geometric completion of the tensor $G^{\mu\nu}$ into $S^{\mu\nu}$.

9.3.I: Reduction of the general gravitation theory to a relativistic field theory.
We now study the reduction of the proposed general theory (9.3.18) to a conventional field theory in Minkowski space (the reduction to an isominkowski space is studied in Sect. II.9.5).

This problem was studied by Yilmaz [25]. He notes that the contracted Bianchi identity in Eq.s (9.3.12) and the expression

$$\partial\,(\sqrt{-g}\;U^{\nu}{}_{\mu}) = 0, \qquad (9.3.20)$$

imply the property

$$\tfrac{1}{2}\,\partial_{\nu}\,g_{\alpha\beta}\,\tau^{\alpha\beta} = t^{\nu}{}_{\mu\,|\,\nu}. \qquad (9.3.21)$$

Since the covariant divergence of a stress–energy tensor is the volume force $\tau du/ds$, the preceding equation reduces to the *field equations*

$$\tau\,du_{\mu}\,/\,ds = \tfrac{1}{2}\,\partial_{\mu}\,(\,g_{\alpha\beta}\,\tau^{\alpha\beta}\,), \qquad (9.3.22)$$

which achieves the desired reduction.

For our needs we note that the above expression can be subjected to a double formulation in Minkowskian and Riemannian spaces. In fact, by factorizing the Riemannian metric in the form $g_{\alpha\beta} = T_{\alpha}{}^{\rho}\eta_{\rho\beta}$, $T_{\alpha}{}^{\rho} = T^{\alpha}{}_{\rho}$ where

$\eta_{\alpha\rho}$ is the Minkowskian metric, Eq.s (9.3.22) can be entirely expressed in Minkowski space

$$\tau \, du_\mu \, / \, ds \; = \; \tfrac{1}{2} \, \partial_\mu \, (\eta_{\alpha\beta} \, Y^{\alpha\beta}) \, , \qquad\qquad Y^{\alpha\beta} \; = \; T^\alpha{}_\rho \, W^{\rho\beta} \, . \qquad (9.3.23)$$

Unfortunately, the above reduction depends on property (9.3.20) which is incompatible with the Freud identity itself, Eq.s (9.3.8) because it assumes the equations without source $G_{\mu\nu} = 0$, while for the Freud identity $G_{\mu\nu} \neq 0$.

To prove this point, consider the expression derived by Rund [26]

$$\partial_\nu \, U_\mu{}^\nu \; = \; E_{\alpha\beta}[\hat{L}] \, \partial_\nu \, g^{\alpha\beta} \, , \qquad \hat{L} = \sqrt{-g} \, \tilde{G}, \qquad (9.3.24)$$

where $E_{\alpha\beta}(\hat{L})$ represents the familiar Euler–Lagrange equations in the density $\hat{L}$,

$$E_{\alpha\beta}(\hat{L}) \; = \; \partial_\rho \, \frac{\partial \hat{L}}{\partial_\rho \, g^{\alpha\beta}} \; - \; \frac{\partial \hat{L}}{\partial \, g^{\alpha\beta}} \, . \qquad (9.3.25)$$

On the other side, the conventional Lagrangian $L = \sqrt{-g}R$ admits the decomposition $L = \hat{L} + D$, where D is a pure divergence. Since $E_{\alpha\beta}(D) \equiv 0$ for any divergence, we have

$$E_{\alpha\beta}(\hat{L}) \; = \; E_{\alpha\beta}(\,L + D\,) \equiv E_{\alpha b}(L) \, . \qquad (9.3.26)$$

Eq.s (9.3.24) can then be equivalently written [26]

$$\partial_\nu \, U_\mu{}^\nu \; = \; E_{\alpha\beta}(\hat{L}) \, \partial_\mu \, g^{\alpha\beta} \; = \; E_{\alpha\beta}(L) \, \partial_\mu g^{\alpha\beta} \; = G_{\alpha\beta} \, \partial_\mu \, g^{\alpha\beta} \, . \qquad (9.2.27)$$

As a result, the *expression* $\partial_\nu \, W_\mu{}^\nu = 0$ *is derived under the "assumption" of Einstein's equations in vacuum* $G_{\alpha\beta} = 0$. The problematic aspects of the latter, and their need to have a nowhere–null source in the r.h.s. then imply the following correct form

$$\partial_\nu \, W_\mu{}^\nu \; = \; (\, k \, / \, \sqrt{-g} \,)(\, \tau_{\alpha\beta} \, + \, t_{\alpha\beta} \,) \, \partial_\mu \, g^{\alpha\beta} \, , \qquad (9.3.28)$$

under which field equations (9.3.22) and (9.3.23) are no longer valid.

Nevertheless, the same methods can be used for the same reduction under the correct expression (9.3.28). This task is left as an exercise for the interested reader.

It is unfortunate that Yilmaz never considered in his studies the electromagnetic origin of the $\tau^{\mu\nu}$ tensor [16], the geometric incompleteness of Einstein's tensor [27], and Logunov's conception of gravitation as admitting a dual representation in Riemannian and Minkowskian spaces [18–20].

We should finally mention that, contrary to his followers, Albert Einstein was fully aware of the lack of final character of his gravitation. In fact, he

compared the l.h.s. of his gravitational equations to the left wing of a house made of "fine marble" and the r.h.s. to the right wing of a house made of "bare wood".

9.4: ISOGRAVITATION, GENOGRAVITATION AND THEIR ISODUALS

In this section we study the interior gravitational models submitted by this author [27–29] which: 1) are based on the isotopies, genotopies and isodualities of the Riemannian geometry in (3+1)–dimension; 2) are arbitrarily nonlinear, nonlocal–integral and nonlagrangian when projected in our space–time, although isolinear, isolocal and isolagrangian when considered in their appropriate isospace; and 3) are divided into *isogravitation* for the characterization of the interior astrophysical bodies composed of matter when considered closed-isolated from the rest of the Universe, *genogravitation* for interior gravitational problem of matter in open–irreversible conditions and *isodual isogravitation* or *isodual genogravitation* for the corresponding problems of antimatter.

9.4.A: The isoriemannian geometry and its direct universality. Let $\Re(x,g,R)$ be a conventional (3+1)–dimensional Riemannian space with nonsingular, real–valued and symmetric metric g(x) over the reals $R(x,+,\times)$. The isotopies of Class 1 permit the construction of the so–called *isoriemannian spaces* (Sect. I.5.6.B)

$$\hat{\Re}(x,\hat{g},\hat{R}): \quad \hat{g} = T(x, \dot{x}, \ddot{x}, \mu, \tau, \omega, ...) g(x), \qquad \hat{1} = T^{-1} > 0, \qquad (9.4.1)$$

which evidently contain conventional spaces as a particular case when the isounit $\hat{1}$ recovers the conventional unit $I = $ diag. (1, 1, 1, 1).

Since we have assumed the Riemannian geometry to be *exact* in vacuum, *all isounits* $\hat{1}$ *of the isoriemannian spaces are restricted hereon by the condition of recovering the conventional unit* I *in vacuum*, which can be realized, e.g., under the condition of null density

$$\hat{1}_{\mu=0} = 1, \qquad \hat{g}_{\mu=0} = g, \qquad \hat{\Re}(x,\hat{g},\hat{R})_{\mu=0} = \Re(x,g,R). \qquad (9.4.2)$$

This condition evidently guarantees *ab initio* that all interior isotopic models recover conventionally *Riemannian* (but not necessarily Einsteinian) models in the interior problem in vacuum.

The infinitely possible values of the isounits $\hat{1}$ deserve a comment. Recall that the Riemannian geometry applies for all infinitely possible values of the mass M. Along much similar lines, the covering isoriemannian geometry applies for all infinitely possible isounits $\hat{1}$ *for each given value of the mass* M. This is due to the fact that the isounit geometrizes the interior of astrophysical masses. Each of them can be realized in nature in an infinite number of different sizes,

densities, temperatures, chemical compositions, etc., thus calling for different numerical values of their geometrization.

To put it differently, the assumption of only one isounit $\hat{1}$ would imply the existence of only one speed of light within physical media which is contrary to physical evidence. Similarly, the assumption of the same isounit $\hat{1}$ for all astrophysical bodies with the same mass M would imply that the speed of light within physical media varies with the mass, which is also contrary to experimental evidence.

Also recall that general relativity cannot possibly predict the numerical value of the mass on purely theoretical grounds, which mass must be derived from experimental measures of weight, curvature, and the like. Along exactly the same lines, no theory can possibly predict the numerical value of its own unit, which must be derived from experimental measures, this time, on density, temperature, etc. However, the availability of sufficient experimental data on interior conditions does indeed permit the identification of the unique numerical value of the isounit for each astrophysical body, as we shall see in Vol. III.

The lifting $\Re(x,g,R) \to \hat{\Re}(x,\hat{g},\hat{R})$ implies a step-by-step isotopic generalization of the Riemannian geometry called *isoriemannian geometry,* first submitted by this author in [27], studied in detail in [27,28] and outlined in Sect. I.5.6.

A primary function of the isoriemannian geometry is the direct representation via the isometric $\hat{g}$ of the locally varying speed of light in interior conditions, as well as the inhomogeneity and anisotropy of interior media this time in a curved space.

This can be achieved via the diagonal forms of the metrics and isometrics

$$g = \text{diag.} (g_{11}, g_{22}, g_{33}, g_{44}), \quad T = \text{diag.} (n_1^{-2}, n_2^{-2}, n_3^{-2}, n_4^{-2}) > 0, \quad n_\mu > 0,$$

$$\hat{g} = (g_{11} n_1^{-2}, g_{22} n_2^{-2}, g_{33} n_3^{-2}, g_{44} n_4^{-2}), \tag{9.4.3}$$

where the $n_\mu(x, \dot{x}, \ddot{x}, \mu, \tau, ...)$ are called the *gravitational characteristic functions of the medium considered.* In most applications of Vol. III they are averaged into the *gravitational characteristic constants* $n^\circ_\mu = \text{Aver.} [n_\mu(x, \dot{x}, ...)] = \text{const.}$

It is then evident that the term $\hat{g}_{44} = g_{44}/n_4^2$ permits a direct representation of the local variation of the speed of light in interior conditions while the expression $\hat{g}_{44} = g_{44}/n^\circ_4$ characterizes the average speed of light throughout the medium considered. Simple invariance arguments then require the evident extension of the n_4-quantity to the remaining space components.

Rules (9.4.3) apply to all conventional Riemannian metrics, and therefore provide a simple method for the explicit construction of interior models from given exterior ones. As an example, the *isoschwartzschild's line element* can be written

$$ds^2 = n_s^{-2} [(1 - 2M/r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] - n_4^{-2}(1 - 2M/r) dt^2$$

$$\tag{9.4.4}$$

with the understanding that a more rigorous formulation is that with isodifferentials (Sect. II.8.4.A) in terms of the isospherical angles (Sect. II.5.5). Note finally that in practical applications, the characteristic quantities n_μ are averaged to constants n°_μ because measures of interior astrophysical effects are evidently done from the outside, thus requiring their average over the entire medium.

Representations (9.4.3) are manifestly compatible with the locally *isominkowskian* structure of Ch. II.8. In fact, under *normal coordinates* $\bar{x}(x)$ for which the Riemannian (but not the isoriemannian) metric $g(x)$ reduces to the Minkowski metric η, the isoriemannian metric $\hat{g}$ recovers the isominkowskian metric $\hat{\eta} = T\eta$ identically.

This assures *ab initio* that isogravitation admits the isospecial relativity of the preceding chapter in the local tangent isominkowskian space.

The *equivalence* of the Riemannian and isoriemannian geometries has been studied in Vol. 1. In particular, the two geometries coincide at the abstract, realization-free level by construction according to a mechanism now familiar, i.e., jointly with a given deformation of the curvature, the unit is deformed of the inverse amount thus preserving the original curvature.

The physical differences between the Riemannian and the isoriemannian geometries have also been discussed in Vol. I with the understanding that they appear only when the latter is projected in the space of the former, i.e.., the equations are written in $\Re(x,g,R)$, rather than $\Re(x,\hat{g},\hat{R})$.

Their most important physical difference is that, while the Riemannian geometry necessarily restricts the dependence of the metric to the local coordinates only, $g = g(x)$, the isoriemannian covering leaves such functional dependence completely unrestricted, $\hat{g} = \hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau, \omega, ...)$. This is precisely the reason which renders the isoriemannian geometry particularly suited for quantitative treatment of interior nonlinear–nonlocal–nonpotential effects which are simply beyond any hope of treatment via the Riemannian geometry.

This illustrates *the direct universality of the isoriemannian geometry*, that is, its capability of admitting as particular cases all infinitely possible, signature-preserving generalizations of the Riemannian geometry (universality), directly in the considered x-variables (direct universality).

Stated in a nutshell, the central issue is here: *why work on a Riemannian space* $\Re(x,g,R)$ *with metric* $g(x)$ *when one can work on the covering isoriemannian space* $\Re(x,\hat{g},\hat{R})$ *under the same geometric axioms but with an unrestricted functional dependence of the isometrics* $\hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau, \omega, ...)$?

9.4.B. The fundamental theorem of the interior gravitation for matter. The *isogeneral theory of relativity* or *isogravitation* for short, is the interior theory of gravitation for matter characterized by the isoriemannian geometry.

In order to formulate it, we have to recall the notion of *isolagrangian theory* from Sect. II.I.4. Consider a theory which is of Lagrangian order higher than the first in a conventional metric space, $L = L(g, \partial g, \partial\partial g, ...)$. By embedding all

terms of order higher than the first in the isotopic element, the isotopies essentially permit the reduction of the latter Lagrangian to a first-order one provided that it is written in the appropriate isospace. We shall then write L(g, ∂g, ∂∂g,...) ≡ L(ĝ, ∂ĝ).

A simple example in Euclidean space E(r,δ,R) is given by the representation of drag forces via a multiplicative factor in the kinetic energy for which L(r, ṙ, r̈, ...) = T(r, ṙ, r̈, ...) ½ ṙ² which can be identically rewritten as the first-order isolagrangian L̂(ṙ) = ½ ṙ² = ½ ṙ * ṙ = ½ ṙ T(r, ṙ, r̈, ...) ṙ. The same result holds in a curved space.

By using the basic laws and identities of the isoriemannian geometry as presented in Sect. 1.5.6 and the isocalculus on isomanifolds (Sect. 11.8.4.A), the isogravitation is characterized by the following:[123]

Theorem 9.4.1 (Fundamental theorem of isogravitation for matter [26-28]): *In a (3+1)-dimensional isoriemannian space ℜ(x,ĝ,R) of Class I, the most general possible isolagrange equations Ê^{μν} = 0 verifying the properties: 1) symmetric condition Ê^{μν} = Ê^{νμ}, 2) the null value of the covariant isoderivative Ê^{μν}ᵣ_ν = 0, and 3) the isofreud identity (1.5.6.29) are characterized by the isovariational principle*

$$
\hat{\delta}\hat{A} = \hat{\delta} \int \hat{L}(\hat{g}, \partial\hat{g}, \hat{\epsilon}, \mathfrak{F})\,\partial x = \hat{\delta}\int\hat{\Delta}^{\frac{1}{2}}[\lambda(\hat{R}+\Theta) + 2\Lambda + \rho(\hat{\epsilon}+\mathfrak{F})]\,dx = 0 ,
$$

(9.4.5)

where Δ̂ = det(-ĝ), λ, Λ, and ρ are constants, ε^{μν} is a source tensor representing the electromagnetic origin of mass, 𝔉^{μν} is a stress-energy tensor, R̂ is the curvature isoscalar (1.5.6.17), and Θ is the isoscalar (1.5.6.18). For the case λ = p = 1 and Λ = 0, the isoriemannian geometry characterizes the following fundamental field equations of the interior isogravitation for matter

$$
\hat{R}^{\mu\nu} - \tfrac{1}{2}\hat{g}^{\mu\nu}\hat{R} + \tfrac{1}{2}\hat{g}^{\mu\nu}\Theta = k(\hat{\epsilon}^{\mu\nu}_{elm} + \mathfrak{F}^{\mu\nu}) ,
$$

(9.4.6a)

$$
\hat{\epsilon}^{\mu\nu}_{elm}|_\nu = 0 .
$$

(9.4.6b)

A few comments are now in order. The first important aspect in the above theorem originates from the very structure of the isotopies which imply the appearance of the isoscalar term Θ in the l.h.s. of the field equations. The second important aspect is the appearance of two tensors in the r.h.s. satisfying the following identification

$$
\hat{M}^{\mu\nu}_{matter} = \hat{\epsilon}^{\mu\nu}_{elm} + \mathfrak{F}^{\mu\nu} .
$$

(9.4.7)

---

[123] Theorem 9.4.1 can also be proved via step-by-step isotopies of the exterior theorem in vacuum of Lovelock and Rund [9], p. 321, plus the addition of the Freud Identity.

The conventional mass tensor is therefore completely eliminated and replaced by the fields originating matter itself. In particular, the tensor $\mathfrak{F}^{\mu\nu}$ originates from the short range interactions (Sect. II.9.3B) plus other geometric contributions from the isofreud identity. We therefore have the following:

Corollary 9.4.1.A: *Isogravitation is a theory on the origin of the gravitational field for matter.*

Note that the tensor $\mathcal{E}^{\mu\nu}$ is exactly the electromagnetic tensor of ref. [16]. However, the stress–energy tensor $\mathfrak{F}^{\mu\nu}$ *does not* coincide with that of ref.s [21–25], first of all, because it is formulated in isoriemannian spaces and, secondly, because of the extra term in the isofreud identity which must be added and subtracted to the r.h.i. to achieve the property $\mathcal{E}^{\mu\nu}{}_{|\nu} = 0$.

Corollary 9.4.1.B: *Isogravitation is an interior gravitational theory of matter with null isotorsion on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$ and non–null torsion when projected on $\mathfrak{R}(x,g,R)$.*

Since the isotopies are axiom–preserving the conventional null character of torsion in vacuum is preserved, i.e., we have the null *isotorsion*

$$\tau_h{}^s{}_k = \Gamma^2{}_h{}^s{}_k - \Gamma^2{}_k{}^s{}_h \equiv 0 \quad \rightarrow \quad \hat{\tau}_h{}^s{}_k = \hat{\Gamma}^2{}_h{}^s{}_k - \hat{\Gamma}^2{}_k{}^s{}_h \equiv 0. \qquad (9.4.8)$$

However the projection of the isotorsion in the conventional space $\mathfrak{R}(x,g,R)$ requires the following reinterpretation of the covariant isoderivative

$$X^i{}_{\hat{|}} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_h{}^i{}_k T^h{}_r X^r \equiv X^i{}_{|k} = \frac{\partial X^i}{\partial x^k} + \hat{\Gamma}^2{}_r{}^i{}_k X^r, \qquad (9.4.9a)$$

$$\hat{\Gamma}^2{}_r{}^i{}_k = \hat{\Gamma}^2{}_h{}^i{}_k T^h{}_r \neq \hat{\Gamma}^2{}_k{}^i{}_r, \qquad (9.4.9b)$$

from which one can see that the conventional torsion is not null.

Alternatively we can say that, starting with a symmetric isoconnection $\hat{\Gamma}_h{}^i{}_k$ on $\hat{\mathfrak{R}}(x,\hat{g},\hat{R})$, its projection $\hat{\Gamma}_r{}^i{}_k$ on $\mathfrak{R}(x,g,R)$ is no longer necessarily symmetric. The above property was first reached by Gasperini [30] via *conventional* differential forms on a *conventional* Riemannian space.[124] The geometrization of the property into a symmetric isotorsion was achieved by the author in ref. [27].

The physical significance of Corollary 9.4.1.B is the following. A null torsion is important in the exterior problem to represent the *stability* of the orbits, while, if preserved in interior problems, the same null torsion implies excessive approximations, such as the acceptance of the perpetual motion within a physical environment.[125]

---

[124] A review of Gasperini's studies can be found in ref. [29].

Note that the above property stresses the need for *two different, but compatible theories: one for the exterior gravitational problem with null torsion, and one for the interior gravitational problem with null isotorsion but non-null torsion.* A central feature of the isotopies of Riemann is that of permitting the representation of both exterior and interior gravitational problems via one single set of geometric and physical axioms.

Corollary 9.4.1.C: *The limit of isogravitation (9.4.5) to a conventional Riemannian space $\Re(x,g,R)$ yields exterior field equations (II.9.3.18) where the sources $\mathcal{E}^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ are nowhere null even for a body with null total charge, electric and magnetic moments.*

We therefore reach the conclusion that the isogravitation yields as exterior limit in vacuum *identical* to the field equations derived via the conventional Riemannian properties including the Freud identity. Einstein's gravitation can be obtained only as a particular case by putting to zero the sources $\mathcal{E}^{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ and the isoscalar Θ. This would however raise all the problematic aspects of Sect. II.9.3, prevent a theory on the origin of the gravitational field, prohibit quantitative studies of antimatter, and have all the other shortcomings identified in the literature.

Note that isogravitation can directly represent via its isometric:

1) The radial variation of the density $\mu(x)$ of the interior medium (inhomogeneity);

2) The existence of a preferred direction in the interior medium due to intrinsic angular momenta or other reasons (anisotropy);

3) The local variation of the speed of light $c = c(x, \mu, \tau, ...)$;

4) The local variation of the index of refraction $n(x, \mu, \tau, ...)$;

5) The maximal local causal speed which, in interior conditions, is no longer the speed of light (Sect. II.8.5);

and all other features of the tangent isospecial relativity.

9.4.C: Isodual isoriemannian geometry and the fundamental theorem for interior gravitation of antimatter. We consider now the *isodual isoriemannian geometry* (Sect. I.5.6) for the characterization of antimatter. It is the geometry of the isodual isospace on isodual isoreals

$$\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d): \quad \hat{g}^d = T^d g = -\hat{g}^d, \quad \hat{1}^d = (T^d)^{-1} = -\hat{1}. \qquad (9.4.10)$$

As such, antimatter has *negative-definite energy, time and other physical quantities, although referred to the negative-definite unit of the isodual isofield* $\hat{R}^d(\hat{n}^d,+,*^d)$ (Sect. I.2.5). We therefore have the following:

---

[125] Note that the. null torsion of Einstein's gravitation is one additional reason, besides those of Sect. II.9.3, for its inapplicability to interior gravitational problems among others we could not mention for brevity.

Theorem 9.4.2 (Fundamental theorem of isodual isogravitation for antimatter [26-28]): *The isodual isogravitation for antimatter is the antiautomorphic image of the isogravitation of Theorem 9.4.1 under isoduality to the isospace* $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$.

The above characterization of antimatter permits truly *novel* advances, that is, advances beyond the possibilities of the Riemannian geometry, let alone Einstein's gravitation. We are here referring to quantitative studies suitable for direct experimental verification with current technology of *antigravity* [31], this time, studied at the level of its possible origin, the interior gravitational problem on a curved isospace.

To study these aspects, we first have the following

Theorem 9.4.3 (Fundamental theorem for gravity of antimatter [28-29]): *The interior problem of antimatter verifies Theorem 9.4.1 under the following isodual maps*

Basic unit	$\hat{1} \rightarrow \hat{1}^d = -\hat{1}$,
Isotopic element	$T \rightarrow T^d = -T$,
Isometric	$\hat{g} = Tg \rightarrow \hat{g}^d = -\hat{g}$,
Isoconnection coefficients	$\hat{\Gamma}_{klh} \rightarrow \hat{\Gamma}^{d\,l}_{klh} = -\hat{\Gamma}^{l}_{klh}$,
Isocurvature tensor	$\hat{R}_{lijk} \rightarrow \hat{R}^d_{lijk} = -\hat{R}_{lijk}$,
Isoricci tensor	$\hat{R}_{\mu\nu} \rightarrow \hat{R}^d_{\mu\nu} = -\hat{R}_{\mu\nu}$,
Isoricci scalar	$\hat{R} \rightarrow \hat{R}^d = \hat{R}$, (9.4.11)
Isoeinstein tensor	$\hat{G}_{\mu\nu} \rightarrow \hat{G}^d_{\mu\nu} = -\hat{G}_{\mu\nu}$,
Isotopic scalar	$\Theta \rightarrow \Theta^d = \Theta$,
Completed isoeinstein tensor	$\hat{S}_{\mu\nu} \rightarrow \hat{S}^d_{\mu\nu} = -\hat{S}_{\mu\nu}$,
Electromagnetic potentials	$\hat{A}_\mu \rightarrow \hat{A}^d_\mu = -\hat{A}_\mu$,
Electromagnetic field	$\hat{F}_{\mu\nu} \rightarrow \hat{F}^d_{\mu\nu} = -\hat{F}_{\mu\nu}$,
Elm energy-mom. tensor	$\hat{T}_{\mu\nu} \rightarrow \hat{T}^d_{\mu\nu} = -\hat{T}_{\mu\nu}$,
Stress-energy tensor	$\hat{t}_{\mu\nu} \rightarrow \hat{t}^d_{\mu\nu} = -\hat{t}_{\mu\nu}$,

where we have used the isodual rules, including the isodual isoderivative

$$\partial^d/\partial^d x = \hat{1}^d \partial/\partial x = -\partial/\partial x.$$

In summary, we have the following characterizations: **Riemannian geometry** for the exterior problem of matter; **isodual Riemannian geometry** for the exterior problem of antimatter; **isoriemannian geometry** for the interior problem of matter; and **isodual isoriemannian geometry** for the interior problem of antimatter.

The study of the remaining general properties of antimatter via the isodual Riemannian and isoriemannian geometries is left to the interested reader.

Riemannian and isoriemannian geometries is left to the interested reader.

9.4.D: Antigravity. Recall that in conventional Riemannian spaces the energy is necessarily positive–definite (as a consequence of the very definition of the energy–momentum tensor). Under isoduality we however have the following

Corollary 9.4.3.A: *The total energy of isodual isogravitation is negative–definite.*

The above property evidently ensures the representation of antimatter according to the desired characteristics. In conventional Minkowskian and Riemannian spaces the electromagnetic potentials and fields do change sign for antiparticles, but the energy–momentum tensor remains the same. The latter changes sign only when computed in isodual spaces $\Re^d(x,g^d,R^d)$ and $\hat{\Re}^d(x,\hat{g}^d,\hat{R}^d)$.

Next, it is easy to see the following important property

Corollary 9.4.3.B: *The action of the isodual isogravitational field of antimatter on antimatter is attractive.*

In fact, the curvature isotensor is indeed *negative–definite*. But it is defined with respect to a *negative–definite isounit*, thus resulting in attraction.

It is then equally easy to see the following important property

Corollary 9.4.3.C: *The projection of an elementary antiparticle (such as a positron) in the gravitational field of matter is attractive, while isoselfdual bound states of particles and antiparticles are attracted by both matter and antimatter.*

The first part of the abocve property is a direct consequence of Theorem 9.4.3, while the latter part is due to the positive–definite character of the total energy of isoselfdual bound states studied in Sect. 8.7.C. The above results therefore confirm a central prediction of hadronic mechanics, the possibility of reversing the sign of gravity in a form experimentally verifyable with current technology.

Note that in *both electromagnetic and gravitational cases* we have: particles represented in spaces or isospaces and antiparticles represented via the isodual spaces or isospaces; and, again in both cases, we have *attraction* for particles in the field of antiparticles and viceversa.

In particular, the replacement of charge with mass yields the equivalence between the Coulomb law $F = q(\pm q)/r^2$ and the Newton law $F = GM(\pm M)/r^2$, where the negative values are referred to isodual spaces under the identification of the electromagnetic and gravitational fields discussed earlier.

The prediction of the *space–time machine* [32] which is consequential to antigravity will be studied in Sect. II.9.7.

9.4.E: Genogravitation and its isodual. A central property of isogravitation is that it represents closed–isolated systems with reversible center-of–mass trajectories. In order to study *irreversibility* we have to consider the more general *Riemann–admissible (or genoriemannian) spaces* of Class I (Sect. I.7.7)

$$<\hat{\Re}>(x,<\hat{g}>,<\hat{R}>): \quad <\hat{g}> = <T>g, \quad <\hat{1}> = (<T>)^{-1}, \quad \hat{1}>(<\hat{1})^{\dagger}, \quad (9.4.12)$$

which are now defined on the genofields $<\hat{R}>$ $(<\hat{n}>,+,<>)$.

As one can see, the most direct way of performing the transition from isoriemannian to genoriemannian manifolds is via *nonhermitean* generalizations of the isotopic element T much along the case of the genominkowskian spaces of the preceding chapter.

The *genoriemannian geometry* is the geometry of genospaces $<\hat{\Re}>(x,<\hat{g}>,<\hat{R}>)$. The identification of its main properties is left to the interested reader for brevity. *Genogravitation* is the image of isogravitation for nonhermitean isotopic elements. Its axiomatic characterization of matter in irreversible conditions is then evident. The corresponding characterization of antimatter is then done via isoduality.

9.5: ISOMINKOWSKIAN CHARACTERIZATION OF GRAVITY AND ITS ISODUAL

In the preceding section we have studied the interior gravitational problem on a *curved* isospace $\hat{\Re}(x,\hat{g},\hat{R})$. In this section we shall study what is undoubtedly the most important contribution of isotopies to both exterior and interior gravitation, its formulation on an isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ which, as studied in Vol. I, is both *isoflat* and *isocurved*, thus permitting a *symbiotic unification of the general and special relativities with ensuing universal symmetry for both, and unique quantizationof gravity without the Hamiltonian.*

9.5.A: Statement of the Problem. The special relativity was built on the solid grounds of the Poincaré symmetry $P(3.1) = SL(3.1)×T(3.1)$ which is *universal* for all possible exterior, relativistic, classical and quantum problems of point–particles moving in in the homogeneous and isotropic vacuum. The special relativity therefore resulted to be exactly valid in the above specified arena .

On the contrary, the general relativity was built without a single universal symmetry holding for all possible metrics with consequential lack of guidance for the achievement of a structure that would resist the test of time.

The above scenario has been altered by the isotopies of the Poincaré symmetry $\hat{P}(3.1)$ of the preceding chapter. In fact, as we shall see in this section,

the isopoincaré symmetry is the universal symmetry for classical or operator, and interior or exterior gravitation. The theory of gravitation therefore emerges from these studies as possessing the same solid grounds of the special relativity and therefore as having the same rigid guidelines of a universal symmetry.
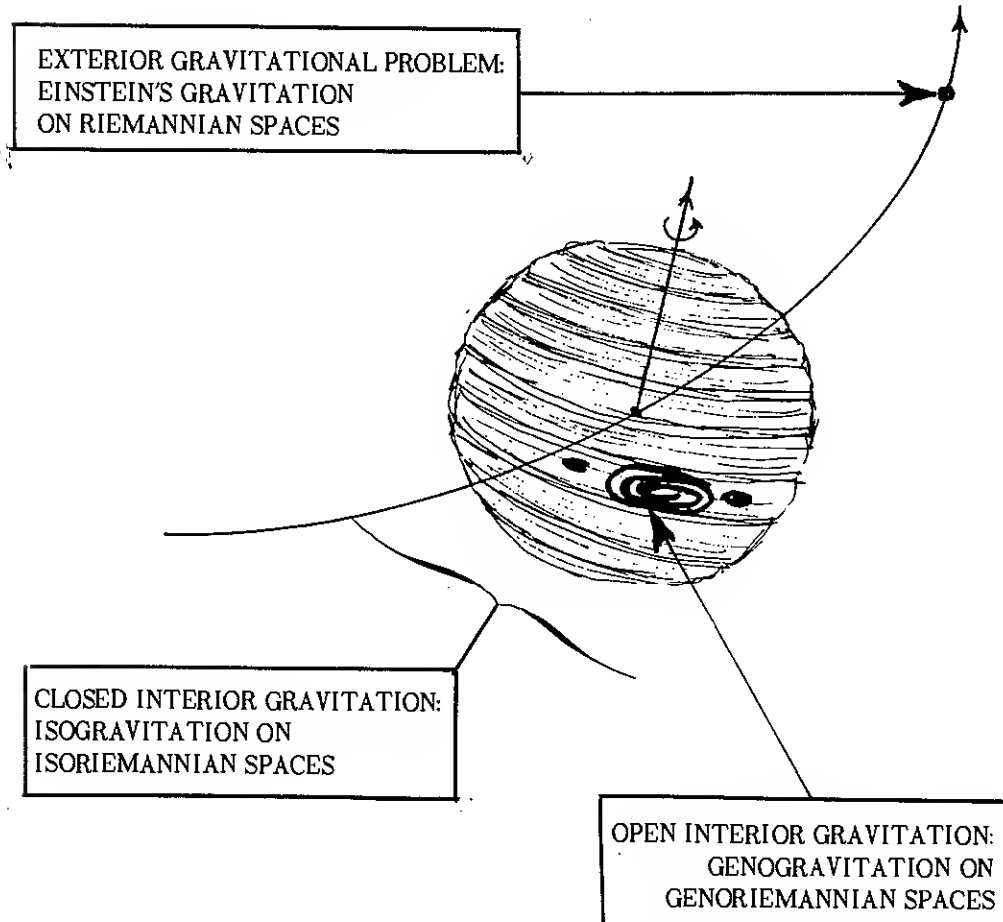
GRAVITATION, ISOGRAVITATION AND GENOGRAVITATION

FIGURE 9.4.1: A schematic view of the conception of gravitation presented in this chapter. The birth of all relativities for the exterior problem can be identified with the first visual observation of the *Jovian system* by Galileo Galilei in 1906. the birth of the new isorelativities for the interior problem can be identified with the visual observation, this time, of *Jupiter's structure*, as offered by contemporary telescopes or by the recent NASA planetary missions. Such a visual observation reveals the following physical evidence:

a) Jupiter is a stable system when considered as a whole and, thus, it verifies conventional total conservation laws when isolated from the rest of the Universe;

b) Jupiter's interior dynamics is essentially nonconservative, as established by

vortices with continuously varying angular momenta. This implies the existence of internal exchanges of energy and other quantities, but always in such a way that they balance each other resulting in total conservation laws.

c) Jupiter's interior dynamics is intrinsically irreversible, but in such a way that its center-of-mass trajectory verifies in full the time reversal symmetry.

In order to achieve a quantitative representation of the above physical evidence, this author constructed a *dual* generalization of conventional Galilean, special and general relativities, the first of *isotopic* character based on a Lie-isotopic algebraic structure, and the second of the yet more general *genotopic* character whose basic algebraic structure is of the covering Lie-admissible type (Ch. 1.7). The isogravitation of this chapter is conceived to represent *closed-isolated nonhamiltonian systems* [13], i.e., systems satisfying total conventional conservation laws under generalized internal forces. The emerging theory is then *structurally reversible*, that is, it is reversible for reversible Hamiltonians and isotopic elements. The complementary genotopic formulations are conceived to represent the most general possible *open-nonconservative systems with consequential structurally irreversibility*.

Moreover, by recalling that the isopoincaré symmetry is isomorphic to the conventional one, $\hat{P}(3.1) \approx P(3.1)$, the isotopies permits the achievement of a geometric unification of relativistic and gravitational theories[126] for all the above indicated classical or operator and external or internal levels. All novel predictions of the theory, such as antigravity, the space-time machine, etc., can then be reduced to the primitive space-time symmetry $\hat{P}(3.1)$ and its isodual $\hat{P}^d(3.1)$. All these results are centrally dependent on the formulation of gravity in *isoflat* isospaces. In fact, the isopoincaré symmetry cannot be defined in isorie-mannian spaces and requires the use of the flat isominkowskian spaces.

9.5.B: Equivalence of Riemannian and isominkowskian spaces. It is generally believed that a Rieannian space is *necessary* to represent gravitation. This belief has been disproved by the isotopic methods in 1988 [27]. In fact, Riemannian metrics $g(x)$ can be *identically* interpreted as being the metric of the isoflat isominkowski spaces $\hat{\eta}(x)$ thus achieving a symbiotic unification of both, the Minkowskian and Riemannian spaces as studied in Vol. I.

We shall study below only the isoflat profile of gravitation on $\hat{M}(x,\hat{\eta},\hat{R})$. The isocurved profile is essentially the same as that via the Riemannian geometry, only referred to a different unit, and it will be ignored for brevity.

Let us consider first exterior problems in vacuum. All Riemannian spaces

---

126 The isotopic unification of Einstein's special and general relativity *is not* possible because of the problematic aspects of the latter identified in Sect. 11.9.3. For instance, the special relativity is form-invariant under the Poincaré symmetry, while the general relativity is not invariant under the isopoincaré symmetry (see, later on, Corollary 11.9.5.2.A). The unification is instead possible if the general relativity is structurally revised to avoid said problematic aspects.

$\mathfrak{R}(x,g,R)$ admit the Minkowskian factorization of the metric

$$g(x) = T_{gr}(x)\,\eta, \qquad T_{gr} > 0, \tag{9.5.1}$$

where the positive–definiteness of the *gravitational isotopic element* $T_{gr}(x)$ is necessary under the local Minkowskian character of the space.

As a result, all possible conventional Riemannian spaces admit an equivalent formulation in terms of isominkowskian spaces. But the latter are locally isomorphic to the conventional spaces. We therefore have the local equivalence properties

$$\mathfrak{R}(x,g,R) \approx \hat{M}(x,\hat{\eta},\hat{R}) \approx M(x,\eta,R): \quad \hat{\eta} = T(x)\,\eta \equiv g(x), \quad \hat{1} = [\,T(x)\,]^{-1}. \tag{9.5.2}$$

The above property is at the foundation of the universality of the isopoincaré symmetry $\hat{P}(3.1)$ for gravitation studied below. The property also demands that *any correct theory of gravitation must admit an equivalent formulation in Minkowskian and in isominkowskian spaces* in full agreement with the results of Sect. II.9.3.C.

This is readily possible for exterior gravitational theories (II.9.3.18) including Logunov's gravitation (9.3.6). First, gravitation (II.9.3.18) can be *identically* interpreted as occurring in isominkowskian spaces, that is, the field equations remain unchanged (because of the identity of the metrics), and they are merely referred to a space with gravitational isounit $\hat{1}_{gr} = [T_{gr}(x)]^{-1}$. This eliminates the curvature. Once the equations are reformulated in isominkowski space, their further reformulation in conventional Minkowski space merely requires factorization (9.5.1).

It is easy to see that we have similar results for the interior gravitational problems. In fact, the isometrics of the isoriemannian spaces $\mathfrak{R}(x,\hat{g},\hat{R})$ and those of the isominkowskian spaces $\hat{M}(x,\hat{\eta},\hat{R})$ have *the same arbitrary functional dependence*. As a result, we can assume the identification $\hat{g} \equiv \hat{\eta}$, $\hat{g} \in \mathfrak{R}(x,\hat{g},\hat{R})$, $\hat{\eta} \in \hat{M}(x,\hat{\eta},\hat{R})$, which implies that *the intervals in isoriemannian and isominkowskian spaces can be assumed to coincide*. Note however that the *isounits* of the two spaces are different. In fact, the isounit of $\mathfrak{R}(x,\hat{g},\hat{R})$ is $\hat{1} = T^{-1}$, $\hat{g} = Tg$, $g \in \mathfrak{R}(x,g,R)$, while that of $\hat{M}(x,\hat{\eta},\hat{R})$ is $\hat{1}_{gr} = (T_{gr})^{-1}$, $\hat{\eta} = T_{gr}\eta$, $\eta \in M(x,\eta,R)$.

We can therefore write the single isoseparation between two points x and y for both spaces of Class I $\mathfrak{R}(x,\hat{g},\hat{R})$ and $\hat{M}(x,\hat{\eta},\hat{R})$, $\hat{g} \equiv \hat{\eta}$, in the diagonal form

$$x^2 = [\,(x^1-y^1)\,T_{11}(x,\dot{x},\ddot{x},\mu,\tau,...)\,(x^1-y^1) + (x^2-y^2)\,T_{22}(x,\dot{x},\ddot{x},\mu,\tau,...)\,(x^2-y^2) +$$
$$+ (x^3-y^3)\,T_{11}(x,\dot{x},\ddot{x},\mu,\tau,...)\,(x^3-y^3) - (x^4-y^4)\,T_{22}(x,\dot{x},\ddot{x},\mu,\tau,...)\,(x^4-y^4)\,]\,\hat{1} \tag{9.5.3a}$$

$$\hat{1} = \text{diag.}\,(\,T_{11}^{-1}, T_{22}^{-2}, T_{33}^{-1}, T_{44}^{-1}\,) = \hat{1}^{\dagger} > 0, \quad T_{\mu\mu} > 0, \tag{9.5.3b}$$

with consequential isotopic equivalence

$$\mathfrak{R}(x,\hat{g},\hat{R}) \approx \hat{M}(x,\hat{\eta},\hat{R}), \quad \hat{g}(x,\dot{x},...) \equiv \hat{\eta}(x,\dot{x},...), \quad \hat{g} \in \mathfrak{R}(x,\hat{g},\hat{R}), \hat{\eta} \in \hat{M}(x,\hat{\eta},\hat{R}), \tag{9.5.4}$$

which implies that *interior gravitational models must admit an equivalent formulation in the isoriemannian and isominkowskian spaces.*

It is also easy to see that *all structural distinctions between interior and exterior gravitation are lost at the level of the isominkowskian representation.* In fact, such a distinction is merely reduced to the different functional dependence $\hat{\eta}(x)$ and $\hat{\eta}(x, \dot{x}, \ddot{x}, ...)$.

An illustration of the isominkowskian representation is in order. The familiar Schwartzschild's exterior line element [5] (for $c_0 = 1$)

$$ds^2 = (1 - 2M / r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 - (1 - 2M / r) dt^2, \quad (9.5.5)$$

can be interpreted as belonging to an isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ with isotopic element

$$T = \text{diag.} (T_s, T_t), \quad T_s = (1 - 2M/r)^{-1} \text{diag} (1, 1, 1), \quad T_t = (1 - 2M/r). \quad (9.5.6)$$

Similarly, the isoschwartzschild's line element (11.9.4.4) here assumed for simplicity to verify the space–isotropy

$$ds^2 = n_s^{-2} [(1 - 2M / r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2] - n_4^{-2} (1 - 2M / r) dt^2, \quad (9.5.7)$$

can be equally assumed as belonging to the isominkowski space with isotopic element

$$T = \text{diag.} (T_s, T_t), \quad T_s = n_s^{-2} (1 - 2M/r)^{-1} \text{diag} (1, 1, 1), \quad T_t = n_4^{-2} (1 - 2M/r). \quad (9.5.8)$$

Note how only the functional dependence is changed in the transition from the exterior (9.5.6) to the interior representation (9.5.8), while the underlying isominkowskian character remains the same.

9.5.C: Universal isopoincare´ symmetry for gravitation. The isorotational symmetry $\hat{SO}(3)$ of Ch. 11.6, the isolorentz symmetry $\hat{L}(3.1)$ of Sect. 11.8.3 and the isopoincaré symmetry $\hat{P}(3.1)$ of Sect. 11.8.4 have been constructed precisely as isosymmetries of the generic invariant (9.5.3). As a result, they are isosymmetries of classical exterior and interior gravitation (see later on for the operator counterpart). Let us recall that the isopoincaré symmetry is composed of

A) The *isorotations* (11.6.2.20), e.g., in the (1, 2)–plane which we rewrite for isoseparation (9.5.3)

$$x' = x \cos (T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_3) - y T_{11}^{-\frac{1}{2}} T_{22}^{\frac{1}{2}} \sin (T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_3)$$
$$y' = x T_{11}^{\frac{1}{2}} T_{22}^{-\frac{1}{2}} \sin (T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_3) + y \cos (T_{11}^{\frac{1}{2}} T_{22}^{\frac{1}{2}} \theta_3),$$
$$z' = z, \quad x'^4 = x^4, \quad (9.5.9)$$

where θ_3 is a conventional Euler angle. The most general possible isorotation in

three–dimension is then obtained via the isotopies of the conventional combination of rotations on all three Euler angles $\{\theta_1, \theta_2, \theta_3\}$ as studied in Ch. II.6.

B) The *isolorentz transforms* (II.8.3.19) in the (3, 4)–plane which we can write

$$x' = x^1 , \quad y' = y ,$$

$$z' = z \cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) - x^4 T_{33}^{-\frac{1}{2}} T_{44}^{\frac{1}{2}} \sinh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v)$$

$$= \hat{\gamma}(x^3 - \beta T_{33}^{-\frac{1}{2}} T_{44}^{\frac{1}{2}} x^4) ,$$

$$x^4 = z T_{33}^{\frac{1}{2}} T_{44}^{-\frac{1}{2}} \sinh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) + x^4 \cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v)$$

$$= \hat{\gamma}(x^4 - \beta T_{33}^{\frac{1}{2}} T_{44}^{-\frac{1}{2}} x^3) \tag{9.5.10}$$

where

$$\beta = v / c_o, \quad \beta = v_k T_{kk}^{\frac{1}{2}} / c_o T_{44}^{\frac{1}{2}} ,$$

$$\sinh(T_{33}^{-\frac{1}{2}} T_{44}^{-\frac{1}{2}} v) = \beta \hat{\gamma} , \quad \cosh(T_{33}^{\frac{1}{2}} T_{44}^{\frac{1}{2}} v) = \hat{\gamma} = |1 - \beta^2|^{-\frac{1}{2}} , \tag{9.5.11}$$

C) The *isotranslations* (II.8.3.34)

$$x' = x + a B^{-2}(x, \dot{x}, \ddot{x}, ...) , \tag{9.5.12}$$

where the B–functions are computed via Eq.s (II.8.3.35), plus *isoinversions* (II.8.34c) and (II.8.3.34d).

The verification that the above isotransforms leave invariant arbitrary isoseparations (9.5.3) is instructive. An important result of this volume can be formulated via the following:

Theorem 9.5.1 (Universality of the isopoincare´ symmetry for classical gravitation): *Consider all infinitely possible* (3+1)–*dimensional isoriemannian spaces* $\hat{\Re}(x,\hat{g},\hat{\Re})$ *with local coordinates* x *and Hermitean nonsingular isometrics* $\hat{g}(x, \dot{x}, \ddot{x}, ...)$ *under the isotopic decomposition* $\hat{g}$ = $T_{gr}\eta$, *where* η *is the local Minkowskian metric. Introduce the reinterpretation in isominkowskian spaces* $\hat{M}(x,\hat{\eta},\hat{\Re})$ $\hat{\eta}$ = $T_{gr}\eta \equiv \hat{g}$ *over the isofield* $\hat{\Re}(\hat{n},+,*)$ *of real isonumbers with isounit* $\hat{1}_{gr}$ = $T_{gr}^{-1} > 0$. *Then the symmetry which is directly universal of all infinitely possible isoriemannian–isominkowskian separations* (x – y)$^t\hat{\eta}$(x – y) *is the isopoincaré symmetry* $\hat{P}(3.1)$ *with gravitational isounit* $\hat{1}_{gr}$.

In fact, the decomposition $\hat{g}$ = $T_{gr}\eta$ is always possible for all infinite (3+1)–dimensional isoriemannian metrics from their local Minkowskian character. The theorem then follows from the Lie–isotopic theory.

Corollary 9.5.1.A: *Under the conditions of the theorem the isopoincaré symmetry* $\hat{P}(3.1)$ *of all possible isoriemannian line elements* (x – y)$^t\hat{\eta}$(x – y) *is locally isomorphic to the conventional Poincaré symmetry* P(3.1) *of the*

Minkowskian line element $(x - y)^t \eta (x - y)$, $\hat{P}(3.1) \approx P(3.1)$.

In fact, the conditions of nondegeneracy plus that of locally Minkowskian character imply $T_{gr} > 0$. The local isomorphism $\hat{P}(3.1) \approx P(3.1)$ then follows (see, Sect. II.8.3 for details). Note that Theorem 9.5.1 has been expressed for the most general possible interior conditions, but it includes the exterior Riemannian space as an evident particular case.

The verification of the invariance of isoseparation (9.5.3) under the $\hat{P}(3.1)$ symmetry is elementary. In particular *there is nothing to compute* but just plug the $T_{\mu\mu}$ elements of the factorization $\hat{g}(x) = T_{gr}(x)\eta$ in the isotransforms. The invariance is then guaranteed by Theorem 9.5.1. For example, the symmetry of *Schwartzchild's exterior separation* (9.5.5) is given by merely plotting in isotransforms (9.5.8) and (9.5.9) the values

$$T_{11} = T_{22} = T_{33} = (1 - 2M/r)^{-1}, \quad T_{44} = (1 - 2M/r). \tag{9.5.13}$$

Similarly, the symmetry of *Krasner anisotropic exterior separation* (see, e.g., ref. [8])

$$ds^2 = t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2 - dt^2, \quad p_1 + p_2 + p_3 = 1, \tag{9.5.14}$$

is given by plotting the identifications in isotransforms (9.5.8) or (9.5.9)

$$T_{11} = t^{2p_1}, \quad T_{22} = t^{2p_2}, \quad T_{33} = t^{2p_3}, \quad T_{44} = 1. \tag{9.5.15}$$

The symmetry of *Yilmaz exterior separation* [25] in vacuum

$$ds^2 = e^{2\phi} (dx^2 + dy^2 + dz^2) - e^{-2\phi} dt^2, \tag{9.5.16}$$

is given by the identifications

$$T_{kk} = \delta_{ik} e^{2\phi}, \qquad T_{44} = e^{-2\phi}. \tag{9.5.17}$$

A similar situation occurs for all other exterior gravitational theories (see, e.g., ref.s [7-11]).

The above applications are however only the beginning of the possibilities of the isosymmetry $\hat{P}(3.1)$. As an illustration for interior cases, the invariance of the *isoschwartzschild separation* (9.5.6) is given by plotting in isotransforms (9.5.8) or (9.5.9) the quantities

$$T_{kk} = (1 - 2M/r)^{-1} n_k^{-2}, \quad T_{44} = (1 - 2M/r) n_4^{-2}, \tag{9.5.18}$$

where one should keep in mind that the n's have a generally *integral* character.

The above results are specifically and solely intended for *matter*. The *isodual isopoincaré symmetry* $\hat{P}^d(3.1)$ is the universal symmetry for *antimatter*, as characterized by the antiautomorphic map $\hat{1}_{gr} \rightarrow \hat{1}_{gr}^d = -\hat{1}_{gr}$ (see Sect. II.8.3).

Note the necessary *loss* of the curvature/Riemannian space in the construction of the universal isopoincaré symmetry and the use of the flat space $\hat{M}(x,\hat{\eta},\hat{R})$ locally isomorphic to the Minkowski space $M(x,\eta,R)$. In fact, the curved representation of gravitation in the Riemannian or isoriemannian spaces implies the impossibility of even defining the isopoincaré symmetry.

9.5.D: Universal covariance of gravitational fields under the iso-poincare´symmetry.

The isosymmetries $\hat{P}(3.1)$ and $\hat{P}^d(3.1)$ permit a structural revision of gravitation based on universal covariance laws much similar to those of the special relativity.

Consider an electromagnetic field of matter with energy–momentum tensor $\mathcal{E}_{elm}(x, ...)$ depending on space–time coordinates x and other quantities (e.g., the velocities) represented in *conventional* Minkowski space $M(x,\eta,R)$. Its transformation laws under the *conventional* Lorentz symmetry are given by the familiar form

$$U^{-1}\, \mathcal{E}^{\mu\nu}{}_{elm}(x, ...)\, U \;=\; \Lambda^{\mu}{}_{\alpha}\, \Lambda^{\nu}{}_{\beta}\, \mathcal{E}^{\alpha\beta}{}_{elm}(x', ...)\,, \qquad (9.5.19)$$

where $x' = \Lambda x$, Λ is a Lorentz transform and the additional functional dependence of $\mathcal{E}_{elm}$ is transformed accordingly.

Consider now the isominkowskian representation $\hat{M}(x,\hat{\eta},\hat{R})$ of the Riemannian space $R(x,g,R)$, $g(x) = T_{gr}(x)\eta \equiv \hat{\eta}$, $\hat{1}_{gr}(x) = [T_{gr}(x)]^{-1}$; construct the isolorentz symmetry $\hat{L}(3.1)$ for the gravitational isounit $\hat{1}_{gr}(x) > 0$; and focus the attention on the reformulation $\hat{\mathcal{E}}^{\mu\nu}{}_{elm}(x, ...)$ on $\hat{M}(x,\hat{\eta},\hat{R})$ of the energy–momentum tensor $\mathcal{E}^{\mu\nu}{}_{elm}(x, ...)$ on $\Re(x,g,R)$. It is evident that the invariance law (9.5.19) is lifted into the isotopic form

$$\hat{U}^{-1} * \hat{\mathcal{E}}^{\mu\nu}{}_{elm}(x, ...) * \hat{U} \;=\; \hat{\Lambda}^{\mu}{}_{\alpha}(x') \, T_{gr}{}^{\alpha}{}_{\rho}(x^\gamma)\hat{\Lambda}^{\nu}{}_{\beta}(x') \, T_{gr}{}^{\beta}{}_{\sigma}(x') \, \hat{\mathcal{E}}^{\rho\sigma}{}_{elm}(x', ...) \;=$$

$$=\; \tilde{\Lambda}^{\mu}{}_{\rho}(x') \, \tilde{\Lambda}^{\nu}{}_{\sigma}(x') \hat{\mathcal{E}}^{\rho\sigma}{}_{elm}(x', ...), \qquad (9.5.20)$$

where $x' = \hat{\Lambda}*x = \hat{\Lambda} T_{gr}x \equiv \tilde{\Lambda}x$, Λ, $\tilde{\Lambda}$ and T_{gr} are x–dependent, and the final result is computed in x'. Note that we have used the factorization $\hat{\Lambda} = \hat{\Lambda}\hat{1}_{gr}$ to simplify the calculations and reduce them in our space–time, with the understanding that the mathematically correct expression is that in isospace with isotopic product.

The transition to the interior gravitation is then elementary and merely requires the elimination of any restriction on the functional dependence of the isotopic element T_{gr}. In this case we have an isoriemannian space $\hat{\Re}(x,\hat{g},\hat{R})$ with isometric $\hat{g}(x, \dot{x}, \ddot{x}, \mu, \tau...)$ which is reformulated in terms of an isominkowskian space $\hat{M}(x,\hat{\eta},\hat{R})$ again under the condition $\hat{g} \equiv \eta$. Then the isometric is subjected to the factorization $\hat{g} = \hat{\eta} = T_{gr}(x, \dot{x}, \ddot{x}, \mu, ...)\eta$ with ensuing isounit $\hat{1}_{gr} = T_{gr}^{-1}$. The isolorentz symmetry is now constructed with respect to the latter isounit $\hat{1}_{gr} > 0$. The transformation law is the same as law (9.5.20), although with a more general

functional dependence. The same isotransforms hold for all other isotensors.Corresponding antiautomorphic isotranforms hold under the isodual isopoincaré symmetry. We therefore have the following important property:

Theorem 9.5.2 (Universal isocovariance law under isopoincaré symmetry: *All gravitational theories of matter (antimatter) must be isocovariant under the isopoincaré symmetry (isodual isopoincaré symmetry) in isominkowskian (isodual isominkowskian) representation.*

The implications of the above results for Einstein's gravitation in vacuum, Eq.s (II.9.3.1) are made clear by the following:

Corollary 9.5.2.A: *The Schwartzschild's metric is not form-invariant under the isopoincaré symmetry.*

The need for a structural revision of Einstein's gravitation in vacuum then follows exactly along the lines of Sect. II.9.3.[127]

Virtually all results of this chapter can be derived, directly or indirectly from the above universal law. In fact, basic properties such as the isocomposition of speeds (II.8.5.7), the isodoppler law (II.8.5.9), the isoequivalence (principle (II.8.5.10 etc., are a direct consequence of the above universal law. The numerical representations of astrophysical and other data studied in Vol. III are also a consequence of the above universal covariance law.

9.5.E: Isotopic quantization of gravity. We now outline the novel isotopic quantization of gravity without the Hamiltonian presented by this author under the name of *quantum isogravity* at the *Seventh Marcel Grossman Meeting* on *General Relativity,* Stanford University, July 1994 [33] (see also [48]).

Beside achieving such a novel quantizationof gravity, the main result emerging from these studies is that *relativistic hadronic nmechanics is jointly isoflat and isocurved,* e.g., it possesses locally isocommuting momenta, yet it preserves curvatures in its entirety. In the following we shall only study the isoflat character for brevity and rdefer the isocurved operator formalism to a more specialized literature.

The main idea of quantum isogravity is *to embed gravitation in the unit of a conventional relativistic quantum theory.* This is permitted by the isotopic methods via the now familiar factorization of any given Riemannian metric in the form $g(x) = T_{gr}(x)\eta$ and the lifting of the unit $1 = $ diag. (1, I, 1, I) of any given relativistic quantum theory into the *gravitational isounit* $\hat{1}_{gr} = [T_{gr}(x)]^{-1}$.

Recall that the essential gravitational elements are contained in T_{gr} and not in η. Recall also that $T_{gr}(x)$ is always positive-definite from the locally

[127] Despite these results, we shall continue to use the Schwartzschild metric because of its unquestionable *approximate* character.

Minkowskian character of $\Re(x,g,R)$ and, therefore, it can always be diagonalized. Recall finally that the metric for raising and lowering the indices in $\hat{M}(x,\hat{\eta},\hat{R})$ is $\hat{\eta}(x) \equiv g(x)$, and $\hat{1} = (\hat{1}^\mu{}_\nu) = (\hat{1}_\mu{}^\nu) = (\hat{1}_{\mu\nu}) = (\hat{1}^{\mu\nu})$.

A consistent isoquantization of gravity then requires the lifting of the totality of the mathematical structure of relativistic quantum mechanics into that of relativistic hadronic mechanics of the preceding chapter, such as the liftings: $R(n,+,\times) \to \hat{R}(\hat{n},+,*)$, $M(x,\eta,R) \to \hat{M}(x,\hat{\eta},\hat{R})$, $\xi \to \hat{\xi}$, $\mathcal{JC} \to \hat{\mathcal{JC}}$, $P(3.1) \to \hat{P}(3.1)$, etc.

The following properties are particularly significant for these studies: 1) the operator description of gravitation via relativistic hadronic mechanics is invariant under its own time evolution and derivable from first principles; 2) relativistic hadronic mechanics admits the conventional theory as a particular case for $\hat{1}_{gr} \equiv I$; and 3) relativistic quantum and hadronic mechanics coincide at the abstract level in which (from $T_{gr} > 0$) $\hat{R}(\hat{n},+,*) \equiv R(n,+,\times)$, $\hat{M}(x,\hat{\eta},\hat{R}) \equiv M(x,\eta,R)$, $\hat{\xi} \equiv \xi$, $\hat{\mathcal{JC}} \equiv \mathcal{JC}$, $\hat{P}(3.1) \equiv P(3.1)$, etc. In turn, these abstract identities assure the mathematical and physical consistency of quantum isogravity.

The main assumption of quantum isogravity is therefore that a *consistent operator form of gravity already exists. It did creep in un-noticed until now because it is embedded in the unit of conventional relativistic quantum mechanics.*

For clarity, let us reinspect the main lines of Sect. II.8.4 from a gravitational viewpoint. Relativistic hadronic mechanics already provides an operator formulation of gravity as presented in the preceding chapter. This is due to the fact that the theory has ben constructed for an arbitrary functional dependence of the isounit, including a nonlinear dependence ion the local coordinates x. The inclusion of gravitation in relativistic hadronic mechanics is therefore a mere question of realization of the isounit.

In particular, relativistic hadronic mechanics provides a full operator description of *exterior gravitation* when the isounit is assumed to be that of exterior gravitational problems,

$$\hat{1}_{gr}(x) = [\, T_{gr}(x)\,]^{-1} > 0\,, \quad g(x) = T_{gr}(x)\,\eta\,, \quad g \in \Re(x,g,R),\ \eta \in M(x,\eta,R)\,. \quad (9.5.21)$$

As an example, an operator formulation of Schwartzschild's line element is given by relativistic hadronic mechanics with isotopic element (II.9.5.6).

An operator formulation of the *interior gravitation* is given instead by relativistic hadronic mechanics with unrestricted functional dependence of the isounit

$$\hat{1}_{gr}(x,\dot{x},\ddot{x},\mu,\tau,...) = [\, T_{gr}(x,\dot{x},\ddot{x},\mu,\tau,...)\,]^{-1} > 0\,,$$

$$\hat{g}(x,\dot{x},\ddot{x},\mu,\tau,...) = T_{gr}(x,\dot{x},\ddot{x},\mu,\tau,...)\,\eta\,, \quad \hat{g} \in \hat{\Re}(x,\hat{g},\hat{R}),\ \eta \in M(x,\eta,R)\,. \quad (9.5.22)$$

As an example, the interior isoschwartzschild's metric admits an operator formulation for isotopic element (II.9.5.8).

Let us also recall that the isoexpectation value of an operator A in

relativistic hadronic mechanics is given by $\hat{<}A\hat{>} = <\,|\,T\,A\,T\,|\,>/<\,|\,T\,|\,>$. We therefore have the following important property.

Lemma 9.5.1 [33]: *The isoexpectation values of all possible exterior or interior gravitational isounits are the conventional unit.*

$$\hat{<}\hat{1}\hat{>} = \frac{<\,|\,T\,T^{-1}\,T\,|\,>}{<\,|\,T\,|\,>} = 1.$$
(9.5.23)

The above result is another way of expressing the embedding of gravitation in the basic unit of the theory. The above lemma also explains the reason for selecting the term "quantum" isogravity.

The momentum operator for quantum isogravity in diagonal form is[128]

$$p_\mu * | > \; = \; p_\mu\,T(x,\,...)\,| > \; = \; -\,i\,\hat{\partial}_\mu\,| > \; = \; -\,i\,\hat{1}_{\mu\mu}\,(x,\,...)\,\partial_\mu\,| > \;\text{(no sum)}.$$
(9.5.24)

The fundamental isocommutation rules are then given from Eq.s (11.8.14)

$$[\,p_\mu\,\hat{,}\,x_\nu\,]*| > \; = p_\mu\,T(x,\,...)\,x_\nu \; - \; x_\nu\,T(x,\,...)\,p_\mu\,)\,T(x,\,...)\,| > = -\,1\,\eta_{\mu\nu}\,| >,$$
(9.5.25a)

$$[\,p_\mu\,\hat{,}\,p_\nu\,]*| > \; \equiv \; 0, \qquad [\,x_\mu\,\hat{,}\,x_\nu\,]*| > \; \equiv \; 0.$$
(9.5.25b)

We reach in this way the additional property:

Lemma 9.5.2 [33]: *The components of the linear momentum isocommute for all possible exterior and interior gravitational theories in isominkowskian representation.*

As it is well known, a basic property of gravitation in a *curved* space is that *the components of the linear momentum do not commute.* The above lemma establishes that we have indeed a genuine representation of gravitation in a *flat* space because said components commute. This result is achieved by representing via the isotopic element the quantity truly expressing gravitation, the term T_{gr} in the factorization $g = T_{gr}\eta$.

The isocommutation rules of *the gravitational isopoincaré algebra* can be written from Eq.s (11.8.4.23) in terms of the *conventional* generators $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ and p_μ

$$[\,M_{\mu\nu}\,\hat{,}\,M_{\alpha\beta}\,]*|\psi> \; = i\,(\eta_{\nu\alpha}\,M_{\beta\mu} - \eta_{\mu\alpha}\,M_{\beta\nu} - \eta_{\nu\beta}\,M_{\alpha\mu} + \eta_{\mu\beta}\,M_{\alpha\nu})*|\psi>$$
(9.5.26a)

$$[\,M_{\mu\nu}\,\hat{,}\,p_\alpha\,]*|\psi> = i\,(\eta_{\mu\alpha}\,p_\nu - \eta_{\nu\alpha}\,p_\mu)*|\psi>, \quad [\,p_\mu\,\hat{,}\,p_\nu\,]*|\psi> \; = 0,$$
(9.5.26b)

where the η's in the r.h.s. is the *conventional* Minkowski metric.

[128] The nondiagonal extension should be formulated via the isodifferential and isoderivatives of Sect. II.8.4.A.

The *gravitational isocasimirs* are then given by

$$\hat{C}^{(0)} * | \hat{\Phi} > \ = \ \hat{1} * | \hat{\Phi} > = | \hat{\Phi} >, \tag{9.5.27a}$$

$$\hat{C}^{(2)} * | \hat{\Phi} > \ = \ P^2 * | \hat{\Phi} > \ = \ \hat{\eta}^{\mu\nu} P_\mu * P_\nu * | \hat{\Phi} > \ = P_\mu * P^\mu * | \hat{\Phi} >, \tag{9.5.27b}$$

$$\hat{C}^{(4)} * | \hat{\Phi} > \ = \ \hat{W}^2 * | \hat{\Phi} > \ = \ \hat{\eta}^{\mu\nu} \hat{W}_\mu * \hat{W}_\nu * | \hat{\Phi} >, \quad \hat{W}_\mu \ = \tfrac{1}{2} \epsilon_{\mu\alpha\beta\rho} M^{\alpha\beta} * P^\rho, \tag{9.5.27c}$$

In particular, the second–order isocasimir explicitly reads

$$p^2 * | \hat{\Phi} > \ = \hat{\eta}^{\mu\nu}(x, ...) \, p_\mu T(x, ...) \, p_\nu \, T(x, ...) | \hat{\Phi} > \ = \ \text{inv.} \tag{9.5.28}$$

where one should remember that *the metric* $\hat{\eta}$ *of the isominkowski space is the gravitational metric,* i.e., $\hat{\eta} = g(x)$ for the exterior problem and $\hat{\eta} = \hat{g}(x, \dot{x}, \ddot{x}, ...)$ for the interior problem.

The transformations on $\hat{M}(x, \hat{\eta}, \hat{R})$ are of the isotopic type $x' = \hat{A} * x = \hat{A} T(x, ...)x = \tilde{A}x$, $\hat{A} = \tilde{A}\hat{1} \in \hat{P}(3.1)$. The connected component of the *gravitational isopoincaré group* can then be written in terms of the original generators $X = \{M_{\mu\nu}, p_\mu\}$ and parameters $w = \{w_k\} = \{(\theta, v), a\}$

$$\hat{A}(\hat{w}) \ = \ \textstyle\prod_k \hat{e}^{i X * \hat{w}} \ = \ (\textstyle\prod_k e^{i X T w})\hat{1}, \tag{9.5.29}$$

where $\hat{e}^A = (e^{AT})\hat{1}$. The explicit form of the transformations is then given again by Eq.s (11.9.5.9)–(11.9.5.12)

Theorem 9.5.3 (Universality of the isopoincaré symmetry for operator gravitation): *The isopoincaré symmetry of relativistic hadronic mechanics is the invariance of all possible exterior or interior, operator gravitational theories in their isominkowskian formulation.*

Note the complete equivalence of the isocommutation rules of the classical and operator formulations of the isopoincare symmetry. The ensuing universality of the isopoincaré symmetry for the characterization of classical or operator and exterior or interior gravitation has implications so deep to warrant a structural revision of the contemporary formulation of gravitation. Evidently, in this volume we can only initiate its study.

To begin, let us recall that the proof of total conservation laws in general relativity over a Riemannian space is rather complex indeed (see, e.g., ref.s [8]). In isogravitation this complex procedure is replaced by a mere *visual inspection.* By recalling that the generators of the isopoincaré symmetry are the *conventional* ones and that all symmetries, whether conventional or isotopic, imply the conservation of their own generators, we have the following:

Corollary 9.5.3.A: *The isopoincaré covariance of gravitation in isominkowskian representation ensures the validity of the conventional*

total conservation laws.

Moreover, the relativistic limit of gravitational conservation laws is quite controversial. In fact, as shown by Yilmaz [21–25] such a limit yields only the conservation laws of the *rest* energy and not that of the *total* energy. This controversy is resolved in its very foundations by the isogravitation. In fact, we have the following:

Corollary 9.5.3.B: *The isopoincaré covariance of gravitation on isominkowskian spaces assures that all conventional relativistic total conservation laws are verified at the limit* $\hat{1}_{gr} \rightarrow 1 = diag. (1, 1, 1, 1).$

Quantum isogravity can also be embedded in conventional relativistic equations. In fact, we can write the following *isogravitational Dirac equation*

$$(\hat{\gamma}^\mu * p_\mu - i \hat{m}) * | > = [\hat{\gamma}^\mu T_{gr} \hat{\eta}_{\mu\nu} p^\nu - i m \hat{1}_{gr}] T_{gr} | > = 0, \qquad (9.5.30)$$

under the isotopic generalization of the conventional gamma matrices γ^μ called *isogamma matrices* characterized by

$$\{ \hat{\gamma}^\mu \hat{,} \hat{\gamma}^\nu \} = \hat{\gamma}^\mu T_{gr} \hat{\gamma}^\nu + \hat{\gamma}^\nu T_{gr} \hat{\gamma}^\mu = 2 \hat{\eta}^{\mu\nu}, \quad \hat{\gamma}^\mu = T_{\mu\mu}^{1/2} \gamma^\mu \hat{1}, \quad (9.5.31)$$

The important point is that at the abstract level the conventional and isogravitational Dirac's equations coincide, $(\gamma^\mu p_\mu + im)| > \equiv (\hat{\gamma}^{\mu}*p_\mu + i\hat{m})*| >.$

Note that *the anticommutator of the isogamma matrices yields (twice) the Riemannian metric* $g(x) \equiv \hat{\eta}(x)$, thus confirming the full embedding of gravitation. A similar isotopic realization of gravity can be formulated for any other relativistic equation as studied in Ch. II.10. As an example, the *Dirac–Schwartzschild equation* presented for the first time in ref. [33] is given by Eq.s (9.5.31) with

$$\hat{\gamma}_k = (1 - 2M/r)^{-1/2} \gamma_k \hat{1}_{gr}, \qquad \hat{\gamma}_4 = (1 - 2M/r)^{1/2} \gamma_4 \hat{1}_{gr}. \quad (9.5.32)$$

Similarly one can construct the *Dirac–Krasner equation* and others.

Another fundamental implication of the isotopic treatment of gravity is expressed by the following:

Corollary 9.5.C: *Classical or operator, exterior or interior and relativistic or gravitational theories are unified by the isopoincaré symmetry when expressed in flat isominkowskian spaces.*

In contemporary formulations the special and general relativities have *geometrically different* treatments, the former being in a flat space and the latter in a curved space. The isominkowskian representation of gravitation eliminates this distinction, thus permitting a unified treatment.

UNIVERSALITY OF THE ISOPOINCARE' SYMMETRY

EXT. GALILEI'S RELATIVITY ON TANGENT EUCLIDEAN PLANE: GALILEI'S SYMMETRY $G_\delta(3.1)$

EXT. SPECIAL RELATIVITY ON TANGENT MINKOWSKI PLANE: POINCARE' SYMMETRY $P_\eta(3.1)$

EXT. GENERAL RELATIVITY ON RIEMANNIAN SPACES: ISOPOINCARE' SYMMETRY $\hat{P}_g(3.1)$

P[3.1]

INT. ISOGENERAL RELATIVITIES ON ISORIEMANNIAN SPACES: ISOPOINCARE' SYMMETRY $\hat{P}_g(3.1)$

INT. ISOSPECIAL RELATIVITIES ON TANGENT ISOMINKOWSKIAN SPACES: ISOPOINCARE' SYMMETRY $\hat{P}_{\hat\eta}(3.1)$

INT. ISOGALILEAN RELATIVITIES ON TANGENT ISOEUCLIDEAN SPACES; ISOGALILEI SYMMETRY $\hat{G}_\delta(3.1)$

FIGURE 3.1: A reproduction of Figure IV.3.1, ref. [29], p. 297, illustrating the universal character of the isopoincaré symmetry for systems which are: linear or

nonlinear, local-differential or nonlocal-integral, potential-Lagrangian or nonpotential-nonlagrangian, interior or exterior, classical or operator and relativistic or gravitational, with the inclusion of corresponding Galilean formulations under the isotopic contraction (ref. [29], Ch. VI). Also, *quantum isogravity permits a novel approach to the unification of weak, electromagnetic and gravitational interactions via the embedding of gravity in the unit of conventional unified gauge theories* (see App. 11.9.c) We can therefore submit the conjecture that gravitation is *already* contained in the existing unified gauge theories. It did escape identification until now because it is embedded in the *unit* of the theory. The study of this *iso-grand-unification* is left to specialized contributions.

9.5.F: Isoequivalence principle.

The *isoequivalence principle* can be best formulated in *isominkowskian* (rather than isoriemannian) treatments and it is based on the use of the conventional normal coordinates $\bar{x}(x)$ under which $g(x) \rightarrow \eta$.

For the exterior problems we have the reduction of the gravitational to the Minkowskian unit $\hat{1}_{gr}(x) \rightarrow 1 = \mathrm{diag.}\,(1, 1, 1, 1)$. In this case there are no remnants representing curvature, but the nowhere null source of the gravitational field in vacuum persists.

For interior problems we have again the preservation of the source of the gravitational field and the absence of terms representing curvature, but the terms representing internal nonlinear-nonlocal-nonlagrangian effects in flat spaces persist, and we shall write $\hat{1}_{gr}(x, \dot{x}, \ddot{x}, ...) \rightarrow \hat{1} = \mathrm{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2}, b_4^{-2})$, where the b's are the isominkowskian characteristic functions of Ch. II.8.

9.5.G: Isotopic units of space and time.

The isominkowskian formulation of gravity is not a mere mathematical curiosity because it has rather deep *experimental* implications. In fact, it predicts a characterization of space and time structurally more general than that of the general relativity.

Recall that the *gravitational isounits* of space and time components in $\hat{M}(x,\hat{\eta},\hat{R})$ are given by $\hat{1}_{\mu gr} = T_{\mu\mu}^{-1}$ and that the projection of the isominkowskian representation into $M(x,\eta,R)$ yields the *space-time isounits* (Sect. II.8.H)

$$\hat{1}_{s\,gr} = \mathrm{diag.}\,(\,T_{11}^{-\frac{1}{2}}, T_{22}^{-\frac{1}{2}}, T_{33}^{-\frac{1}{2}}\,), \quad \hat{1}_{t\,gr} = T_{44}^{-\frac{1}{2}}. \quad (9.5.33)$$

As one can see, space and time are *local quantities* according to the above isounits, that is, they assume different values for observers with the same speed with respect to an inertial frame but different local gravitational fields.

As an explicit example, space and time for an observer in an exterior gravitational field characterized by Schwartzchild's exterior metric have the isounits

$$\hat{1}_{s\,gr} = (1 - 2M/r)^{1/2}\{1, 1, 1\}, \quad \hat{1}_{t\,gr} = (1 - 2M/r)^{-1/2}, \quad r > 2M, \quad (9.5.34)$$

Needless to say, Eq.s (9.5.33) merely illustrate the principle of identification of the

space–time units which therefore applies also for other metrics.

The fundamental *experimental* question raised by isogravitation is therefore whether here on Earth we live in conventional time or in isotime. Equivalently, the experimental issue is to see whether our time here on Earth is the same or different than the time on other planets with significantly different gravitational fields such as Jupiter. Still equivalently, the basic issue is whether the gravitational representation occurring in the physical reality is that via Riemannian or isominkowskian spaces.

This issue has been suggested for experimental verification in ref. [31] by sending a certain probe to Jupiter, and will be discussed in more details in Vol. III. If the time evolves on Jupiter in the same way as on Earth after the conventional corrections, this would establish that the physical description is provided by Riemannian spaces, while the isominkowskian representation has a mere mathematical value. On the contrary, if the measures show a different in time behaviour, this would establish the inverse occurrence, that the physical description is that provided by the isominkowskian geometry, with the Riemann geometry being mathematical in character.

It should be indicated that all gravitational aspects studied by this author suggest the preference of the isominkowskian over the Riemannian formulation of gravitation. The first and most dominant aspect is that the universal isopoincaré symmetry can be solely constructed in the former and not in the latter. But there are numerous additional elements all leading to the same conclusion, such as the need for a theory to admit weight in *Minkowski* space and others studied below.

9.5.H: Isotopic formulation of gravitational horizons and singularities. One of the central occurrences which stimulated the construction of the isotopies is the lack of exact character of the Riemannian geometry in gravitational collapse, black holes, big bang, and all that.

The best approach for quantitative representation of internal nonlinear–nonlocal–nonlagrangian effects is again that via the *isominkowskian* (rather than the isoriemannian) geometry. We provide below the essential lines for detailed studies in the specialized literature.

As it is well known (see, e.g., ref. [34]), the *gravitational horizon* can be introduced as the value r = 2M for which Schwartzchild's metric becomes singular *in Riemannian space*. However, the region of space in the vicinity of the horizon is far from being empty, being composed of hyperdense chromospheres and other media. More realistic studies of the horizon therefore require the inclusion of contributions from the locally varying speed of light, the inhomogeneity and anisotropy of the medium, and other internal properties.

Under isominkowskian representation we then have the following important:

Lemma 9.5.3 [33]: *Gravitational horizons are the zeros of the space isounits, while geometric singularities are the zeros of the time isounit.*

In fact, for the isominkowskian formulation of the Schwartzchild's metric we can write

$$\text{Gravitational Horizon: } \hat{1}_{s\,gr} = (1-2M/r)\{1, 1, 1\} = 0, \qquad (9.5.35a)$$

$$\text{Gravitational singularities: } \hat{1}_{t\,gr} = (1 - 2M/r)^{-1} = 0. \qquad (9.5.35b)$$

But the above quantities have a sole dependence on the coordinates and, as such, they are insufficient to represent realistic interior conditions. It is at this point where the restriction of the isominkowskian spaces to a sole dependence on the local coordinates becomes manifestly unwarranted. In fact, the isometric is defined for an unrestricted functional dependence achieving the direct universality for interior conditions indicated earlier. We therefore write

$$\text{Gravitational horizons: } \hat{1}_s(r, \dot{r}, \ddot{r}, \mu, \tau, \omega, n, \dots) = \{ T_{11}^{-1}, T_{22}^{-1}, T_{33}^{-1} \} = 0,$$

$$\text{Gravitational singularities: } \hat{1}_t(r, \dot{r}, \ddot{r}, \mu, \tau, \omega, n, \dots) = T_{44}^{-1} = 0. \qquad (9.5.36)$$

Deeper insights in the problem of gravitational collapse, black holes and all that are then expected from explicit models of isogravitation.

Note that in the simplest possible cases the isotopies essentially provide *multiplicative factors* to conventional metrics, as expressed in Eq.s (11.9.4.3). Under a space–isotropy this essentially implies the recovering of known results, and their rescaling to different values due to the isotopic factor. In turn, such rescaling represents actual speeds of light different than that in vacuum. This and other aspects are left for detailed studied to the interested reader.

9.6: ISOCOSMOLOGY AND GENOCOSMOLOGY.

9.6.A: Basic postulates. As it is well known, current astrophysical observations cannot distinguish whether galaxies, quasars and other distant astrophysical bodies are made up of matter or antimatter. Also, antimatter has a positive–definite energy in current cosmologies. These occurrences therefore de-emphasize the need to study the distribution between matter and antimatter.

The isospecial and isogeneral relativities imply a novel cosmology, here called *isocosmology*, which is based on the following postulates:

Postulate I. *The Universe has equal amounts of matter and antimatter.*

This postulate requires that for every aggregate of galaxies, quasars and other bodies made up of matter in a given region of the Universe there is another aggregate of the same mass composed of antimatter in some other region.

Postulate II. *The Universe possesses an isodual structure with null total physical characteristics of energy, time, charge, linear momentum, angular momentum, etc., when computed in our space-time.*

This postulate first requires that the Universe is composed of a *matter warp* with positive-definite physical characteristics evolving forward in time and represented by the isoriemannian geometry, plus a hitherto unknown an*timatter warp* coexisting with our own warp with negative-definite physical characteristics, evolving backward in time and represented by the isodual isoriemannian geometry. The postulate then requires that total physical characteristics defined as the sum of the characteristics of all particles and antiparticles are null *when computed in our space-time.* More specifically, the latter requirement implies that the energy of antiparticles is projected in our space-time thus resulting to be negative-definite with respect to our unit +1. The assumption of an equal amount of matter and antimatter then implies null total energy, and the same situation occurs for all other physical characteristics.[129]

Postulate III. *The Universe is a closed system in stable-reversible conditions.*

This requires that matter (antimatter) possesses a structure describable by the isoriemannian (isodual Riemannian-isotopic) geometry without need for their genoriemannian coverings for the axiomatization of open-irreversible conditions.

Postulate IV. *The isoriemannian geometry, its isominkowskian reformulation and their isoduals apply everywhere and at all times.*

Recall that the isoriemannian geometry was constructed for *interior* gravitational problems. The above postulate therefore assumes that the entire Universe is an interior system, as plausible from the fact that space can be considered perfectly empty only in a local approximation, e.g., for planetary distances. At intergalactic distances space becomes an ordinary medium (thus activating the isoriemannian geometrization) because it is filled up with dust, electromagnetic waves, particles, etc.

Postulate V. *The Universe is invariant under the isosymmetry* $P(3.1) \times P^d(3.1)$, *where the isopoincaré symmetry applies for the matter warp, and the isodual isopoincaré symmetry applies for the antimatter warp.*

The latter postulate is the most dominant with novel, far reaching implications, such as the prediction of causal speeds higher than the speed of light in vacuum, reduction of the dimension of the measured Universe, prediction of antigravity, prediction of the space-time machine, prediction of a new form of subnuclear energy, and others.

Postulate V also implies that *the isocosmology provides a characterization of the Universe in a "flat" isospace* along the lines of Sect. II.9.5. In fact, as now familiar, the isopoincaré symmetry cannot be formulated in curved spaces or

---

[129] As indicated in App. II.1.C, A. K. Assis has shown that the total force on a body is null under the assumption that inertial forces are due to gravitational interactions with the rest of the Universe. Note that Assis' proof is valid in *each warp* of matter or antimatter.

isospaces. Note that the fundamental isosymmetry $\hat{P}(3.1) \times \hat{P}^d(3.1)$ is isoselfdual, that is, invariant under isoduality as the π°.

It is evident that the above postulates represent working hypotheses on *limit conditions* of equal distribution of matter and antimatter in stable-reversible conditions which are worth a study. Isocosmology can also be formulated for different distributions of matter and antimatter although preserving the closed–reversible conditions.

The novelty of the isocosmology should be pointed out. First, contemporary cosmologies are based on non–null total physical characteristics of energy, etc. (see ref. [8] and literature quoted therein). The assumption of null total physical characteristics is new because dependent on the new isodual geometries. Also, contemporary cosmologies have no universal symmetry at all. In particular, the structure of the universal symmetry of Postulate V is a novelty per se inasmuch as it is essentially Lie–isotopic. Yet another novelty is the "flat" character of the isogeometry while preserving a full representation of the Riemannian curvature as shown in the preceding section.

The broader *genocosmology* is based on the genoriemannian geometry and its isodual, and characterizes an *open Universe in irreversible conditions with an arbitrary distribution of matter and antimatter*. In this latter case the basic symmetry is given by the Lie–admissible genopoincaré symmetry and its isodual $<\hat{P}>(3.1) \times <\hat{P}>^d(3.1)$ (see Sect. 11.8.5). Total physical quantities are then conserved in genospaces, but their projections in our space–time are nonconserved.

The genocosmology has therefore rather deep implications, such as the prediction that *the Universe is not closed*. A number of cosmological studies of Lie–admissible character have already been conducted, although all in the conventional spaces, such as the studies by Gasperini [34], Jannussis [35], Gonzalez–Diaz [36] and others.

The mathematical foundations of the new cosmologies have been presented earlier. In this section we point our the essential additional elements.

9.6.B: Characteristic quantities of the Universe. The isominkowskian geometrization of physical media (Sect. 11.8.2) and their classification (Sect. 11.8.5.1) must now be extended to the entire Universe. This implies the validity of the basic postulates of the isospecial relativity, including the isodoppler law, the isoequivalence principle, etc.

The fundamental quantities of the new cosmology are the new units, the isounit $\hat{1}$ for the characterization of matter and its isodual $\hat{1}^d$ for the characterization of antimatter, which then imply the consequential liftings of products, fields, vector spaces, algebras, etc. The departure of $\hat{1}$ from the trivial unit $I = $ diag. $(I, 1, 1, I)$ represents the departure from the Riemannian geometrization of empty space caused by matter and similarly for antimatter.

The novelty of the generalization should be kept in mind. Traditionally, we enlarge our physical description by either increasing the dimension (as in the transition from the Euclidean to the Minkowski space), or by passing from a flat

to a curved space. The new cosmology is based on the *preservation* of the original dimension, flatness or curvature, and in the generalization instead of the basic unit of the theory.

When studying local conditions, such as the interior problem of a matter-quasar, the new units have the most general possible dependence studied earlier, e.g., $\hat{1}(x, \dot{x}, \ddot{x}, \mu, \tau, \omega, ...)$. Postulate III then implies that the new units are Hermitean, with $\hat{1}$ being positive-definite and $\hat{1}^d$ being negative-definite. When studying the Universe as a whole, the new units can be averaged to constants hereon denoted $\hat{1}^\circ > 0$ and $\hat{1}^{\circ d} < 0$.

Hermiticity implies that the new units can be diagonalized in the forms

$$\hat{1} = \text{diag.} \, (\, b^\circ_1{}^{-2}, b^\circ_2{}^{-2}, b^\circ_3{}^{-2}, b^\circ_4{}^{-2} \,), \quad b^\circ_\mu = \text{const.} > 0 \qquad (9.6.1a)$$

$$\hat{1}^d = - \, \text{diag.} \, (\, b^{\circ d}_1{}^{-2}, b^{\circ d}_2{}^{-2}, b^{\circ d}_3{}^{-2}, b^{\circ d}_4{}^{-2} \,), \quad b^{\circ d}_\mu = -b^\circ_\mu \, , \qquad (9.6.1b)$$

which are called the *characteristic constants of the Universe* for the matter and antimatter warps, respectively. A useful alternative formulation is given by $b^\circ_\mu = 1/n^\circ_\mu$, $b^{\circ d}_\mu = 1/n^{\circ d}_\mu$, which is hereon implied.

Since the Universe is not empty, but composed of intergalactic media filling up space plus matter filling up the interior of astrophysical objects, the average *speed of light in the Universe cannot be that in vacuum* c_0, and *it is given instead by* $c = c_0/n^\circ_4$ where $n^\circ_4 = 1/b^\circ_4$ is the average *index of refraction of the Universe* .

The need to generalize the index of refraction n°_4 to the space components has been studied in Ch. 11.8. This implies a necessary generalization of the Euclidean geometry with consequential novel notion of *isodistance* between two points x and y

$$D = [\, (x^1 - y^1)^2 b_1{}^2 + (x^2 - y^2)^2 b_2{}^2 + (x^3 - y^3)^2 b_3{}^2]^{1/2} \, . \qquad (9.6.2)$$

The isocosmology therefore implies that the conventional Euclidean notion for the computation of the distance of an astrophysical body is no longer applicable and must be replaced by expression (9.6.2). This implies different values of intergalactic distances between the isocosmology and the conventional cosmology.

A fundamental problem of the isocosmology is the identification of preliminary estimates of the characteristic constants b°_μ because they would provide important general features of the Universe. This problem will be studied in Vol. III.

9.6.C: Isotopic representation of the "missing mass". As it is well known, the conjecture of the "missing mass" in our Universe (see, e.g., ref. [34]) has been reason for considerable uneasiness in the physis community which has

stimulated several models avoiding the conjecture of a large unidentified mass. At any rate, recent astrophysical measures by Henriksen and Mamon [37] appear to disprove the need for a large "missing mass".

It is intriguing to note that the isocosmology can reduce the value of the "missing mass" or eliminate it altogether. This is a consequence of the *isotopic principle of mass-energy equivalence* (II.8.5.10),

$$E = M\,c^2 = M\,c_0{}^2\,b^\circ_4{}^2\,.\qquad(9.6.3)$$

To understand the issue, we must study first the difference between the global and local isominkowskian representations. By recalling Postulate IV, the total energy of the universe is *not* the familiar expression $E_{tot} = M_{tot}\,c_0{}^2$, but rather the form

$$E_{tot} = M_{tot}\,c_0{}^2\,b^\circ_4{}^2\,,\qquad(9.6.4)$$

Since b°_4 is expected to be smaller than 1 for intergalactic media (yet close to 1), this could suggest a total energy of the Universe *smaller* than currently estimated.

The fallacy of the above argument is due to the sole use of characteristic functions b°_4 for the *intergalactic space* without any consideration for the much bigger *interior* contributions.

In fact, in the transition to an interior problem, say, that of a matter–quasar, the situation is fundamentally different because we pass from an extremely rarefied medium to a medium of extremely large density. In this latter case, the *interior characteristic constant* b°_4 is bigger than 1, as indicated by all available experimental data (see Vol. III). As a result, *the isocosmology predicts that the total energy of the Universe is bigger than currently assumed.*

It then follows that *the missing mass–energy is merely characterized by the difference between the isotopic and conventional energy*

$$M_{miss} = E_{miss}\,/\,c_0{}^2 = M_{tot}\,(\,b^\circ_4{}^2 - 1\,)\,,\qquad(9.6.5)$$

Note that the same approach can be used for the *reduction* of the "missing mass", that is, for its identification part with dark matter and part via law (9.6.5).

Note finally that estimates of the total mass and of the "missing mass" would yield a numerical estimate of the characteristic quantity b°_4 according to the rule

$$b^\circ_4 = \{\,(\,M_{tot} + M_{miss}\,)\,/\,M_{tot}\,\}^{\frac{1}{2}}\,.\qquad(9.6.6)$$

In turn, such knowledge would permit the identification of the main dynamical characteristics of the Universe as in classification of Fig. II.8.5.I. The expectation

is that the Universe is an isominkowskian medium of Type 9.

9.6.D: Origin of the Universe. According to current cosmological views, the Universe originated in the "big bang" (see, e.g., ref. [34] and literature quoted therein), which essentially implies the "creation" of its very large, positive, total energy *literally from nothing*. This perspective is also altered by the new cosmology in various ways.

First, the new cosmology avoids the necessary "creation" of a large energy from nothing in a primordial explosion because the total energy of the Universe, *if equally composed of matter and antimatter*, can be indeed null when projected in a our space–time.

Perhaps deeper revisions are permitted by the complete isotopic interpretation of the cosmological redshift of galaxies and their possible reduction to at rest conditions studied in Vol. III via the use of the isodoppler law. In fact, the new cosmology permits (among other possibilities) a novel conception of the Universe as being unlimited with local creation of matter and antimatter resulting in galaxies generally at rest with respect to each other (or moving at low, thus ignorable speeds).

Note that the local creation of matter (only) is along the studies by Arp [38] for the representation of the cosmological quasar redshift. Arp's studies are then turned from a theory admitting the creation of matter from nothing, into a theory in which matter and antimatter are created in such a way to keep the original null total value.

But perhaps the deepest departures from conventional lines exist in cosmological considerations pertaining to time. To begin, the isocosmology predicts the creation of two opposing time evolutions, one for matter and another for antimatter warp, which are such to preserve the originally null time evolution prior to creation. Even when considering each individual warp, the conventional notion of, say, "age of the Universe" has no meaning for the isocosmology because time has a local quantity. The only notion which can be formulated is that of "average age" of matter in the Universe.

It is hoped the considerations of this chapter convey the viewpoint that, rather than having achieved terminal character, our knowledge of the structure of the Universe is at its first infancy.

9.7: SPACE–TIME MACHINE

The *time machine*, defined as a *closed time–like loop in the forward light cone*, has been studied by a number of authors (see, e.g., papers [39,40] and references quoted therein).

These studies have essentially shown that closed loops are *not* possible in

the Minkowskian space–time because of the time–like causality condition and other reasons. Studies [39,40] were therefore based on a complex generalization of space–time. The possibility of reaching closed time–like loops in our physical space–time was studied via quantum tunneling and other means.

In this section we shall review a novel formulation of the *space–time machine* submitted by this author [32] for closed loops in space–time under the time–like condition of causality. The proposed experimental verifications, which are feasible with current technology, will be studied in Vol. III jointly with indirect experimental support. The inability to have closed time–like loops in Minkowski space can be traced back to the very foundations of the special relativity, in particular, to: the basic interval and related causality condition

$$x \, x + y \, y + z \, z - t \, c_0^2 t < 0 ; \qquad (9.7.1)$$

the preservation of the same interval for both particles and antiparticles; and Einstein's *relativity of space and time* [3] according to the familiar expression of *time dilation* and *space contraction*

$$\Delta r' = \Delta r \left(1 - v^2 / c_0^2 \right)^{\frac{1}{2}}, \quad \Delta t' = \Delta t / \left(1 - v^2 / c_0^2 \right)^{\frac{1}{2}}, . \qquad (9.7.2)$$

The space–time machine is rendered theoretically conceivable by the covering isospecial relativity (Ch. II.8) in our physical environment without complex extensions because of the following features:

1) Line element (9.7.1) and related causality condition are generalized into the isotopic form for particles (where $c = c_0 b_4$)

$$[x \, b_1^2(t, r, ..) x + y \, b_2^2(t, r, ...) y + z \, b_3^2(t, r, ...) z - t \, c^2(t, r, ...) t \,]\hat{1} < 0 . \quad (9.7.3)$$

2) The isodual form is assumed for antiparticles

$$[-x \, b_1^2(t, r, ..) x - y \, b_2^2(t, r, ...) y - z \, b_3^2(t, r, ...) z + t \, c^2(t, r, ...) t \,]\hat{1}^d < 0. \quad (9.7.4)$$

3) Space and time coordinates have the isotopic behaviour with speed

$$\Delta r' = \Delta r \left(1 - v^k b_k^2 v^k / c^2 \right)^{\frac{1}{2}}, \quad \Delta t' = \Delta t / \left(1 - v^k b_k^2 v^k / c^2 \right)^{\frac{1}{2}}, \qquad (9.7.5)$$

and have the isounits for particles and antiparticles, respectively,

$$\hat{1}_s = \text{diag.} \left(b_1^{-1}, b_2^{-1}, b_3^{-1} \right), \hat{1}_t = b_4^{-1} , \qquad (9.7.6)$$

and corresponmding isoduals $\hat{1}^d{}_s = \text{diag.} \left(-b_1^{-1}, -b_2^{-1}, -b_3^{-1} \right), \hat{1}^d{}_t = - b_4^{-1}$. For instance, a particle in the exterior field of antimatter characterized via the *isodual Schwartzchild's geometry* in spherical coordinates has the space and time isodual isounits

$$\hat{1}_s^d = -\{ (1 - 2 M / r)^{1/2} \text{diag.} \{ I, I, I \}, \hat{1}_t^d = -(1 - 2 M / r)^{-1/2}, \ r > 2M. \quad (9.7.8)$$

Note also that an isoselfdual particle can reverse the sign of its time evolution via merely passing from the field of matter to that of antimatter (Sect. II.7.8.C). Note finally that, in our notation, the space and time coordinates do not change under isoduality, $M(x,\eta,R) \rightarrow M^d(x,\eta^d,R^d)$. Only their units change sign.

It is easy to see that the above main characteristics of the isospecial relativity permit the theoretical formulation of the space–time machine in our environment without complex extensions. The main idea is the conception of motion in isospace and isotime, rather than in conventional space and time. In fact, the isounits $(\hat{1}_s, \hat{1}_t)$ can be altered in value by a sufficient gravitational field of matter, and can then be reversed in sign via the field of antimatter, while the generalization of the *unit* of the invariant permits the preservation of causality.

The most elementary form of the space–time machines submitted in ref. [32] can be formulated as follows:

BASIC HYPOTHESIS 9.7.1: The "space-time machine" consists of an isoselfdual particle such as the π° which:

STEP 1: Starts at a given space point P;

STEP 2: Is immersed in a sufficiently intense gravitational field of antimatter at a time t which causes motion in space as well as motion backward in time, and then

STEP 3: Is immersed in a gravitational field of matter in a location and with intensity such to permit the return to the original space-time point via motion forward in time.

Note the inapplicability of conventional objections due to causality. In fact, such objections refer to motion backward in time under the tacit assumption that it occurs in our space–time and, thus, with respect to the unit +1. On the contrary, motion backward in time for the isodual isospecial relativity occurs in a fundamentally different space–time with a negative unit −1.

In fact, the causal time–like condition is preserved throughout the entire process,

$$\text{Step 1: } (x\, x + y\, y + z\, z - t\, c_0^2\, t)(+1) < 0,$$

$$\text{Step 2: } (-x\, n_1^{-2}\, x - y\, n_2^{-2}\, y - z\, n_3^{-2}\, z + t\, c_0^2\, n_4^{-2}\, t)\hat{1}^d < 0, \qquad (9.7.9)$$

$$\text{Step 3: } (x\, n_1^{-2}\, x + y\, n_2^{-2}\, y + z\, n_3^{-2}\, z - t\, c_0^2\, n_4^{-2}\, t)\hat{1} < 0.$$

Note that the above space–time machine implies the gravitational attraction for both fields of matter and antimatter, owing to the use of an isoselfdual test particle, in which case we only have the reversal of the sign of time. More general realizations via the use of a particle or an antiparticle and the reversal of both gravitation and time are left to the interested reader.

The above space–time machine is experimentally verifiable with current

technology, as studied in detail in Vol. III. Here we note that the first experimental resolution is that *whether we evolve in conventional time or in isotime* (Sect. II.9.G) which can be resolved by sending a suitable probe to Jupiter. The second experimental issue is *whether a particle in the gravitational field of antimatter (or viceversa) experiences a gravitational repulsion* which can be resolved via the tests of Sect. II.8.7. The existence of antigravity would then imply the necessary reversal of time for antiparticles.

THE SPACE–TIME MACHINE
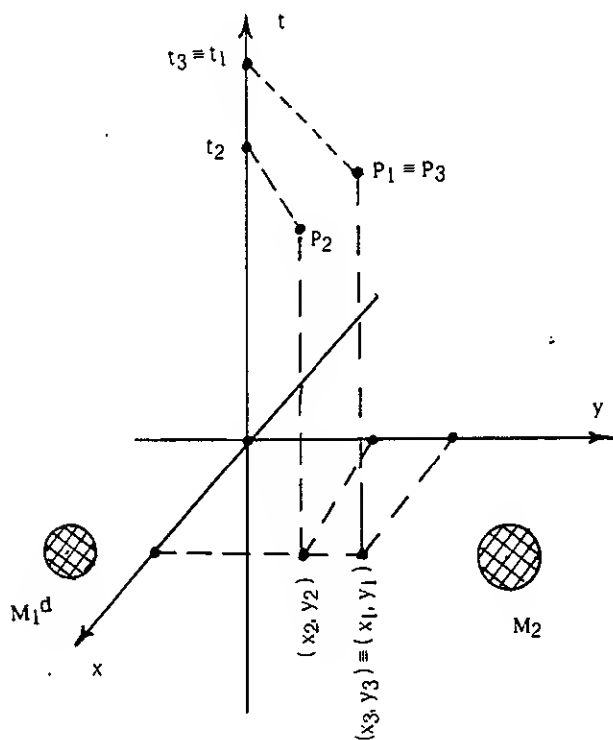
FIGURE 9.7.1: A schematic view of the simplest possible version of the proposed "space-time machine" via a neutral particle such as the π° which is expected to move backward in time when immersed in the gravitational field of antimatter M_1^d and then return to the original time via immersion in the gravitational field of matter M_2. In this simplest possible case motion in space is attractive under both fields because the π° is isoselfdual.

APPENDIX 9.A: AXIOMATIZATION OF THE ELLIS-MAVROMATOS-NANOPOULOS LIE-ADMISSIBLE MODEL OF BLACK HOLES

There is nowadays a general consensus that the structure of black holes requires a *nonunitary image of quantum mechanics* (see, e.g., the various contributions in proceedings [41]). The central issue is therefore the selection of such nonunitary image which: 1) is form–invariant under the most general possible nonunitary time evolution; 2) preserves the Hermiticity and therefore the observability at all times of operators representing physical quantities; and 3) can provide an axiomatic characterization of irreversibility in each of the four different arrows of time (see below).

In this volume we have shown that hadronic mechanics is indeed a nonunitary image of quantum mechanics satisfying all the above requirements. By comparison, other generalizations (such as the so–called q–deformations, nonlinear theories or generalizations via nonassociative envelopes) are generally unable to verify requirements 1), 2) and 3) for various reasons identified earlier.

In a recent paper J. Ellis, N. E. Mavromatos and D. V. Nanopoulos [42] have introduced a novel structure model of black holes based on non–critical string theory, here called *E-M-N black hole model*, which requires a structural generalization of quantum mechanics, the special and the general relativities because of its essential nonunitary and irreversible structure. The model is based on the earlier Lie–admissible formulations by this author (see, e.g., the review in ref. [13] of 1983) which permit the identification of the *origin* of the nonunitary, irreversible, Lie–admissible statistics by Misra, Prigogine and Courbage (Sect. II.7.9) at the ultimate level of the structure of matter, that at the level of elementary particles in open conditions.

The E-M-N black hole model does verify the necessary and sufficient conditions for a Lie–admissible structure, but it is formulated on conventional Hilbert spaces and fields, thus lacking a form–invariant structure, the Hermiticity–observability of physical quantities and an axiomatic characterization of irreversibility.

In this appendix we indicate that the E-M-N black hole model can be *identically* expressed in the formalism of hadronic mechanics with the consequential full achievement of the above properties. Moreover, we indicate that *the appropriate Lie–admissible structure already contains an irreversible operator version of gravitation which is hidden in the basic units of the theory.* These features evidently signal possible fundamental advances in the study of black holes which are here encouraged.

The E-M-N black hole model initiates with Hawking [43] abandonment of the conventional S–matrix formalism for the transformation of incoming into outgoing states

$$\rho_{\text{out }D}^{C} \neq S^{CB}_{DA} \rho_{\text{inc }B}^{A}, \qquad (9.A.1)$$

due to the lack of factorization of the (super) scattering matrix into the product of S and $S^\dagger$ elements

$$S^{CB}_{DA} \neq S^{C}_{A} (S^\dagger)^{B}_{D}. \qquad (9.A.2)$$

But the integration of the conventional Liouville equation yields the conventional S–matrix. The abandonment of the latter therefore implies a necessary modification of the Liouville equation which is written in ref. [42], Eq. (3), in the form

$$\partial_t \rho = i[\rho,H] + \mathcal{S}H\rho. \qquad (9.A.3)$$

Ref. [42] then proves that the extra term $\mathcal{S}H\rho$ is such to verify certain symmetric conditions for Lie–admissibility identified by this author back in 1978 [13].

The problematic aspects of the class of equations (9.A.3), *the Liouville equation with an external term,* when defined over a conventional Hilbert space $\mathcal{H}$ over a conventional field $C(c,+,\times)$, have been studied in detail in Ch. I.7, such as: a) lack of form–invariance under their own time evolution; b) general loss of the Hermiticity–observability of the original operators of the equation without the external term; c) lack of preservation of the Hermiticity–observability under time evolution; d) lack of a well defined enveloping algebra with consequential lack of uniqueness of the exponentiation and related physical laws (e.g., the uncertainties); e) loss of the measurement theory due to the lack of a well defined left and right unit in the envelope; f) violation by the bracket $(\rho, H) = \rho H - H\rho - i\mathcal{S}H\rho$ of the conditions to characterize any algebra (violation of the left scalar and distributive laws); g) loss of fundamental space–time symmetries, such as SU(2), SL(2.C), etc., jointly with the inability to replace them with covering symmetries; h) consequential loss of physical meaning of conventional notions, such as "particles with spin $\frac{1}{2}$", and others.

The E–M–N black hole model can avoid *all* the above problematic aspects via the *identical* reformulation of Eq. (9.A.3) in terms the axioms of the *Lie-admissible branch of hadronic mechanics* as identified in preceding chapters, thus leaving the main results of ref. [42] essentially unchanged.

The first step is the rewriting of Eq. (9.A.3) in the operator Lie–admissible form [12,13] according to the rules

$$-i \frac{\overset{<\partial>}{\rho}}{\overset{<\partial>}{t}} = (\rho, H) = \rho H - H\rho - i\mathcal{S}H\rho = \rho^{<}T H_0 - H_0 T^{>}\rho = \rho < H_0 - H_0 > \rho$$

$$^{<}T H_0 = H, \qquad H_0 T^{>} = H - i\mathcal{S}H, \quad H = H^\dagger, \ H_0 = H_0^\dagger, \qquad (9.A.4)$$

with formal solution

$$^\smile T \ = \ ^\triangleleft 1_t{}^\triangleright H\, H_0{}^{-1}, \quad T^\smile \ = \ ^\triangleleft 1_t{}^\triangleright H_0{}^{-1} (\, H \, - \, i\, \delta\, H \,), \qquad (9.A.5)$$

where: H_0 is generally assumed to be the original, Hermitean, conserved, Hamiltonian of the Liouville equation without external term; the nonhermitean *space genotopic elements* $^\triangleleft T$, $T^\triangleright$ and $^\triangleleft T\pm T^\triangleright$ are nonsingular; $^\triangleleft\partial^\triangleright / ^\triangleleft\partial^\triangleright t = {}^\triangleleft 1_t{}^\triangleright \partial/\partial t$ are the *genotime derivatives* with $^\triangleleft 1_t{}^\triangleright$ being the *time genounit* and $^\triangleleft T_t{}^\triangleright = (^\triangleleft 1_t{}^\triangleright$ being the *genotopic element*; the space genotopic elements are generally operators, while the time genotopic elements are generally complex functions; and we indicate hereon both directions of time with the unified notation $<>$ for simplicity with the understanding that only one of them can be used.

Reformulation (9.A.4) permits: A) the characterization of an algebra as commonly understood in mathematics by the brackets of the theory $(\rho, H) = \rho{<}H - H{>}\rho$ (which now verify the left and right scalar and distributive laws); B) the identification of two directions of time, *forward to future time* denoted with the symbol $>$ and *forward from past time* denoted with the symbol $<$; D) the identification of two unique enveloping operator genoalgebras, that for motion forward to future time $\xi^\triangleright$ whose elements are the same operators A, B, ... of the envelope ξ of the original equation without external term, although equipped with the genoassociative product $A{>}B = A T^\triangleright B$, $(A{>}B){>}C = A{>}(B{>}C)$, $T^\triangleright$ fixed, with conjugate envelope $^\triangleleft\xi$ with the same elements A, B, ... but new genoassociative product $A{<}B = A ^\triangleleft T B$, $^\triangleleft T$ fixed.

Despite the above advances, reformulation (9.A.4) is still basically insufficient to resolve all the problematic aspects indicated above when formulated over a conventional Hilbert space $\mathcal{H}$ over $C(c,+,\times)$. In fact, it is easy to see that H_0 as well as all other physical quantities do not preserve Hermiticity-observability under time evolution, thus preventing physical applications.

The resolution of the latter problematic aspects requires the identification of the new units of the theory, the *space genounit for motion forward to future time and forward from past time*

$$1^\triangleright = (\, T^\triangleright \,)^{-1} \ = \ ^\triangleleft T_t{}^\triangleright (\, H - i\, \delta\, H \,)^{-1} H_0, \quad {}^\triangleleft 1 \ = \ ^\triangleleft T_t{}^\triangleright (\, ^\triangleleft T \,)^{-1} \ = \ H_0\, H^{-1}, \quad (9.A.6)$$

which are independent from the *time genounits* $1_t{}^\triangleright$ and $^\triangleleft 1_t$ as indicated earlier.

It is easy to see that the generalized units $^\triangleleft 1$ and $1^\triangleright$ are the correct left and right units of their respective genoenvelopes $^\triangleleft\xi$ and $\xi^\triangleright$, i.e., $1^\triangleright{>}A = A{>}1^\triangleright \equiv A$ and, separately, $^\triangleleft 1{<}A = A{<}^\triangleleft 1 \equiv A$ for all possible elements A.

The next step is the *dual* generalization of the entire formalism of quantum mechanical into that of the covering hadronic mechanics for each of the two directions of time $<>$. The generalizations are essentially such to admit the genounits $^\triangleleft 1^\triangleright$ as the correct left and right generalized units. This includes the dual lifting of: fields; vector, metric and Hilbert spaces; algebras, groups and symmetries; geometries; functional analysis; etc., as studied in the preceding

chapters. We here merely recall the genofields $^<\hat{C}^>(^<\hat{c}^>,+,<>)$ with genonumbers $^<\hat{c}^> = c^<\hat{1}^>$ where c is complex and consequential generalized operations (Sect. I.7.4); the genohilbert spaces $^<\mathcal{H}^>$ with genostates $<|$ and $|>$, composition $<|^<T^>| >^<\hat{1}^> \in {}^<\hat{C}^>(^<\hat{c}^>,+,<>)$ and expectation value $<|^<T^>A^<T^>| >/<|{}^<T^>|>$ (Sect. I.7.8); the genoderivatives $^<\partial^>/^<\partial^>t = ^<\hat{1}_t^>\partial/\partial t$; etc. In this formalism the final expression of the Lie-admissible Liouville equation can be written

$$i \, {}^<\partial^> \rho / {}^<\partial^> t = (\rho, H) = \rho <H_{\hat{1}} - H>_{\rho_{\hat{1}}}>, \qquad (9.A.7)$$

where we have indicated for clarity the genounit for each genoproduct.

As proved earlier, when Eq. (9.A.7) is expressed on a genohilbert space $^<\mathcal{H}^>$ over a genofield $^<\hat{C}^>(^<\hat{c}^>,+,<>)$ it does indeed resolve all remaining problematic aspects identified earlier. In fact, the equation becomes invariant under its own time evolution, all operators which are originally Hermitean-observable in the absence of the external term, remain Hermitean-observable at all times; there is now a fully consistent *dual* measurement theory for motions $<>$ based on the corresponding genounits $^<\hat{1}^>$; etc.

Note the achievement of *Hermiticity-observability of the energy under its nonconservation* $i\partial_t H_0 = (H_0, H_0) = H_0({}^<T - T^>)H_0 \neq 0$, $H_0 = H_0^\dagger$. Note also that the conservation law $i\partial_t H_0 \equiv 0$ is readily admitted as the Lie-isotopic subcase for which $^<T = T^> = T = T^\dagger$.

Moreover, after Eq. (9.A.3) is reformulated in terms of Eq. s (9.A.7), the E-M-N black hole model admits two genotopic representations of gravitation, one per each direction of time, which are essentially embedded in the genounits $^<\hat{1}^>$ (see Sect. II.9.5 for details).

We should also mention that the alternative formulation used in Ref. [42], Eq. (148), i.e.,

$$\partial_t = -i(H\rho - \rho H^\dagger), \quad H \neq H^\dagger, \qquad (9.A.8)$$

possesses the *same* problematic aspects a)–h) of Eq. (9.A.3) (see again Ch. I.7 for details).[130]

Nevertheless, Eq. (9.A.8) can also be *identically* reformulated in the formalism of hadronic mechanics via the rules

$$^<\partial^>_t \rho = i(\rho H^\dagger - H\rho) = i(\rho <H_0 - H_0>\rho) = (\rho, H_0), \qquad (9..A.9a)$$

$$H = {}^<\hat{1}_T^> H_0 T^>, \quad H^\dagger = {}^<T H_0 \hat{1}_T^>, \quad H_0 = H_0^\dagger. \qquad (9.A.9b)$$

---

[130] For instance, the brackets are now the *triple system* $\partial_t\rho = i(\rho, H, H^T) = i(\rho H - H^\dagger\rho)$ evidently because H is now distinct from $H^\dagger$ which, as such, does not characterize an *algebra* as currently understood in mathematics. This implies that all Lie algebras, let alone SU(2), cannot be even formulated, with consequential lack of mathematical and physical significance of the spin as well as other problematic aspects.

In closing, the reader should be aware that, while a solution of Eq.s (9.A.2) does not exist in the conventional Hilbert space $\mathcal{H}$, it does indeed exist in the genospace $\hat{\mathcal{H}}^>$, yielding the *nonpotential scattering theory* studied in Ch. II.12. In fact, the forward integration of the Lie–admissible Liouville equation (9.A.4) implies the *forward genoscattering theory*

$$\rho_{out\ D}^{\ C} = \hat{S}^>_{DA}{}^{CB} > \rho_{inc\ B}^{\ A},$$

(9.A.10)

with the decomposition

$$\hat{S}^>_{DA}{}^{CB} = \hat{S}^>_A{}^C > (\hat{S}^{>\dagger})^B_D,$$

(9.A.11)

where the $\hat{S}^>$–matrix is now *genounitary* (Sect. I.7.78), i.e.,

$$\hat{S}^> > \hat{S}^{>\dagger} = \hat{S}^{>\dagger} > \hat{S}^> = \hat{1}^>,$$

(9.A.12)

with similar results for motion forward from past time (see Ch. II.12 for all details, including the invariance of the theory under the forward Lie–admissible genopoincaré symmetry $\hat{P}^>(3.1)$).

We finally recall that the isodual antiautomorphic maps $\hat{1}^> \to \hat{1}^{>d} = -\hat{1}^>$ and $\hat{1}_t^> \to \hat{1}_t^{>d} = -\hat{1}_t^>$ characterize two additional directions in time, the *backward from future time* $>^d = ->$ and the *backward toward past time* $<^d = -<$, with two consequential additional classes of structures, including: isodual genofields $\hat{C}^{>d}(\hat{c}^{>d},+,<>^d)$, isodual genospaces $\hat{\mathcal{H}}^{>d}$, etc. Since isoduality is equivalent to charge conjugation, the latter two structures represent antiparticles (see Ch. II.10).

In summary, the reformulation of the E–M–N black hole model in terms of the Lie–admissible branch of hadronic mechanics leaves the results of ref. [42] essentially unchanged while avoiding all the problematic aspects a)–h) identified earlier. Also, such reformulated model admits a covering scattering theory, contains in its generalized unit a representation of gravity, and achieves the axiomatic characterization of irreversibility in all four possible time arrows as proposed by this author via the four possible genoreal numbers[131]

$\hat{R}^<(\hat{t}^<,+,<)$		$\hat{R}^>(\hat{t}^>,+,>)$	
$^d\hat{R}^<(^d\hat{t}^<,+,^d<)$	0	$\hat{R}^{>d}(\hat{t}^{>d},+,>d)$	time

---
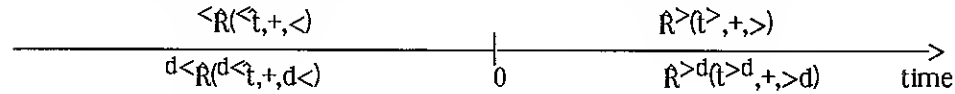
[131] The reader should be aware that, as indicated earlier, a full axiomatization of irreversibility requires the additional condition $T^> = (^<T)^\dagger$ which is necessary for the preservation of Hermiticity, Eq.s (II.3.3.8), the conjugation from motion forward from past time to future time, $> = <^\dagger$, and other reasons. In this way all four time arrows can be connected via Hermiticity and isoduality. As one can see from Eq.s (9.A.4.b), this latter condition is not verified by the E–M–N model [42]. Unlike the preceding reformulation, the verification of the condition $T^> = (^<T)^\dagger$ requires a structural revision of the model.

Note that the four possible times are given by $^{<}t^{>} = t^{<}1^{>}$ and $^{<}t^{>d} = t^{<}1_t^{d} = -^{<}t^{>}$ where the measured time t remains the conventional real one as in *Jannussis complex time* (Fig. 11.6.7.1). The axiomatization is characterized by the fundamental quantities of the theory, the time genounits $^{<}1_t^{>}$ and their isodual $^{<}1_t^{>d}$.

APPENDIX 9.B: ISOTOPIC UNIFICATION OF WEAK, ELECTRO-MAGNETIC AND GRAVITATIONAL INTERACTIONS

The *Lie-isotopic generalization of contemporary gauge theories*, here called *isogauge theories*, was initiated in two pioneering papers of M. Gasperini [44] of 1983. The isogauge theories were then studied by M. Nishioka [45], G. Karayannis, A. Jannussis and L. Papaloukas [46] and others. A review can be found in monograph [47], App. A. We are aware of no contributions available at this writing on the broader *Lie-admissible generalization of gauge theories*, or *genogauge theories*.

In this appendix we shall briefly review the main lines of the isogauge theory. We shall then point out realistic possibilities for a novel unified theory of weak, electromagnetic *and gravitational* interactions which is permitted by the embedding of gravitation in the generalized *unit* of the theory [27,33].

By following the original derivation by Gasperini [loc. cit.], consider a theory on an isohilbert space $\hat{\mathcal{H}}$ over an isofield $\hat{C}(\hat{c},+,*)$ with isotopic element T and isounit $\hat{1} = T^{-1}$ which possesses a global invariance under an n-dimensional, compact, connected, Lie-isotopic group $\hat{G}$. The isotransforms can be written

$$\hat{\psi}' = \hat{U} * \hat{\psi} = \hat{U}T\hat{\psi}, \tag{9.B.1}$$

where $\hat{U}$ is a isounitary operator

$$\hat{U} = \hat{e}^{-iX_k * \theta_k} = \{e^{-iX_k T\theta_k}\}\hat{1} = \hat{U}\hat{1} = \hat{U}^\dagger = \hat{U}^{-\hat{1}}, \tag{9.B.2}$$

$\hat{e}$ is the now familiar isoexponentiation, $\theta_k = \theta_k\hat{1}$ are the parameters and X_k are the generators of $\hat{G}$, k = 1, 2, ..., n.

From the isounitarity of $\hat{U}$ on $\hat{\mathcal{H}}$, the global $\hat{G}$-isoinvariance explicit reads

$$\hat{\psi}'^\dagger * \hat{\psi}' = \hat{\psi}^\dagger * \hat{U}^\dagger * \hat{U} * \hat{\psi} = \hat{\psi}^\dagger * \hat{\psi}. \tag{9.B.3}$$

In order to introduce the local invariance for $\theta = \theta_k(x)$ and T=T(x), Gasperini [loc. cit.] introduces the *isocovariant derivative*

$$\hat{D} = (\partial_\mu - i g A^k_\mu * X_k)1 = \hat{D}_\mu 1, \qquad (9.B.4)$$

where A^k_μ are conventional gauge potentials, with isotransforms

$$\hat{D}'_\mu = \hat{U} * \hat{D}_\mu * \hat{U}^{-1}, \qquad \hat{D}'_\mu * \hat{U} * \hat{\psi} = \hat{U} * \hat{D}_\mu * \hat{\psi}, \qquad (9.B.5a)$$

$$A'^k_\mu * X_k = \hat{U} * A^k_\mu * X_k * \hat{U}^{-1} - i(\partial_\mu \hat{U}) \hat{U}^{-1} g^{-1}. \qquad (9.B.5b)$$

By using the infinitesimal forms

$$\hat{U} \cong 1 - i \epsilon^k * X_k, \qquad \hat{U}^{-1} \cong 1 + i \epsilon^k * X_k, \qquad (9.B.6)$$

one gets to first-order the variations

$$\delta A^k_\mu * X_k = g^{-1} \partial_\mu (\epsilon^k T X_k) + i [A^k_\mu * X_k, \epsilon^i * X_i], \qquad (9.B.7)$$

which can be written

$$A^k_\mu T X_k = - g^{-1} (\partial_\mu e^i T) X_i + i A^j_\mu \epsilon^k T [X_j \hat{,} X_k], \qquad (9.B.8)$$

where the expressions

$$[X_i \hat{,} X_k] = X_i T X_k - X_k T X_i = \hat{C}_{jk}^i * X_i, \qquad (9.B.9)$$

are the isocommutation rules of $\hat{G}$.

Gasperini finally defines the *isotopic Yang-Mill strength* $\hat{F}_{\mu\nu}$ via the expressions

$$\hat{F}_{\mu\nu} * \hat{\psi} = i g^{-1} [\hat{D}_\mu \hat{,} \hat{D}_\nu] * \hat{\psi}, \qquad (9.B.10)$$

which transforms covariantly under the isosymmetry $\hat{G}$ with explicit rules

$$F^k_{\mu\nu} * X_k = (\partial_\mu A^k_\nu - \partial_\nu A^k_\mu) * X_k +$$

$$+ A^k_\alpha (\delta^\alpha_\nu \partial_\mu T - \delta^\alpha_\mu \partial_\nu T) X_k - i g A^j_\mu A^k_\nu T [X_j \hat{,} X_k]. \qquad (9.B.11)$$

For the particular case in which T (ordinarily) commutes with all X_k, Gasperini [loc. cit.] identifies a structure which, in the language of these volumes, is that characterized by the *Klimyk rule* (1.4.7.34), and essentially implies the isorenormalization of the coupling constant

$$g \rightarrow \tilde{g} = g T. \qquad (9.B.12)$$

As one can see, the isotopic lifting of the gauge theory, particularly in its

general form, is not trivial because the gauge potentials are now coupled to the isotopic element T, thus implying a structural generalization of the entire theory.

The unified gauge theory of weak electromagnetic and gravitational interactions suggested by this author [33] for study is essentially given by: 1) the factorization of an arbitrary Riemannian metric $g(x) = T_{gr}(x)\eta$, where η is the conventional Minkowski metric; 2) the identification of the isotopic element $T(x)$ of the isogauge theory with the gravitational isotopic element $T_{gr}(x)$ and consequential coupling to the gauge potential of the weak and electromagnetic interactions indicated earlier; and 3) the formulation of the theory via the isoderivatives of Sect. II.8.4.A.

Note the abstract identity of conventional and isotopic unified gauge theories, particularly when the latter are formulated in terms of the isoderivatives, yet with the full inclusion of the term truly representing curvature, the isotopic element $T_{gr}(x)$.

REFERENCES

1. B. RIEMANN, Gött. Nachr. **13**, 133 (1868) and *Collected Works*, H. Weber, Editor, Dover, New York (1953)
2. A. EINSTEIN, Ann. Phys. **49**, 769 (1916)
3. J. STACHEL, Editor, *Collected Papers of Albert Einstein*, Princeton Univ. Press, Vol.s 1-5 (1987 on), Princeton, N.J., USA
4. K. SCHWARTZSCHILD, Sitzber. Deut. Akad. Wiss. Berlin. Kl. Math.–Phys. Tech., 189 (1916).
5. K. SCHWARTZSCHILD, Sitzber. Deut. Akad. Wiss. Berlin. Kl. Math.–Phys. Tech., 424 (1916)
6. P. FREUD, Ann. Math. **40**, 417 (1939)
7. W. PAULI *Theory of Relativity*, Pergamon Press, New York, 1958.
8. C. W. MISNER, K. S. THORNE and J. A. WHEELER, *Gravitation*, Freeman, San Francisco, USA (1970)
9. D. LOVELOCK and H. RUND, *Tensors, Differential Forms and Variational Principles*, Wiley, New York, 1975.
10. P. G. BERGMANN, *Introduction to the Theory of Relativity* (with *Preface* by Albert Einstein), Dover Publ., New York, 1942.
11. C. MØLLER, *Theory of Relativity*, Oxford University Press, Oxford, 1972.
12. R. M. SANTILLI, *Foundations of Theoretical Mechanics*, Vol. I: *The Inverse Problem in Newtonian Mechanics* (1978), Springer–Verlag, Heidelberg , Germany
13. R. M. SANTILLI *Foundations of Theoretical Mechanics*, Vol. II: *Birkhoffian Generalization of Hamiltonian Mechanics* (1982), Springer–Verlag, Heidelberg , Germany

14. R. M. SANTILLI, in *Proceedings of the International Conference on Quantum Statistics and Foundational Problems of Quantum Mechanics*, Hadronic J. Suppl. 1 (1985), 662–685.

15. G. F. WEISS, *Scientific, Ethical and Accountability Problems in Einstein's Gravitation*, Andromeda, Bologna, 1990.

16. R. M. SANTILLI, Ann. Phys. 83, 108 (1974)

17. D. F. LOPEZ, in Proceedings of the XIII International Conference on High Energy Physics and Field Theory, A. A. Logunov et al Editors, IHEP, Protvino, Russia (1994); and Hadronic J. 16, 429 (1993)

18. A. A. LOGUNOV, Hadronic J. 15, 407 (1982)

19. A. A. LOGUNOV and M. A. MESTVIRISHVILI, Progr. Theor. Phys. 74, 31 (1985)

20. A. A. LOGUNOV and M. A. MESTVIRISHVILI *The Relativistic Theory of Gravitation*, Mir Publ., Moscow, Russia (1989)

21. H. YILMAZ, Phys. Rev. 111, 1417 (1958)

22. H. YILMAZ, Phys. Lett. 27, 1399 (1971)

23. H. YILMAZ, Hadronic J. 2, 1186 (1979); 3, 1478 (1980); 7, 1 (1984); 11, 179 (1988)

24. H. YILMAZ, in *Space–time symmetries – Wigner Symposium* (Y. S. Kim and W. W. ZACHARY, Editors), North–Holland, Amsterdam, 1989, 373–410.

25. H. YILMAZ, Hadronic J. 12, 305 (1989)

26. H. RUND, *Algebras, Groups and Geometries* 8, 267 (1991)

27. R. M. SANTILLI, Hadronic J. Suppl. 4B, issue no. 4 (1988)

28. R. M. SANTILLI, *Isotopic Generalization of Galilei's and Einstein's Relativities*, Vol. 1: *Mathematical Foundations*, Second Edition, Academy of Sciences of Ukraine, Kiev, Ukraine, in press

29. R. M. SANTILLI, *Isotopic Generalization of Galilei's and Einstein's Relativities*, Vol. II: *Classical Isotopies*, Second Edition, Academy of Sciences of Ukraine, Kiev, Ukraine, in press;

30. M. GASPERINI, *Hadronic J.* 7, 650 and 971 (1984), and 8, 52 (1985)

31. R. M. Santilli, Hadronic J. 17, 257 (1994)

32. R. M. SANTILLI, Hadronic J. 17, 285 (1994)

33. R. M. SANTILLI, in *Proceedings of the Seventh Marcel Grossmann Meeting on General Relativity*, M. Keiser et al., Editors, World Scientific, Singapore, in press

34. R. M. SANTILLI, Chine J. Syst. Eng. ~ Electr. 6, issue no. 4 (1995)

35. A. JANNUSSIS and I. TSOHANTJIS, in *Hadronic Mechanics and Nonpotential Interactions* (M. Mijatovic, Editor), Nova Science, New York, 1990.

36. P. F. GONZALEZ–DIAZ, *Hadronic J.* 9, 199 (1986)

37. M. J. HENRIKSEN and G. A. MAMON, Astrophys. J. 421, L63 (1994)

38. H. ARP, in *Frontiers of Fundamental Physics* (F. Selleri and M. Barone, Editors), Plenum, New York (1994)

39. D. DEUTCH, Phys. Rev. D 44, 3197 (1991)

40. L.–X. LI, Phys. Rev. D 48, 4735 (1993)

41. M. KEISER et al, Editors, *Proceedings of the Seventh Marcel Grossmann*

Meeting on General Relativity, M. Keiser et al., Editors, World Scientific, Singapore, in press

42. J. ELLIS, N. E. MAVROMATOS and D. V. NANOPOULOS, CERN preprint TH.7195/94 (revised version), and in *Erice Summer School, 31–st Course: From Superstrings to the Origin of Space–time*, Erice, July 1993, World Scientific, Singapore

43. S. HAWKINS, Comm. Math. Phys .**87**, 395 (1982)

44. M. GASPERINI, Hadronic J. **6**, 935 and 1462 (1983)

45. M. NISHIOKA, Lettere Nuovo Cimento **40**, 309 (1984); Nuovo Cimento A **85**, 331 (1985); Hadronic J. **7**, 1636 (1984); **9**, 87 (1986); **10**, 255 (1987)

46. G. KARAYANNIS, A. JANNUSSIS and L. PAPALOUKAS, Hadronic J. **7**, 1342 (1985); G. KARAYANNIS, Lettere Nuovo Cimento **43**, 23 (1985), and "Lie–isotopic Lifting of Gauge Theories", Ph. D. Thesis, University of Patras, Greece (1985); G. KARAYANNIS, and A. JANNUSSIS, Hadronic J. 9, 203 (1986)

47. A. K. ARINGAZIN, A. JANNUSSIS, D. F. LOPEZ, M. NISHIOKA and B. VELJANOSKI, *Santilli's Lie–isotopic Generalizations of Galilei's and Einstein's Relativities*, Kostarakis Publisher, Athens, Greece (1991)

48. R. M. SANTILLI, Isotopic unification of gravitation and relativistic quantum mechanics and its universal isopoincaree symmetry, in *Gravitation, Particles and Space–Time*, (Ivanenko Memorial Volume), P. 1. Pronin and J. V. Sardanashvily, Editors, World Scientific, Singapore (1996)

NOTES ADDED IN THIS SECOND EDITION

The universal isopoincaré symmetry for gravitation is also studied in

R. M. SANTILLI, J. Moscow Phys. Soc., **3**, 255 (1993),

The isotopic representation of gravitational syngularities is studied in

R. M. SANTILLI, Comm. Theor. Phys. **3**, 47 (1994),

and the isodualk representation of antiparticles is studied in

R. M. SANTILLI, Comm. Theor. Phys. **3**, 153 (1994.

10: ISOTOPIES, GENOTOPIES AND ISODUALITIES OF RELATIVISTIC FIELD EQUATIONS

10.1: STATEMENT OF THE PROBLEM

In this chapter we study the isotopies and isodualities of conventional relativistic equations (see, e.g., ref.s [1,2] and quoted references), including topics such as: the representation of antiparticles via isodual spaces [3,4]; the relativistic equations for particles and antiparticles fully immersed within a hadronic medium [5,6]; the isospinorial covering $\mathscr{P}(3.1) = S\hat{L}(2.\hat{C}) \times \hat{T}(3.1)$ of the isopoincaré symmetry $\hat{P}(3.1)$ and its isodual [7]; the so-called "Dirac's generalization of Dirac's equation" [8,9] and others. For a review of the basic methods one may consult the recent monograph [10].

The notion of antiparticle was historically born from the negative-energy solutions of relativistic equations (see, e.g., ref.s [1,2] for general lines and the historical paper [11]). As an example, the familiar *Klein-Gordon equation* for a charged particle under an electromagnetic field can be written

$$(i \partial_t - e A)^2 \psi(x) = [(- i c_0 \nabla - e A)^2 + m^2 c_0^4] \psi(x), \quad (10.1.1)$$

thus admitting solutions with both *positive-definite and negative-definite energies*

$$i \partial_t \psi(x) = \pm [(- i \nabla - e A)^2 + m^2 c_0^4]^{\frac{1}{2}} \psi(x) = \pm E \psi(x). \quad (10.1.2)$$

These solutions exist already at the semiclassical level where they cause no problem because of the continuity of the transformation of the energy under which a state with positive energy remains such.

However, in the transition to quantum field theories, the negative-energy solutions caused significant problematic aspects because of the possible discrete exchanges of energy and the finite transition probability between positive- and negative-energy solutions. Also, the negative-energy solutions had negative-

definite probability densities at small distances, which are contrary to the basic axioms of quantum mechanics.

These difficulties forced the construction of rather artificial models, which have never been accepted as final by the entire scientific community, such as the conjecture of the existence of many different infinite seas of negative-energy states with infinite density, one per each antiparticle, whose "holes" represent antiparticles [1,2], and others. One can see the uneasiness caused by these theories, e.g., because they are afflicted by the same criticism moved against the ether theory, lack of drag effects for matter moving in these infinitee seas.

In this chapter we study a fundamental hypothesis of hadronic mechanics whose quantitative study is permitted by the isotopic techniques, *the characterization of antiparticles via isodual numbers, isodual spaces, isodual symmetries, etc.*

The main contention is that the preceding difficulties are due to the fact that the negative-energy solutions are treated with the same methods as those for ordinary particles, that is, antiparticle are assumed to exist in the same space of ordinary particles and are therefore referred to the *same basic unit* +1. If, on the contrary, antiparticles are assumed to exist in a space different than that of particles, specifically build for their description via *isoduality*, then particles and antiparticles are referred to *different units of opposite signs* in which case all conventional difficulties are resolved without need of infinite seas of undetectable states.

These considerations are referred to particle in the ordinary *exterior relativistic problem in vacuum*. Our first studies will therefore be conducted via the isodual image of *conventional* quantum mechanics, or *isodual quantum mechanics* studied in Ch.s II.3 and II.4. The transition to interior problems of antiparticles, and their study via the isodual hadronic mechanics will be considered thereafter.

On historical profiles, the hypothesis that antiparticle have negative-definite energy and move backward in time is, by no means, new because it dates back to the time of the discovery of antiparticles (Stueckelberg and others). The use of the same space-time for both particles and antiparticles can be seen beginning from the historical papers by Dirac, Heisenberg, Kramers and others [11]. Recent studies have also shown the necessity of using negative masses for antiparticles within a conventional relativistic framework *even when represented in the same space-time of particles* (see the comprehensive presentation by Recami and Ziino [12]).

In short, the novelty of this chapter is the representation of antiparticles in a new space-time distinct from our own, first, with negative-definite units for exterior conditions in vacuum and, second, with arbitrary negative-definite units for interior conditions.

The second topic studied in this chapter is the isotopies of relativistic equations which constitutes a central methodological tool for the applications of

hadronic mechanics of Vol. III.

Recall that Dirac [1] conceived his celebrated equation for the representation of the electron in the structure of the hydrogen atom or under *external* electromagnetic interactions at large. Most importantly, Dirac's equation *does not* describe the hydrogen atom as a closed-isolated bound state.

Because of these structural features, the *isotopies of Dirac's equation,* called *isodirac equation* for short, are ideally suited for the representation of an electron, this time, *under external, short range, nonlinear-nonlocal-noncanonical interactions due to full immersion within a hadronic medium.*

The conventional Dirac equation represents an electron when moving in the homogeneous and isotropic vacuum under action-at-a-distance interactions. On the contrary, the isodirac equation represents the same electron when immersed in the hyperdense, inhomogeneous and anisotropic medium in the interior of hadrons.

More generally, hadronic mechanics implies that *Dirac's equation describes a particle under external electromagnetic interactions, while the isodirac equations describes the same particle under external strong interactions.*

As we shall see, the isotopies of Dirac's equations imply a necessary *alteration of the intrinsic characteristics of the particle* called *mutation* [5]. This fundamentally novel characteristic of hadronic mechanics is physically due to the deformation of the structure of an elementary particle when immersed within a hadronic medium. Mathematically, the mutations are due to the isotopic deformations of the Casimir invariant of the Poincare' symmetry of Ch. 11.8.

The third topic studied in this chapter is the isotopies of the spinorial covering of the Poincaré symmetry $\mathcal{P}(3.1) = S\hat{L}(2.\hat{C})\times\hat{T}(3.1)$ as well as their isoduals $\mathcal{P}^d(3.1) = S\hat{L}^d(2.\hat{C})\times\hat{T}^d(3.1)$ which are characterized in a natural way by the isotopies of Dirac's equation, as it occurs in the conventional case.

The transition from *particles to isoparticles* is therefore characterized by the lifting $\mathcal{P}(3.1) \rightarrow \hat{\mathcal{P}}(3.1)$. The mutation of the intrinsic characteristics is then a mere consequence. The transition from *antiparticles to isoantiparticles* is then characterized by the isodual lifting $\mathcal{P}^d(3.1) \rightarrow \hat{\mathcal{P}}^d(3.1)$ with similar consequences.

The interpretation of antiparticles via isodual spaces and symmetries was first submitted by this author in memoir [3] of 1989, and then treated in more detail in the recent paper [4].

The hypothesis of the mutation of the electron under interior condition was first submitted by this author in the original proposal to build hadronic mechanics, ref. [5], Sect. 4.20, pp. 798-806, where the mutated electron was called *eleton*. The notion was introduced via a generalization of Dirac's equation due to the addition of variationally nonselfadjoint (nonpotential) interactions which are known to be velocity- dependent, e.g., of the type

$$(\gamma^\mu p_\mu + i m) \psi(x) = 0 \rightarrow (\gamma^\mu p_\mu + \Gamma^\mu p_\mu + i m) \hat{\psi}(x) = 0, \quad F_{NSA} = \Gamma^\mu p_\mu.$$
$$(10.1.3)$$

where the Γ-quantities, in their simplest possible form, can be assumed as scalar multiples of the γ-matrices, e.g., $\Gamma^\mu = f(x, \dot{x}, ...) \gamma^\mu$.

The main point of ref. [5] is that the generalization of Dirac's equation with nonpotential-nonlagrangian, velocity-dependent interactions implies a *necessary alteration of Dirac's gamma matrices* $\gamma_\mu \to \hat{\gamma}^\mu = \gamma^\mu + \Gamma^\mu$, which, in turn, implies a *necessary* alteration of the *intrinsic* characteristic of particles.

By recalling that any interaction implies a renormalization, we can say that the above mutations are *isorenormalizations* originating from the *nonlagrangian* character of the interactions which, as we shall see, are fully confirmed by the isotopies of $\mathcal{P}(3.1)$.

A notion of fundamental relevance for the applications of hadronic mechanics to nuclear and particle physics, which was submitted in ref. [5], p. 803, Eq.s (4.20.16), is the *mutation of the intrinsic magnetic and electric moments of the electron in interior conditions*

$$\mu_k \approx -(1+f)\frac{e\hbar}{2mc_o}\sigma_k, \quad \hat{m}_k \approx i(1+f)\frac{e\hbar}{2mc_o}\sigma_k. \qquad (10.1.4)$$

These mutations essentially express the expectation that, when immersed in a hyperdense hadronic medium, the particles experiences a deformation of their charge distributions, as necessary from the inhomogeneity and anisotropy of the medium itself. In turn, such a deformation implies a consequential, necessary mutation of the magnetic and electric moments as requested from classical electrodynamics. As we shall sew in Vol. III, these properties permit the achievement of the first exact-numerical representation on record of the total magnetic moments of few body nuclei which have not been exactly represented via conventional quantum mechanics despite over half a century of attempts.

More general mutations of intrinsic characteristics then permit additional fundamentally novel applications also studied in Vol. III, such as: a novel structure model of unstable hadrons as the chemical synthesis of ligther hadrons generally identifiable in their decay with the lowest mode (which therefore becomes a tunnel effect of their constituents) [13]; the "construction" of fractional charges for quarks, when assumed as composite under hadronic mechanics (which are notoriously extraneous to $\mathcal{P}(3.1)$ but rather natural for the covering isosymmetry $\hat{\mathcal{P}}(3.1)$); the achievement of a quantitative numerical representation of the *attractive* interaction among *two identical electrons* in the Cooper pair in superconductivity [14]; and other applications.

In this introductory section it is important to recall that *the first isotopic generalization of Dirac's equation was introduced by Dirac* [8,9] *himself* in two his last and little known papers of 1971–1972, evidently without his awareness that the structure had an essential isotopic character (in fact, the isotopies were introduced six years later).[132] In the Sect. II.10.7 we shall therefore review

---

[132] At the time of writing memoir [5], this author was unaware of Dirac's papers [8,9] and,

"Dirac's generalization of Dirac's equation", and point out its essential isotopic structure.

As we shall see, with his notoriously brilliant mind, Dirac selected one of the most general possible isotopies, that characterized by a nondiagonal isotopic element T, with a quite intriguing form of degeneracy.

The historical result achieved by Dirac in papers [8,9] is the proof that, in the transition from his conventional to the generalized equation, *the total angular momentum of the electron performs the transition from half-odd-integer values to the null value in the ground state*

$$J^{Tot}{}_{QM} = L + \tfrac{1}{2} \quad \rightarrow \quad J^{Tot}{}_{HM} = 0, \tag{10.1.5}$$

which confirms a fundamental property of the addition of angular momentum and spin of hadronic mechanics (Sect. II.6.12). As we shall see in Vol. III, this property is at the foundation of the chemical synthesis of unstable hadrons from ligther hadrons. The novel hadronic technology studied in Vol. III is based on the above alteration of intrinsic characteristics of particles when in interior conditions.

In closing, we would like to indicate that one of the most interesting studies of Dirac's equation currently available is that by the Tartu School in nonassociative algebras initiated by L. Sorgsepp and continued by J. Lõhmus, E. Paal and others (see their recent monograph [10]) via quaternions and sedenions. We regret the inability of reviewing this approach at this time because it is readily set for its isotopic generalization via the isoquaternions of Sect. 1.2.7. We also regret to be unable to review other approaches for brevity, such as that via Clifford algebras, which can also be isotopically lifted with rather intriguing possibilities.

---

for that reason, they were not quoted. Papers [8,9] were brought to the author's attention by A. Kalnay from Argentina in the occasion of the *First Workshop on Hadronic Mechanics* held at the Institute for Basic Research (then in Cambridge, MA) in 1983. The isotopic structure of "Dirac's generalization of Dirac's equation", which is evident from a mere inspection of the papers, was identified for the first time at that meeting (it is regrettable that A. Kalnay was forced to terminate his research in the field after that meeting, even though he was on the edge of perhaps the most important discoveries of his research life). As we shall see in Vol. III, Dirac's generalization of Dirac's equation has a number of truly intriguing applications, such as a rather unique treatment of the two identical electrons in the Cooper pair in superconductivity with *attractive* interactions, an important role in the construction of unstable hadrons via the chemical synthesis of ligther hadrons, and others.

10.2: EQUIVALENCE OF ISODUALITY AND CHARGE CONJUGATION

As it is well known (see, e.g., ref. [2], pp. 109–110), the *Klein–Gordon equation* for a spin zero particle of charge e under an external electromagnetic field with potential A_μ in Minkowski space $M(x,\eta,R)$), η = diag. (1, 1, 1, –1), $\hbar$ = 1, c_o = 1, ∂_μ = $\partial/\partial x^\mu$,

$$\{ (\partial_\mu + i e A_\mu)(\partial^\mu + i e A^\mu) - m^2 \} \psi(x) = 0 , \qquad (10.2.1)$$

under *charge conjugation*

$$e \to - e , \qquad C \psi(x) = c \psi^\dagger(x), \qquad |c| = 1, \qquad (10.2.2)$$

changes into the form

$$\psi^\dagger(x) \{ (\partial_\mu - i e A_\mu)(\partial^\mu - i e A^\mu) - m^2 \} = 0 , \qquad (10.2.3)$$

while the Lagrangian density

$$\mathcal{L} = - \tfrac{1}{2} \{ (\partial^\mu \psi^\dagger - i e A^\mu \psi^\dagger)(\partial_\mu \psi + i e A_\mu \psi) + m^2 \} , \qquad (10.2.4)$$

is invariant, and the four–current

$$j_\mu = (2 i m)^{-1} [\psi^\dagger \partial_\mu \psi - (\partial_\mu \psi^\dagger) \psi] + e A_\mu \psi^\dagger \psi / m , \qquad (10.2.5)$$

changes sign.

Similarly, let us consider the *Dirac's equation* for a particle of spin 1/2 also under an external electromagnetic field with potential A_μ in Pauli's representation (see [loc. cit.], pp. 176–177),

$$\{ \gamma^\mu [p_\mu - e A_\mu(x) / c_o] + i m \} \psi(x) = 0 , \qquad (10.2.6a)$$

$$\{ \gamma^\mu , \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 \eta^{\mu\nu} , \qquad (10.2.6b)$$

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad \gamma^4 = i \begin{pmatrix} 1_s & 0 \\ 0 & -1_s \end{pmatrix} , \quad 1_s = \text{diag. } (1, 1) \qquad (10..2.6c)$$

Under charge conjugation

$$e \to - e , \qquad C \psi = c S_C^{-1} \bar\psi , \qquad |c| = 1,$$

$$S_C \gamma_\mu S_C^{-1} = - \gamma_\mu{}^t , \qquad \bar\psi = \psi^\dagger \gamma_4 , \qquad (10.2.7)$$

Eq. (10.2.6) transforms into the equation

$$\bar{\psi}(x) \{ \gamma^{\mu} (\partial_{\mu} + e A_{\mu}) - i \ m \} = 0 , \qquad (10.2.8)$$

while the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2} \{ \bar{\psi} \ [\gamma^{\mu} (\partial_{\mu} + i e A_{\mu}) + m] \psi - [\bar{\psi} (\partial_{\mu} - i e A_{\mu}) + m] \psi \}, \qquad (10.2.9)$$

changes sign and the charge current

$$j_{\mu} = \bar{\psi} \ \gamma_{\mu} \psi , \qquad (10.2.10)$$

remains the same.

We now study the behaviour of the above equations under isoduality.

Theorem 10.2.1 [4]: *Isoduality is equivalent to charge conjugation.*

Proof. Recall from Vol. 1 that the field of real numbers $R(n,+,\times)$ is mapped by isoduality into the *isodual isofield* $R^d(n^d,+,\times^d)$ with *isodual real numbers* $n^d = -n$, *isodual multiplication* $n_1^d \times^d n_2^d = -n_1 \times n_2$, *isodual quotient* $n_1^d /^d n_2^d = -n_1/n_2$, *isodual norm* $| n |^d = -| n | < 0$, etc. (see Sect. 1.2.5 for details). Thus, $e^d = -e$, $A_{\mu}^d = -A_{\mu}$, $F_{\mu\nu}^d = -F_{\mu\nu}$, etc.

The field of complex numbers $C(c,+,\times)$ is mapped into the *isodual field* $C^d(c^d,+,\times^d)$ with *isodual complex numbers* $c^d = -\bar{c}$, isodual norm $| c^d |^d = -| c |$, isodual multiplication $c_1^d \times^d c_2^d = -\bar{c}_1 \times \bar{c}_2$, isodual quotient $c_1^d /^d c_2^d = -\bar{c}_1/\bar{c}_2$, etc. (see Sect. 1.2.6 for details). Note that $i^d = -\bar{i} = i$, that is, the imaginary unit is *isoselfdual* (invariant under isoduality).

The Minkowski space $M(x,\eta,R)$ is mapped into the *isodual Minkowski space* $M^d(x,\eta^d,R^d)$, $\eta^d = -\eta$, with *isodual isounit* $1^d = -$ diag. (1, 1, 1, 1) and *isodual separation*

$$x^{2 \ d} = (x^{\mu} \eta^d_{\mu\nu} x^{\nu}) 1^d \in R^d(n^d,+,\times^d) , \qquad (10.2.11)$$

which *coincides* with the conventional separation. Similarly, the basic second-order invariant becomes

$$\{ (p_{\mu} \eta^{\mu\nu} p_{\nu} + m^2) 1 \}^d =$$
$$= (p_{\mu}^d \times^d \eta^{\mu\nu d} \times^d p_{\nu}^d + m^{d \ 2 \ d}) 1^d = p_{\mu} \eta^{\mu\nu} p_{\nu} + m^2 = 0 . \qquad (10.2.12)$$

The *isodual derivatives* on $M^d(x,\eta^d,R^d)$ are given by $\partial^d_{\mu} = \partial^d /^d \partial^d x^{\mu} = -\partial_{\mu}$.

The Hilbert space $\mathcal{H}$ with states $| \psi >, | \phi >, ...,$ and inner product $< \phi | \psi > \in C(c,+,\times)$ is mapped under isoduality into the *isodual Hilbert space* $\mathcal{H}^d$ with *isodual states* $| \psi >^d = -(| \psi >)^{\dagger} = -< \psi |$ and *isodual inner product* $< \phi | \psi >^d =$

$< \psi \, |(-1)| \, \phi >1^d \in C^d(c^d,+,\times^d)$ (see Vol. 1 for additional aspects).

Then, Eq. (10.2.1) for a particle with charge e transforms under isoduality into Eq. (10.2.2) for the antiparticle[133]

$$\{ [(\partial_\mu + i \, e \, A_\mu) (\partial^\mu + i \, e \, A^\mu) - m^2] \times \psi(x) \}^d =$$

$$= [(\partial_\mu{}^d + i^d \times^d e^d \times^d A_\mu{}^d) \times^d (\partial^{\mu \, d} + i^d \times^d e^d \times^d A^{\mu \, d}) - m^{d2d}] \times^d \psi^d =$$

$$= \psi^\dagger(x) [(\partial_\mu - i \, e \, A_\mu) (\partial^\mu - i \, e \, A^\mu) - m^2] =$$

$$= \psi^d(x) [(\partial_\mu - i \, e \, A_\mu) (-1) (- \partial^\mu + i \, e \, A^\mu) - m^2] = 0 , \qquad (10.2.13)$$

while the Lagrangian density becomes

$$\mathcal{L}^d = -\tfrac{1}{2}^d \times^d \{ (\partial^{\mu d} \psi^{\dagger d} - i^d \times^d e^d \times^d A^{\mu d} \times^d \psi^{\dagger d}) \times^d$$

$$\times^d (\partial_\mu{}^d \psi^d + i^d \times^d e^d \times^d A_\mu{}^d \times^d \psi^d) + m^{2d} \} = - \mathcal{L}, \quad (10.2.14)$$

as *necessary* for consistency (because it must be an element of the isodual field). The four-current

$$j_\mu{}^d = (2 \, i \, m)^{-1d} \times^d [\psi^{\dagger d} \times^d \partial_\mu{}^d \psi^d - (\partial_\mu{}^d \psi^{\dagger d}) \times^d \psi^d] +$$

$$+ e^d \times^d A_\mu{}^d \times^d \psi^{\dagger d} \times^d \psi^d /^d m^d = - j_\mu , \qquad (10.2.15)$$

changes sign as for the conventional charge conjugation. This completes the equivalence of isoduality and charge conjugation for the Klein Gordon equation.

In order to study the isoduality of the Dirac equation we first need the following:

Lemma 10.2.1 [4]: *The Dirac gamma matrices are isoselfdual.*

In fact, Eq.s (10.2.6c) satisfy the laws

$$\gamma_k{}^d = - \gamma_k{}^\dagger = \begin{pmatrix} 0 & \sigma_k \\ - \sigma_k & 0 \end{pmatrix} = \gamma_k , \quad \gamma_4{}^d = - \gamma_4{}^\dagger = i^d \times^d \begin{pmatrix} 1_s{}^d & 0 \\ 0 & - 1_s{}^d \end{pmatrix} = \gamma_4 ,$$
$$(10.2.16)$$

As a result of the above property, the gamma matriecs can be written in the following symmetrized form for two-dimensional states and their isoduals

$$\gamma_k = \begin{pmatrix} 0 & - \sigma_k{}^d \\ - \sigma_k & 0 \end{pmatrix} , \quad \gamma_4 = i \times \begin{pmatrix} 1_s & 0 \\ 0 & 1_s{}^d \end{pmatrix} , \qquad (10.2.17)$$

while the wavefunction can also be "symmetrized"

---

[133] One would reach the same results by adding the actual values of h and c_0 which are given by $/^d h^d \times^d c_0{}^d = /\hbar c_0$.

$$\tilde{\psi} = \gamma_4 \times \psi = \gamma_4 \times \begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix} = i \times \begin{pmatrix} \psi_1 \\ \psi_2^d \end{pmatrix} . \qquad (10.2.18)$$

with isodual

$$\tilde{\psi}^d = \psi^d \times^d \gamma_4^d = (\psi_1^d , \psi_2) . \qquad (10.2.19)$$

Therefore, under isoduality Dirac's equation recovers Eq. (10.2.8) identically,

$$[\gamma^{\mu d} \times^d (\partial_\mu^d - e^d \times^d A_\mu^d) + i^d \times^d m^d] \times^d \tilde{\psi}(x)^d =$$

$$= \tilde{\psi}^d(x) \{ \gamma^\mu (\partial_\mu + e A_\mu) - i \ m \} = 0 , \qquad (10.2.20)$$

while the Lagrangian density again changes sign

$$\mathcal{L}^d = \tfrac{1}{2}^d \times^d \{ \tilde{\psi}^d \times^d [\gamma^{\mu d} \times^d (\partial_\mu^d - e^d \times^d A_\mu^d) + m^d] \times^d \psi^d -$$

$$- [\tilde{\psi}^d \times^d (\partial_\mu^d - e^d \times^d A_\mu^d) + m^d] \times^d \psi^d \} = -\mathcal{L} . \qquad (10.2.21)$$

Similarly, the charge current changes sign as in the conventional case

$$j_\mu^d = \tilde{\psi}^d \times^d \gamma_\mu^d \times^d \psi^d = -j_\mu . \qquad (10.2.22)$$

Other conjugation (e.g., that of Weil's equation) follows accordingly. **q.e.d.**

10.3: ISODUAL REPRESENTATION OF ANTIPARTICLES

We now identify a property of the *conventional* Dirac equation which is undetectable by quantum mechanics because of the restriction of all units to the value $+1$, but which is identifiable by the covering hadronic mechanics and actually has fundamental relevance for the applications of the theory.

Consider again Dirac's equation (II.10.2.6) with related gamma matrices in Pauli's representation. Inspection of property (II.10.2.16) reveals that *the isodual spaces are embedded in the very structure of the conventional Dirac's equation in an essential way.* In fact, we have the following:

LEMMA 10.3.1 [4]: *The total unit of Dirac's equation has twelve dimensions, eight for the space-time component and four for the internal part, with the explicit structure*

$$1^{Tot}_{Dirac} = \{ 1_{4\times4}^{orb.} \times I_{2\times2}^{spin} \} \times \{ 1_{4\times4}^{orb. \ isod.} \times 1_{2\times2}^{spin \ isod.} \} . \quad (10.3.1)$$

In fact, the total unit for the intrinsic spin is γ_4, that is,

$$l_{4\times4}{}^{intr} = l_{2\times2}{}^{spin} \times l_{2\times2}{}^{spin\ isod} = I_s \times (-1_s) = l_s \times l_s{}^d . \qquad (10.3.2)$$

For consistency, this necessarily calls for the admission of a corresponding isodual unit for the orbital part.

The next step is the identification of the spaces characterized by the above units. It is evident that $l_{4\times4}{}^{orb}$ characterizes the conventional Minkowski space $M(x,\eta,\mathfrak{R})$, and that $l_{2\times2}{}^{spin}$ characterizes a two-dimensional space for spin $\frac{1}{2}$ of the SU(2) symmetry which we shall write $S(s,\delta,C)$, where $s = \pm\frac{1}{2}$ and $\delta = $ diag. (1, 1).

Corollary 10.3.1.B [loc. cit.]: *The total carrier space of the conventional Dirac equation is the twelve-dimensional space*

$$S^{Tot}{}_{Dirac} = \{ M(x,\eta,R) \times S(s,\delta,C) \} \times \{ M^d(x,\eta^d,R^d) \times S^d(s,\delta^d,C^d) \} . \qquad (10.3.3)$$

The basic symmetry of the Dirac equation has been believed to be the spinorial covering $\mathscr{P}(3.1) = SL(2.C) \times T(3.1)$ of the Poincaré symmetry $P(3.1) = L(3.1) \times T(3.1)$. This belief is not entirely consistent with the fact that the gamma matrices provide a *four*-dimensional representation of spin 1/2, while a *two*-dimensional realization is known not to exist (for massive particles).

The above disparity is resolved by the following:

Theorem 10.3.1 [loc. cit.]: *The total symmetry of Dirac's equation is the tensorial product of the spinorial covering of the Poincaré symmetry and its isodual*

$$G^{Tot}{}_{Dirac} = \mathscr{P}(3.1) \times \mathscr{P}^d(3.1) = \{ SL(2.C) \times T(3.1) \} \times \{ SL^d(2.C^d) \times T^d(3.1) \} . \qquad (10.3.4)$$

The latter property is at the foundation of the representation of antiparticles via isodual methods. It essentially embodies in a primitive symmetry the known property of the Dirac's equation of be "symmetrical" with respect to solution with positive–definite and negative–definite energy.

In fact, Theorem 10.3.1 establishes in a unique and unambiguous way the *isotopic interpretation of antiparticles*. The positive–energy solutions represented by the state ψ_1 must evidently be interpreted with respect to the conventional unit $l_{orb} \times l_{spin}$ and characterize *particles in our space–time*. the negative–energy solutions contained in the state $\psi_2{}^d$ must then be represented as *antiparticles in isodual spaces*. $\mathscr{P}(3.1)$ is the universal symmetry of particles (only) in exterior relativistic conditions (only), while $\mathscr{P}^d(3.1)$ is the universal symmetry of the corresponding antiparticles.

The above occurrence suggests the following redefinition of the *conventional* Dirac equation in terms of the four–dimensional states (11.2.18) with two–dimensional states ψ_1 representing particles and their isoduals $\psi_2{}^d$

representing antiparticles. The isodual Dirac equation is then describes a state $\psi_1{}^d$ representing antiparticles and a second ψ_2 representing particles, and we shall write

$$\{\gamma^\mu(\partial_\mu - e A_\mu) + i\,m\}\,\tilde{\psi}(x) = 0, \qquad \hat{\psi} = \text{Col.}\,(\psi_1, \psi_2{}^d), \qquad (10.3.5a)$$

$$\tilde{\psi}^d(x)\,\{\gamma^\mu(\partial_\mu + e A_\mu) - i\,m\} = 0, \qquad \hat{\psi}^d = \text{Row}\,(\psi_1{}^d, \psi_2). \qquad (10.2.20)$$

Under the above equations, the state ψ_1 represents a particle with energy $E = |E| > 0$, time $t = |t| > 0$, linear momentum p, etc., referred to the unit +1, while $\psi_2{}^d$ represents an antiparticle with energy $E^d = |E|^d = -E < 0$, time $t^d = |t|^d < 0$, etc., referred to the unit −1.

This isodual interpretation of antiparticles implies the elimination of the conjecture of infinite seas of hypothetical undetectable antiparticles with "holes" being the physically observed particles, as well as other conjectures of infinite states in second quantization, as the reader is encouraged to verify.

A further notion important for these studies is that of *isoselfdual particle* which, physically, is a bound state of a particle and its antiparticle such as the π° and, mathematically, is represented via isoselfdual structures of type (11.2.16). The peculiarity of these bound states is that they admit *positive energies and times* when studied in our space–time, and *negative energies and times* when studied in the isodual space–time, as shown in the hadronic bound states of Sect. 11.7.8.

As the reader recalls, $\mathscr{P}(3.1)\times\mathscr{P}^d(3.1)$ is the fundamental symmetry of the novel *isocosmology* introduced in the preceding chapter. It is remarkable that such a *cosmological* symmetry sees its origin in the *conventional* Dirac equation.

10.4: ISOTOPIES OF THE KLEIN-GORDON EQUATION AND THEIR ISODUALS

We now study the isotopies of conventional relativistic equations of Kadeisvili's Class 1 (with isounits that are sufficiently smooth, bounded, nowhere degenerate, Hermitean and positive–definite).

The central objective is to identify a generalization of conventional field equations which is form–invariant under the isopoincaré symmetry $\hat{P}(3.1)$ (Sect. 11.8.4.B) in isominkowski space $\hat{M}(x,\hat{\eta},\hat{\Re})$ (Sect. 11.8.2) which now represents relativistic particles in interior dynamical conditions, e.g., in the core of a star. We shall assume for simplicity the diagonal form of the isotopic element T and isounit $\hat{1} = T^{-1}$ as in Eq.s (11.8.2.2) with an arbitrary functional dependence, including that in the local coordinates, $T = T(x, \dot{x}, \ddot{x}, \psi, \partial\psi, \partial\partial\psi, \mu, \psi, n, ...)$. The

related isoderivatives on $\hat{M}(x,\hat{\eta},\hat{R})$ were identified in Sect. 11.8.4.A.

The identification of the *iso-Klein–Gordon equation* is straightforward from the basic isoinvariant (11.8.4.25), and can be written [6]

$$(\hat{\eta}^{\mu\nu} p_\mu * p_\nu + \hat{m}^2 * \hat{c}^2 \hat{1}) * \psi(x) =$$

$$[\hat{\eta}^{\mu\nu}(x, \dot{x}, ...) p_\mu T(x, \dot{x}, ...) p_\nu + m^2 c_o^4 b_4^4 \hat{1}] T(x, \dot{x}, ...) \psi(x) = 0 , \quad (10.4.1)$$

By recalling the isoquantization of the momentum (11,.8.4.13)

$$p_\mu * \psi(x) = p_\mu T(x, \dot{x}, ...) \psi(x) = - i \partial_\mu \psi(x) =$$

$$= - i \hat{1}_\mu{}^\nu(x, \dot{x}, ..) \partial_\nu \psi(x) = - i b_\mu{}^{-2}(x, \dot{x}, ...) \partial_\mu \psi(x) \quad \text{(no sum)} , \quad (10.4.4)$$

Eq. (10.4.1) acquires the form

$$[\hat{\eta}^{\mu\nu} (b_\mu{}^{-2} \partial_\mu)(b_\nu{}^{-2} \partial_\nu) - m^2 c_o^4 b_4^4) \psi(x) = 0 . \quad (10.4.5)$$

A solution of equation (10.4.1) is the familiar *isoplane waves* [6]

$$\hat{\psi}(x) = N e^{i k^\mu \hat{\eta}_{\mu\nu} x^\nu} = N e^{i(k^k b_k^2 x^k - k^4 b_4^2 x^4)} . \quad (10.4.7)$$

for which

$$(\hat{\eta}^{\mu\nu} p_\mu * p_\nu + \hat{m}^2 * \hat{c}^2 \hat{1}) * \psi(x) = (\hat{\eta}^{\mu\nu} k_\mu k_\nu + m^2 c^4) \psi(x) = 0 , \quad (10.4.8)$$

which is the correct form of the $\hat{P}(3.1)$–isoinvariant.

The isotopies of the remaining aspects of the Klein–Gordon equation (isocurrent, isolagrangian, etc.) are left to the interested reader for brevity. The *isodual iso-Klein–Gordon equation* follows the same lines of the isoduality of the conventional equation but now represent antiparticles in interior relativistic conditions.

We note that, despite the lack of a potential, Eq. (10.4.1) *does not* describe a free particle, but rather a particle under nonlinear–nonlocal–noncanonical interactions, or an electromagnetic wave within an inhomogeneous and anisotropic media.

We finally note that, by conception and realization, *the iso-Klein–Gordon equation incorporates an operator form of gravitation* along the lines of Sect. 9.5.E, i.e., by factorizing any Riemannian metric $g(x)$ in the isominkowskian form $g(x) = T_{gr}(x)\eta$, and then embedding the gravitational isounit $T_{gr}(x)$ in the isotopic element T of the equation. Depending on the selected metric $g(x)$, one therefore has the *iso-Klein–Gordon–Schwartzchild's equation*, the *iso-Klein–Gordon–Krasner equation*, etc. [15]

10.5: ISOTOPIES OF DIRAC'S EQUATION AND THEIR ISODUALS

As recalled in Sect. 11.10.1, a central objective of the conventional Dirac equation is the representation of the electron under the *external* interaction of the proton (exterior problem). One of the central objectives of the *isotopic Dirac equation*, or *isodirac equation*, is the study of the same electron when in moving within a hadronic medium (interior problem). As we shall see in Vol. III, the latter representation permits fundamentally novel structure models of hadrons as the chemical synthesis of lighter hadrons [13] with far reaching implications, including an apparent new form of subnuclear energy called *hadronic energy*.

DIRAC EQUATION
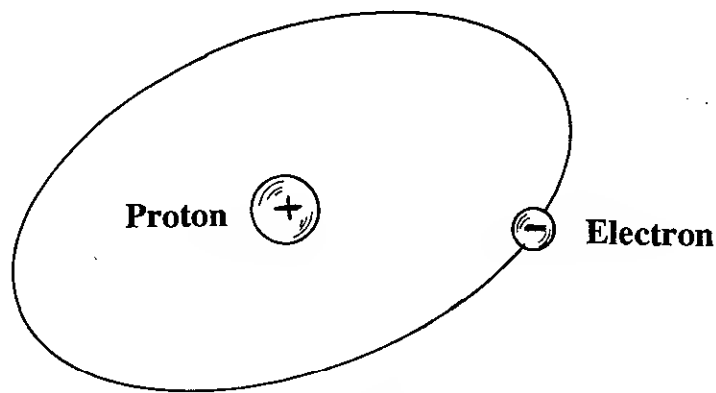
ISODIRAC EQUATION
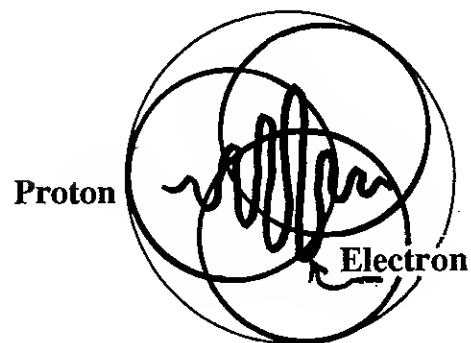
FIGURE 10.5.1: A schematic view of the primary differences between the conventional and isotopic Dirac equation representing the same electron in exterior

and interior conditions, respectively. in the exterior case, the electron moves in the homogeneous and isotropic vacuum under action–at–a–distance interactions. In this case all possible interactions are represented by the external four–potential A_μ. In the interior case the electron is embedded within a hyperdense, inhomogeneous and anisotropic medium by therefore experiencing the conventional action–at–a–distance interactions with four potential A_μ plus novel *contact–nonlagrangian* interactions which, as such, are structurally beyond any descriptive capacity of relativistic quantum mechanics and which are represented in hadronic mechanics via via a generalized unit. The isotopic model represented in this figure is Rutherford's "compression" of the electron inside the proton and consequential chemical synthesis of the neutron [13]. A central problem is a quantitative representation of the fact that the orbital angular momentum of the electron must coincide with the spin of the much heavier proton, to avoid great instabilities due to a particle spinning inside and against the spinning of a much heavier particle. This problem was studied via the isotopies of SO(2), SO(3) and SU(2) symmetries in Sect.s 11.6.5 and 11.6.12. In this section we show how the same result can be obtained via a suitable isotopy of the Dirac equation. In Sect. 11.10.7 we then show how again the same result can be reached via Dirac's studies [8,9].

In studying the topic, the reader is therefore encouraged to abandon the conventional thinking of an electron in vacuum, focus the imagination on the same particle within the densest medium measured in laboratory until now, and therefore seek the highest possible *deviations* from the Dirac equation as an evident *necessary condition* for a representation of such physical differences.

To begin, recall that the isotopies are fundamentally dependent on the assumed unit. The very first step recommended for any isotopy is therefore the identification of the original basic unit, which is then subjected to axiom–preserving lifting. All remaining quantities, including fields, spaces, symmetries, etc., are then constructed in a way to be compatible with the basic generalized units.

The identification of the basic unit of the conventional Dirac equation was done in Eq.s (10.3.1) with related total space given by Eq. (10.3.3)

The central problem in the construction of the isodirac's equation is the *isolinearization* of the second–order isofield equation (10.4.1), that is, a reduction which is of first–order in isospace but which is of arbitrary order when projected in the original space. We shall also add the conditions that: only isotopies of Kadeisvili Class 1 are used; the reduction is properly defined in total isospaces (10.3.3); and the first–order isoequation results to be an isotopy of the corresponding conventional form.

Via self–explanatory notation, we can therefore write

$$(\hat{\eta}^{\mu\nu}\, p_\mu *^{tot} p_\nu + \hat{m}^2)*^{tot} \hat{\psi}(x) \equiv$$

$$\equiv (\hat{\eta}^{\mu\nu}\, \hat{\gamma}_\mu *^{tot} p_\nu + i\,\hat{m})*^{tot} (\hat{\eta}^{\alpha\beta}\, \hat{\gamma}_\alpha *^{tot} p_\beta - i\,\hat{m})*^{tot} \hat{\psi}(x), \quad (10.5.1)$$

where the isolinearized quantities are properly written in their own six-dimensional isospace, e.g., the linear momenta are $p_\mu \times 1_{4\times4}^{intr}$, etc.

Recall from Sect. 11.1.4 that *the isotopies do not alter the dimensionality of the original representation.* Therefore, the generalized quantities $\hat{\gamma}_\mu$, called *isogamma matrices,* remains four-dimensional and are defined by

$$\{ \hat{\gamma}_\mu \,\hat{,}\, \hat{\gamma}_\nu \}^{tot} = \hat{\eta}_\mu \, T^{tot} \, \hat{\gamma}_\nu + \hat{\gamma}_\nu \, T^{tot} \, \hat{\gamma}_\mu = 2 \, \hat{\eta}_{\mu\nu} \hat{1}^{orb}, \qquad (10.5.2)$$

which can be reduced to

$$\hat{\gamma}_\mu = \tilde{\gamma}_\mu \hat{1}^{orb}, \qquad \{ \tilde{\gamma}_\mu \,\hat{,}\, \tilde{\gamma}_\nu \}^{intr} = \tilde{\gamma}_\mu \, T^{intr} \, \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \, T^{intr} \, \tilde{\gamma}_\mu = 2 \, \hat{\eta}_{\mu\nu}. \tag{10.5.3}$$

As we shall see in Sect. 11.10.7, "Dirac's generalization of Dirac's equation" [8,9] belongs precisely to the above class. Nevertheless, the above formulation is excessively general for most of our needs. We shall therefore assume in this section as a first illustration the particularization for which

$$1^{orb} \equiv 1, \qquad T^{orb} \equiv T, \qquad 1^{spin} = 1 = \text{diag.}\,(1, 1), \qquad 1^d = -\text{diag.}\,(1, 1), \quad (10.5.4)$$

under which general conditions (10.5.2) become

$$\{ \hat{\gamma}_\mu \,\hat{,}\, \hat{\gamma}_\nu \} = \hat{\gamma}_\mu \, T \, \hat{\gamma}_\nu + \hat{\gamma}_\nu \, T \, \hat{\gamma}_\mu = 2 \, \hat{\eta}_{\mu\nu} 1, \qquad 1 = T^{-1}, \qquad (10.5.5)$$

with the simplest possible realization

$$\hat{\gamma}_\mu = b_\mu(x, \dot{x}, \ddot{x}, \hat{\psi}, \partial\hat{\psi}, \partial\partial\hat{\psi}, ...) \, \gamma_\mu \, 1 \quad \text{(no sum)}, \, b_\mu > 0, \qquad (10.5.6)$$

where the γ-matrices are the conventional ones.

The *isodirac's equation* for the simplest possible case here considered and without external fields can be written

$$(\hat{\gamma}^\mu * p_\mu + i \hat{m}) * \hat{\psi}(x) = \qquad (10.5.7a)$$

$$(\hat{\eta}^{\mu\nu}(x, \dot{x}, \hat{\psi}, \partial\hat{\psi}, ...) \hat{\gamma}_\mu \, T(x, \dot{x}, \hat{\psi}, \partial\hat{\psi}, ...) p_\nu + i \, m \, 1) \, T(x, \dot{x}, \hat{\psi}, \partial\hat{\psi}, ...) \, \hat{\psi}(x) = 0,$$

$$T = \text{diag.}\,(b_1{}^2, b_2{}^2, b_3{}^2 \, b_4{}^2), \qquad b_\mu > 0, \qquad \hat{m} = m \, 1, \qquad (10.5.7b)$$

$$\hat{\psi} = \hat{\gamma}_4 \, \psi(x) = \text{column}\,(\hat{\psi}_1(x), \hat{\psi}_2{}^d). \qquad (10.5.7c)$$

Despite
its simplicity, the above realization is sufficient to indicate the nontriviality of the isotopy, because it implies an *alteration of the gamma matrices* which is not unitarily equivalent to the conventional ones, i.e., there exist no unitary operator

U such that $\hat{\gamma}_\mu = U\gamma_\mu U^\dagger$. The isolinear character is also evident, because the equation is first order in isospace, i.e., in the first expression of Eq.s (10.5.7a), while it is of arbitrary order when projected in the original space–time, i.e., the last identity in the same equations. It is an instructive exercise for the interested reader to verify that isoplane waves (II.10.4.8) are also a solution of the isodirac equation.

The extension of Eq. (10.5.7) to include *external* electromagnetic interactions is readily given by ($\hbar = 1$)

$$\{\hat{\gamma}^\mu * [\, p_\mu + e\, A_\mu(x)\, /\, c\,] + i\,\hat{m}\,\} * \hat{\psi}(x) = 0 \qquad \hat{m} = m\,\hat{1}, \quad (10.5.8)$$

where the reader should note the crucial replacement of c_o, as in Eq. (II.10.2.6), with $c = c_o b_4$. A step–by–step isotopy of all various aspects of the conventional Dirac's equation [1,2] then follows. We report below only the most essential aspects.

By introducing the isodual $\hat{\psi}^d = \hat{\psi}\dagger * \hat{\gamma}_4 = \text{Row}\,(\hat{\psi}_1{}^d, \hat{\psi}_2)$, the *four isocurrent* and related conservation law are given by the straighforward isotopy of conventional expressions,

$$\mathfrak{J}_\mu = c\,\hat{\psi}^d * \hat{\gamma}_\mu * \hat{\psi}, \qquad \partial^\mu \mathfrak{J}_\mu \equiv 0. \qquad (10.5.9)$$

The *hadronic angular momentum* $\hat{L}$ (Sect. II.6.4) is defined in isoeuclidean space $E(r,\hat{\delta},\hat{R})$, with

$$\hat{\delta} = T\delta = T^{orb}, \qquad \hat{1} = (T^{orb})^{-1}, \qquad T^{orb} = \text{diag.}\,(\, b_1{}^2, b_2{}^2, b_3{}^2\,), \quad (10.5.10)$$

and admits the conventional components as in Sect. II.6.4

$$\hat{L}_k = \epsilon_{kij}\, r_i\, p_j, \qquad (10.5.11)$$

For applications we need an explicit form of the isoeigenvalues of $\hat{L}$. They can be computed via an "old trick" of the isotopies, which consists of introducing a fictitious conventional space $E(\bar{r},\delta,R)$ whose interval coincides with that of $\hat{E}(r,\hat{\delta},\hat{R})$, for which

$$r^2 = r_k\, b_k{}^2\, r_k \equiv \bar{r}_k\, \bar{r}_k = \bar{r}^2. \qquad (10.5.12a)$$

$$r_k = b_k{}^{-1}\, \bar{r}_k, \qquad p_k = b_k{}^{-1}\, \bar{p}_k, \qquad p_k * \hat{\psi} = b_k{}^{-1}\, \bar{p}_k\, \hat{\psi}. \qquad (10.5.12b)$$

Under these conditions we can transform the isotopic algebra in $\hat{E}(r,\hat{\delta},\hat{R})$ into a conventional algebra in $E(\bar{r},\delta,R)$ according to the rules

$$\hat{L}_1 = b_1\, b_2\, L_2, \qquad \hat{L}_2 = b_1{}^{-1}\, b_3{}^{-1}\, L_2, \qquad \hat{L}_3 = b_1{}^{-1}\, b_2{}^{-1}\, L_3, \qquad (10.5.13a)$$

$$s\hat{o}(3): \qquad [\hat{L}_i \,\hat{,}\, \hat{L}_j] = \epsilon_{ijk}\, b_k^{-2}\, \hat{L}_k, \qquad\qquad (10.5.13b)$$

where L_k represents the conventional components in $E(\bar{r},\delta,.R)$. The desired isoeigenvalues for the $L = 1$ on a basis of maximal weight, are then given by

$$\hat{L}^2 * \hat{\psi} = (b_1^{-2} b_2^{-2} + b_2^{-2} b_3^{-2} + b_3^{-2} b_1^{-2})\hat{\psi}, \qquad (10.5.14a)$$

$$\hat{L}_3 * \hat{\psi} = b_1^{-1} b_2^{-1}\hat{\psi}. \qquad\qquad (10.5.14b)$$

with corresponding expressions for higher values of L.

The *hadronic spin* of the isodirac equation can be defined via a simple isotopy of the conventional expression, yielding the following components and related algebra the reader can easily verify

$$s\hat{u}(2): \qquad \hat{S}_k = \tfrac{1}{2}\, \epsilon_{kij}\, \hat{\gamma}_i * \hat{\gamma}_j, \qquad\qquad (10.5.15a)$$

$$[\hat{S}_i, \hat{S}_j] = \epsilon_{ijk}\, b_k^2\, \hat{S}_k \quad \text{(no sum on the k-index)}, \qquad (10.5.15b)$$

The needed isoeigenvalues, also on a basis of maximal weight, are then easily derived from the assumed realization of the $\hat{\gamma}$–matrices

$$\hat{S}^2 * \hat{\psi} = (1/4)(b_1^2 b_2^2 + b_2^2 b_3^2 + b_3^2 b_1^2)\hat{\psi}, \qquad (10.5.16a)$$

$$\hat{S}_3 * \hat{\psi} = \tfrac{1}{2}\, b_1\, b_2\,\hat{\psi}. \qquad\qquad (10.5.16b)$$

An inspection of isoeigenvalues (10.5.14) and (10.5.16) confirms the existence of a nontrivial mutation of the angular momentum and spin, exactly as *desired*. Note also that the structure of the spin isoeigenvalues is different than that for the regular and fundamental representations of s$\hat{u}$(2). This is due to the fact that we have here a different isotopy, that of the four–dimensional representation of SL(2.C), rather than that of the of two–dimensional representation of su(2) as studied in Ch. 11.6. In fact, the isorepresentation is fully aligned with that of the S$\hat{L}$(2.$\hat{C}$) isosymmetry studied in Sect. 11.8.3.

We now study the explicit form of the magnetic and electric moments of the isodirac equation. Introduce the *hadronic spin tensor*

$$\hat{\sigma}_{\mu\nu} = \tfrac{1}{2}(\hat{\gamma}_\mu * \hat{\gamma}_\nu - \hat{\gamma}_\nu * \hat{\gamma}_\mu). \qquad\qquad (10.5.17)$$

and note that the electromagnetic field, being external, is not altered by the isotopies,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad\qquad (10.5.18)$$

The second-order equation then admits the the term

$$e \, (\, \hat{\tilde{\sigma}}_{\mu\nu} * F^{\mu\nu} \,) \, / \, 2 \, m \, c \; = \; e \, (\, \hat{\tilde{\gamma}}_{\mu} * \hat{\tilde{\gamma}}_{\nu} * F^{\mu\nu} \,) \, / \, 2 \, c \, , \tag{10.5.19}$$

which, by using realization (10.5.6) of the isogamma matrices, can be written

$$e \, (\, \hat{\tilde{\sigma}}_k * H_k \; - \; i \, \hat{\tilde{\alpha}}_k * E_k \,) \, / \, 2 \, m \, c \, , \tag{10.5.20}$$

where $\tilde{\sigma}_k = b_k \, \sigma_k$, $\hat{\tilde{\alpha}}_k = b_k \, \alpha_k$ (no sum), and α_k is another symbol for Pauli's matrices.

This yields the desired expression of the *magnetic and electric isodipole moments,* that is, dipole moments computed in isospace $\hat{M}(x,\hat{\eta},\hat{R})$,

$$\hat{\mu}_k \; = \; \frac{b_3}{b_4} \, \mu_k \, , \qquad\qquad \hat{m}_k \; = \; \frac{b_3}{b_4} \, m_k \, , \tag{10.5.21}$$

where the b_4 term originates from the term $c = c_0 \, b_4$, which were first introduced in ref. [5], Eq.s (4.20.16), p. 803, re-expressed in the isotopic language in ref. [6], and applied to the chemical synthesis of protons and electrons in ref. [13].

The computation of the isotopies of the remaining aspects of Dirac's theory is then straighforward and it is here left to the interested reader. In particular, the isotopy of the charge is intriguing for the possible "isotopic construction" of fractional charge of quarks inside hadronic matter (see Vol. III).

The isotopies of Dirac's theory have a number of applications studied in Vol. III. The simplest ones occur for those values of the characteristic quantities b_k which yield a conventional spin $\frac{1}{2}$ but deformed shape of the charge distribution and consequential alteration of the intrinsic magnetic moments. This class of realizations is particularly significant in *nuclear physics* because preserving most of quantum mechanical laws (such as the exclusion principle).

A most important application in *hadron physics* is the achievement of a null total angular momentum under expressions (10.5.14) and (10.5.16) as per Lemma 11.6.10.1. In fact, the lemma here considered implies the conditions

$$\hat{L}^2 \equiv \hat{S}^2, \quad \hat{L}_z \equiv -\hat{S}_z, \tag{10.5.22}$$

i.e.,

$$b_1^{-1} b_2^{-1} \; = \; \tfrac{1}{4} \, b_1 \, b_2 \, , \tag{10.5.23a}$$

$$b_1^{-1} b_2^{-1} \; + \; b_2^{-2} b_3^{-2} \; + \; b_3^{-2} b_1^{-2} \; = \; (1/4) \, (\, b_1^2 \, b_2^2 \; + \; b_2^2 \, b_3^2 \; + \; b_3^2 \, b_1^2 \,) \, , \tag{10.5.123}$$

with the spherically symmetric solution

$$b_1^2 \; = \; b_2^2 \; = \; b_3^2 = \sqrt{2} \; \approx 1.415 \, , \tag{10.5.24}$$

whose numerical values will be used in Vol. III for the study of the chemical synthesis of proton and an electron [13].

In *astrophysics* the isodirac equation is significant because it permits a particular operator description of the gravitational field as embedded in the isotopic element T, along the lines of Sect. II.9.5.E (see also the preceding section for the second-order case) [15]. In fact, by factorizing again the various gravitational metrics $g(x) = T_{gr}(x)\eta$, and by embedding $T_{gr}(x)$ into the isotopic element, we have the *iso-Dirac-Schwartzschild equation, iso-Dirac-Krasner equation*, etc.

The *isodual isodirac equation* is constructed via the same antiautomorphic map of Sect. II.10.2, and can be written

$$\hat{\psi}^d(x)\, T^d(x, ...)\, [\, p_\mu + e^d\, A_\mu{}^d\, /d\, c^d\,]\, T^d(x, ...)\, \hat{\gamma}^m + i\, \hat{m}^d\,] = 0 \,. \quad (10.5.25)$$

where $\hat{\psi}^d = \text{Row}\,(\hat{\psi}_1{}^d, \hat{\psi}_2)$. The rest of the isodual theory follows. Note that the latter equation *is not* needed for the characterization of antiparticles in interior conditions because they are embedded in the structure of the isodirac equation itself.

10.6: ISOTOPIES OF THE SPINORIAL POINCARE′ SYMMETRY AND THEIR ISODUALS

In this section we reduce all the results of this chapter to primitive isosymmetries. It is easy to prove that the following generators

$$\hat{M} = (\hat{M}_{\mu\nu}) = (\, L, N\,), \quad \hat{L}_k = \tfrac{1}{2}\,\epsilon_{kij}\,\hat{\gamma}_i * \hat{\gamma}_j, \quad N_k = \tfrac{1}{2}\,\hat{\gamma}_k * \hat{\gamma}_4\,, \ k = 1, 2, 3,$$
$$(10.6.1)$$

characterize the *isotopic covering* $\hat{SL}(2.\hat{C})$ of the *conventional spinorial covering* SL(2.C) of the *Lorentz symmetry* L(3.1). The isoduals $\hat{L}_k{}^d = -\hat{L}_k$ and $\hat{N}_k{}^d = -\hat{N}_k$ therefore characterize the *isodual isospinorial covering* $\hat{SL}^d(2.\hat{C}^d)$.

By adding the isotranslations $\hat{T}(3.1)$ in $\hat{M}(x,\hat{\eta},\hat{R})$ and their isodual in $\hat{M}^d(x,\hat{\eta}^d,\hat{R}^d)$, the full *isosymmetry of the isodirac equation* is given by

$$\hat{\mathcal{P}}(3.1) \times \hat{\mathcal{P}}^d(3.1) = \{\, \hat{SL}(2.\hat{C}) \times \hat{T}(3.1)\,\} \times \{\, \hat{SL}^d(2.\hat{C}^d) \times \hat{T}^d(3.1)\,\}, \quad (10.6.2)$$

with isocommutation rules for $\hat{\mathcal{P}}(3.1)$

$$[\hat{M}_{\mu\nu}\,\hat{,}\,\hat{M}_{\alpha\beta}] * \hat{\psi} = i\,(\,\hat{\eta}_{\nu\alpha}\,\hat{M}_{\beta\mu} - \hat{\eta}_{\mu\alpha}\,\hat{M}_{\beta\nu} - \hat{\eta}_{\nu\beta}\,\hat{M}_{\alpha\mu} + \hat{\eta}_{\mu\beta}\,\hat{M}_{\alpha\nu}) * \hat{\psi}, \quad (10.6.3a)$$

$$[\hat{M}_{\mu\nu}\,\hat{,}\,P_\alpha] * \hat{\psi} = i\,(\,\hat{\eta}_{\mu\alpha}\,P_\nu - \hat{\eta}_{\nu\alpha}\,P_\mu) * \hat{\psi} = 0, \qquad \mu, \nu = 1, 2, 3, 4\,, \quad (10.6.3b)$$

and isogroup in terms of conventional parameters

$$\hat{g}(\hat{\Lambda}) = \{ \prod_{k=1,2,3} \exp (\tfrac{1}{2} \theta_k \, \hat{\gamma}_k * \hat{\gamma}_4) * \prod_{k=1,2,3} \exp (\tfrac{1}{2} \omega_k \epsilon_{kij} \, \hat{\gamma}_i * \hat{\gamma}_j) \} ,$$
$$(10.6.4)$$

plus the isotranslations (Sect. 1I.8.3), and corresponding structures for the isodual part. The transformation laws of the isowave functions $\hat{\psi}$ are then given by

$$\hat{\psi}'(x\,') = \hat{g}(\hat{\Lambda}) * \hat{\psi} [\hat{\Lambda}^{-1}(x - \hat{x}^o)] , \qquad (10.6.5a)$$

$$\hat{\gamma}^d\,'(x') = \hat{\psi}^d [\hat{\Lambda}^{-1} (x - \hat{x}^o)] * \hat{g}^{-1}(\hat{\Lambda}) . \qquad (10.6.5b)$$

The computation of the remaining isotransformation properties, including isoinversions, is then straightforward

10.7: DIRAC'S GENERALIZATION OF DIRAC'S EQUATION

We now complete this chapter with a review of Dirac's elegant generalization of his own equation exhibiting an essential isotopic structure with null total angular momentum in the ground state [8,9]. We shall present it first in the original notations used by Dirac, and then show its essential structure.

Ref. [8] introduced the following generalization of the conventional Dirac's equation

$$(\alpha_\mu \partial^\mu \; + \; \beta) q \, \psi = 0, \qquad (10.7.1)$$

where

$$\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; \alpha_1 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad (10.7.2a)$$

$$\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \; \alpha_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (10.7.2b)$$

q = Column (q_1 , p_1 ; q_2 , p_2), and ψ is a four-component wavefunction. By assuming $\alpha_4 = 1$, *Dirac's alpha-matrices* are defined by the properties

$$\alpha_\mu \beta \alpha_\nu + \alpha_\nu \beta \alpha_\mu = 2 \beta \eta_{\mu\nu} , \qquad (10.7.3)$$

The spin of Eq. (10.7.1) is represented by

$$s_{ij} = - (\alpha_i \beta \alpha_j - \alpha_j \beta \alpha_i) \, q \, q^T / 8 , \qquad (10.7.4)$$

and possesses the explicit values

$$J^2 = s_{12}{}^2 + s_{23}{}^2 + s_{31}{}^2 = (q_1{}^2 + p_1{}^2 + q_2{}^2 + p_2{}^2)^2/8 - 1/4 =$$

$$= J(J+1),\qquad (10.7.5a)$$

$$J = (q_1{}^2 + p_1{}^2 + q_2{}^2 + p_2{}^2)/4 - 1/2 = (n + n')/2, \qquad (10.7.5b)$$

$$n, n' = 0, 1, 2, ..., \qquad s = 0, \tfrac{1}{2}, 1, ... \qquad (10.7.5c)$$

thus reaching the value $s = 0$ for the ground state.

Dirac essentially submitted the above generalized equation for an intriguing characterization of two oscillators with quantum numbers q_k, p_k, $k = 1$, 2. We refer the interested reader to the original papers [8,9] for all remaining.

The essential (i.e., irreducible) isotopic character of generalized equation (10.7.1) is evident, with the isotopic element given by the nonsingular, but nondiagonal quantity

$$T = \beta, \qquad (10.7.6)$$

Eq. (10.7.1) can therefore be identically written in the isotopic language

$$(\alpha_\mu \partial^\mu + \beta) q \psi \equiv -(\hat{\eta}^{\mu\nu}\alpha_\mu * p_\nu - 1) * \hat{\psi} = 0, \qquad (10.7.7a)$$

$$\hat{\psi} = q\psi, \qquad p_\nu * \hat{\psi} = p_\nu \beta \hat{\psi} = -1^2 \partial_\nu \hat{\psi}, \qquad 1 = T^{-1} = \beta^{-1} \qquad (10.7.7b)$$

Conditions (10.7.3) are also identically preserved in isotopic language, and simply read

$$\{\alpha_\mu \,\hat{,}\, \alpha_\nu\} = \alpha_\mu * \alpha_\nu + \alpha_\nu * \alpha_\mu = 2\hat{\eta}_{\mu\nu}, \qquad (10.7.8)$$

Thus, they belong to the generalized class of isolinearizations (10.5.3), and this illustrates once more the brilliance of Dirac's intuition.

Quite intriguingly, *the isominkowskian space* $\hat{M}(x,\hat{\eta},\hat{\mathfrak{R}})$ *underlying Eq.* (10.7.1) *has a nondegenerate nondiagonal isometric*

$$\hat{\eta} = \beta\eta, \quad \mathrm{Det}\,\beta = 1 \neq 0, \quad \mathrm{Det}\,\hat{\eta} \neq 0, \qquad (10.7.9)$$

but a degenerate isoseparation

$$x^2 = (x^t \beta\eta x)1 = (x^1 x^3 - x^2 x^4 - x^3 x^1 - x^4 x^2)1 = -2 x^2 x^4 1. \qquad (10.7.10)$$

Without his knowledge, Dirac selected an isotopy under which the four-dimensional Minkowski space $M(x,\eta,R)$ is lifted into a degenerate isotopic form with only *two* components. It is this particular structure which permits the mutation of the original total angular momentum $J = n + \tfrac{1}{2}$ into the value $J = 0$ in the ground state, with rather intriguing applications, e.g., to the characterization of the Cooper pairs in superconductivity as done by Animalu [14].

The isotopic reformulation of the remaining aspects (such as the new

isotopic realization of the $\hat{su}(2)$ algebra) is left to the interested reader.

It is instructive to compare Dirac's lifting of the total angular momentum J = n + $\frac{1}{2}$ to a null value and the same result achieved in the preceding section. In essence, in isodirac equation (II.10.5.7) we realize Lemma II.6.10.1 in the form

$$\hat{J}_{tot} = \hat{L} + \hat{S} = 0, \quad \hat{L} = -\hat{S} \neq 0, \tag{10.7.11}$$

that is, when the angular momentum and spin are constrained to coincide but not to be individually null. By comparison, Dirac achieves the same null total value $\hat{J}$ = $\hat{L} + \hat{S} \equiv 0$, via the null value of the individual terms

$$\hat{L} = \hat{S} = 0. \tag{10.7.112}$$

In this latter case Dirac achieves the mutation of the spin via one of the most limiting conditions that are conceivable for the electron collapsing into a degenerate space–time.

We can therefore conclude by saying that *Dirac's intuitive, yet elegant isotopy of his equation confirms a most fundamental aspect of hadronic mechanics and its application to the chemical synthesis of hadrons.*

REFERENCES

1. P. A. M. DIRAC, *Principles of Quantum Mechanics* Oxford Univ. Press (1947)

2. E. CORINALDESI and F. STROCCHI, *Relativistic Wave Mechanics,* North Holland, Amsterdam (1963)

3. R. M. SANTILLI, Hadronic J. Suppl. **4B**, issue no. 1 (1989)

4. R. M. SANTILLI, Comm. Theor. Phys. **3**, 1 (1994)

5. R. M. SANTILLI, Hadronic J. **1**, 574 (1978)

6. R. M. SANTILLI, Hadronic J. Suppl. **4B**, issue no. 3 (1989)

7. R. M. SANTILLI, in *Proceedings of <Deuteron 1993>*, JINR, Dubna, Russia (1994)

8. P. A. M. DIRAC, Proc. Roy. Phys. Soc. **A322**, 435 (1971)

9. P. A. M. DIRAC, Proc. Roy. Phys. Soc. **A328**, 1 (1972)

10. J. LÕHMUS, E. PAAL and L. SORGSEPP, *Nonassociative Algebras in Physics,* Hadronic Press, Palm Harbor, FL (1994)

11. P. A. M. DIRAC, Proc. Cambridge Phil. Soc. **26**, 376 (1930); W. HEISENBERG, Zeits. für Phys. **90**, 209 (1934); H. KRAMERS, Proc. Kgl. Ned. Acad. Wet. **40**, 814 (1937)

12. E. RECAMI and ZIINO, Nuovo Cimento **33A**, 205 (1976)

13. R. M. SANTILLI, Recent theoretical and experimental evidence on the apparent chemical synthesis of the neutron, JINR Communication E3–93–

352, 1993, submitted to Il Nuovo Cimento

14. A. O. E. ANIMALU, Hadronic J. 17, 349 (1994)

15. R. M. SANTILLI, in *Proceedings of the Seventh Marcel Grossmann Meeting on General Relativity,* World Scientific, Singapore (1994)

11: HADRONIC PERTURBATION THEORY

11.1: STATEMENT OF THE PROBLEM

It is generally believed that strong interactions have divergent perturbative expansions because of the high value of their coupling constant. Hadronic mechanics has disproved this additional belief because, as we shall see in this chapter, a divergent perturbative series can be always turned into a convergent form via a suitable isotopic lifting.

This occurrence was indicated in the original proposal of 1978 to build hadronic mechanics [1], but it remained thereafter ignored for several years. The property was first submitted to a preliminary investigation in memoir [2] of 1989 within the context of the isotopies of Class I, confirmed and then applied to the quark theory in paper [3]. The only other contribution of which this author is aware at this writing is paper [4] by Jannussis and Mignani of 1992 which studies the problem of isoconvergence within the broader genotopic (Lie-admissible) theory.

As one can see, the studies are at the very beginning. The objective of this chapter is therefore limited to that of illustrating the mechanism according to which hadronic mechanics can turn a divergent quantum series into a convergent form. Applications will be considered in Vol. 111.

The main idea of the isotopic reconstruction of convergent perturbative series when conventionally divergent, more generally called *hadronic perturbation theory,* is so simple to appears trivial. Recall (see, e.g., ref. [5]) that quantum perturbative treatments have a restricted arena of applicability. In particular, they are effective only when the Hamiltonian has the structure

$$H = H_0 + kV, \quad k \ll 1, \qquad (11.1.1)$$

where the matrix elements of V are comparable in magnitude to those of H_0.

Under sufficient smoothness and regularity conditions generally verified in practical applications, hadronic mechanics can provide convergent perturbative

expansions for all Hamiltonians of the type

$$H = H_0 + k V, \quad k \gg 1,$$
(11.1.2)

where the matrix elements of V are bounded, but arbitrarily larger than those of H_0. The sole condition is that the magnitude of the isotopic element of the theory is sufficiently small for a given Hamiltonian (11.2),

$$|T| \ll 1.$$
(11.1.3)

Convergence is then regenerated.

The origin of the occurrence is the following. In quantum mechanics a physical system is solely described by the Hamiltonian. As such, perturbative series can be convergent or divergent depending on the structure of that given Hamiltonian. In hadronic mechanics we have an infinite number of possible isotopic elements T for each given Hamiltonian H. It is therefore possible to select the operator T such to "compensate" the divergence originating from the Hamiltonian.

The simplest case occurs for a constant isotopic element T and $|V| \approx |H_0|$. In fact, the selection $T = k^{-n}$ provides a form of *isorenormalization of the coupling constant*, i.e., renormalization via isotopic techniques, which essentially implies the lifting

$$k \gg 1 \quad \rightarrow \quad k^{1-n} \ll 1.$$
(11.1.4)

As we shall see, this <regeneration of convergence> is linked to the mechanism of <regeneration of exact symmetry> studied in the preceding chapters, such as the reconstruction of the exact rotational symmetry when believed to be broken by deformations of the sphere, or the reconstruction of the exact Lorentz symmetry when believed to be broken by deformations of the Minkowski metric.

In particular, the isotopy of space–time symmetries is one of the broadest known forms of renormalization, in the sense that it implies a renormalization not only of conventional quantities, but also of the intrinsic characteristics of a given particle. Convergence of perturbative expansions under the isogalilean and isopoincaré symmetries then follows under condition (11.3).

Intriguingly, the application of the isotopic methods to quark theories studied in Vol. III implies an isotopic operator T characterized by the operator formulation of the classical Nambu mechanics for triplets of Sect. 11.2.6. In fact, realization (11.2.6.14c), i.e.,

$$T = H_1^{-1} + H_2^{-1},$$
(11.1.5)

which naturally fulfills condition (11.1.3), because for strong interactions we

naturally have the properties [3]

$$H_1 \gg 1, \quad H_2 \gg 1, \quad |T| \ll 1. \tag{11.1.6}$$

We can therefore anticipate the relevant possibility that the formulation of strong interactions at large, and the isotopies of the quark theories in particular via the hadronic-operator form of Nambu's mechanics, are naturally set to have a convergent perturbative theory.

Despite the limited knowledge in the field lamented earlier, the available results are sufficient to indicate that the divergence of the perturbative expansions of strong interactions appears to be primarily due to their current local-canonical approximation, and that the perturbative series can indeed be turned into a convergent form when the nonlinear-nonlocal-noncanonical character of the strong interactions due to deep weave-overlapping is properly treated.

11.2: TIME INDEPENDENT ISOPERTURBATION THEORY

Consider the conventional, time-independent quantum perturbation theory, e.g., as presented in ref. [5]. The axiomatic structure of hadronic mechanics presented in Ch.s 3 and 4 permits its step-by-step isotopic liftings.

In the following shall study the general case with isofield $\hat{F}_T$, isoenvelopes $\hat{\xi}_T$ and isohilbert spaces $\mathcal{H}_G$ with different isotopic element T and G (Sect. 11.6.2). The main result can be expressed as follows.

Lemma 11.2.1 [2,3]: *Under sufficient smoothness and regularity conditions, given a conventionally divergent, time-independent, quantum perturbative series, there always exists an infinite number of isotopies of Class I of the underlying fields F, enveloping associative algebras ξ and Hilbert spaces*

$$\xi \rightarrow \hat{\xi}_T, \quad F \rightarrow \hat{F}_T, \quad \mathcal{H} \rightarrow \mathcal{H}_G, \tag{11.2.1}$$

characterized by

$$|T| \ll 1 \quad \text{and} \quad |G| \ll 1, \tag{11.2.2}$$

under which the series becomes convergent.

Two inter-related proofs, one of algebraic and the other of analytic character, will be given below and a third proof of group theoretical nature will be given in Vol. III.

Isoalgebraic proof: Consider a conventional, divergent, canonical series in terms of a positive–definite parameter $k > 1$,

$$A(k) = A(0) + k [A, H]_\xi / 1! + k^2 [[A, H]_\xi, H]_\xi / 2! + \dots \Rightarrow \infty, \qquad (11.2.3a)$$

$$[A, H]_\xi = AH - HA, \qquad (11.2.3b)$$

where the operators A and H are Hermitean and sufficiently bounded. Then, the lemma states that under the isotopic lifting the preceding series we have isoconvergence,

$$\hat{A}(k) = \hat{A}(0) + k [A, H]_{\hat{\xi}} / 1! + k^2 [[A, H]_{\hat{\xi}}, H]_{\hat{\xi}} / 2! + \dots \leq |N| < \infty \qquad (11.2.4a)$$

$$[A, H]_{\hat{\xi}} = ATH - HTA, \qquad (11.2.4b)$$

In fact, the simple case in which $T = \epsilon k^{-1}$, where ϵ is a sufficiently small positive–definite constants, the isotopies turn expansion (11.2.3) into the convergent form

$$\hat{A}(k) = \hat{A}(0) + k [A, H]_{\hat{\xi}} / 1! + k^2 [[A, H]_{\hat{\xi}}, H]_{\hat{\xi}} / 2! + \dots$$

$$\equiv \hat{A}(0) + \epsilon [A, H]_\xi / 1! + \epsilon^2 [[A, H]_\xi, H]_\xi] / 2! + \dots \quad k > 1, T \ll 1, \epsilon = kT \ll 1$$
$$(11.2.5)$$

Even though more involved, the same result can also be proved for a positive–definite operator T verifying condition (11.1.3), e.g., when its diagonal elements are very small.

Isoanalytic proof: Consider a Hermitean operator of the type

$$H(k) = H_0 + k V, \quad H_0 | \psi > = E_0 | \psi >, \quad H(k) | \psi(k) > = E(k) | \psi(k) > . \quad k \gg 1.$$
$$(11.2.6)$$

Assume that H_0 has a nondegenerate discrete spectrum. Then, conventional perturbative series are divergent, as well known. In fact, the eigenvalue $E(k)$ of $H(k)$ up to second order is given by (see, e.g., ref. [5])

$$E(k) = E_0 + k E_1 + k^2 E_2 =$$

$$= E_0 + k < \psi | V | \psi > + k^2 \sum_{p \neq n} \frac{|<\psi_p | V | \psi_n >|^2}{E_{on} - E_{op}}, \qquad (11.2.7)$$

But under isotopies for the general case of hadronic mechanics (Ch. 11.3) we have

$$H(k) = H_0 + k V, \qquad H_0 T | \tilde{\psi} > = \tilde{E}_0 | \tilde{\psi} >, \qquad (11.2.8a)$$

$$H(k) \, T \, | \, \psi(k) > \; = \; \tilde{E}(k) \, | \, \psi(k) > \qquad k > 1 \qquad\qquad (11.2.8b)$$

The simple lifting of the conventional perturbation expansion then yields

$$\tilde{E}(k) \; = \; \tilde{E}_0 \, + \, k \, \tilde{E}_1 + k^2 \, \tilde{E}_2 \, + \, \hat{O}(k^2) \; =$$

$$= \; \tilde{E}_0 \, + \, k < \tilde{\psi} \, | \, G \, V \, T \, | \, \tilde{\psi} > \, + \, k^2 \sum_{p \neq n} \frac{| < \tilde{\psi}_p \, | \, G \, V \, T \, | \, \tilde{\psi}_n > |^2}{\tilde{E}_{on} - \tilde{E}_{op}}, \qquad (11.2.9)$$

whose convergence can be evidently reached via the suitable selection of isotopic elements.

As an example, for a positive–definite constant $G = T \ll 1$, expression (10.2.9) becomes

$$\tilde{E}(k) \; = \; \tilde{E}_0 + k T^2 < \psi \, | \, V \, T \, | \, \psi > + k^2 \, T^5 \sum_{p \neq n} \frac{| < \psi_p \, | \, V \, | \, \psi_n > |^2}{\tilde{E}_{on} - \tilde{E}_{op}}, \qquad (11.2.10)$$

This shows that the original divergent coefficients $1, k, k^2, \ldots$ are now turned into the manifestly convergent coefficients $1, \; kT^2, \; k^2 T^5, \ldots$, with $k > 1$ and $T \ll 1$, thus ensuring isoconvergence for a suitable selection of T for each given k and V.

11.3: TIME-DEPENDENT ISOPERTURBATION THEORY

We study now the isotopies of the time–dependent quantum perturbation theory also from ref. [2]. The main result can be expressed via the following predictable extension of Lemma 10.1

Lemma 11.3.1 [loc. cit.]: *Under sufficient smoothness and regularity conditions, given a conventionally divergent, time–dependent, quantum perturbative series, there always exists an infinite number of isotopies of Class I of the underlying field* F, *enveloping associative algebras* ξ *and Hilbert spaces* $\mathcal{H}$ *under which the series becomes convergent.*

Consider the isochrödinger's equation

$$i \hat{1}_t \, \partial_t \, \hat{\psi}(t, r) \; = \; H * \psi(t, r) \; = \; H \, T \, \hat{\psi}(t, r), \qquad\qquad (11.3.1)$$

where $H = H_0 + V$, and

$$H_0 * \hat{u}_n(r) \; = \; E_n \, \hat{u}_n(r). \qquad\qquad (11.3.2)$$

where the quantities with (without) the superscript represent hadronic (quantum) quantities.

The main idea of the *time–dependent hadronic perturbation theory* is that

of studying the expansion

$$\hat{\psi}(t, r) = \sum_n \hat{a}_n(t) \, \hat{u}_n \, e^{-i \, t \, T_t \, E_n} , \qquad (11.3.3)$$

where $\sum$ denotes sum over discrete indices and integral over continuous ones. By substituting expansion (11.3.3) into Eq.s (10.3.1), we have

$$i \sum_n 1_t \frac{d}{dt} \hat{u}_n \, e^{-i \, t \, T_t \, E_n} + \sum_n \hat{a}_n \, E_n \, \hat{u}_n \, e^{-i \, t \, T_t \, E_n} =$$

$$= \sum_n \hat{a}_n \, (H + V) * \hat{u}_n \, e^{-i \, t \, T_t \, E_n} . \qquad (11.3.4)$$

A step–by–step isotopy of the conventional treatment [5] then leads to the expression

$$i 1_t \frac{d}{dt} \hat{a}_k(t) = \sum_n \hat{V}_{kn} \, e^{i \, t \, T_t \, \omega_{kn}} , \qquad (11.3.5)$$

where

$$\hat{V}_{kn} = \int \hat{u}\dagger_k \, G \, V \, T \, \hat{u}_n \, dV , \qquad (11.3.6a)$$

$$\omega_{kn} =- E_k - E_n \quad (\hbar = 1) , \qquad (11.3.6b)$$

The hadronic time–dependent perturbation theory is then based on replacing the interaction term V with kV and in the power series

$$\hat{a}_n = \hat{a}_n^{(0)} + k \, \hat{a}_n^{(1)} + k^2 \hat{a}_n^{(2)} + \ldots , \qquad (11.3.7a)$$

$$\frac{d}{dt} \hat{a}_k^{(0)} = 0, \quad i 1_t \frac{d}{dt} \hat{a}_{k(t)}^{(p)} = \sum_n \hat{V}_{kn} \, \hat{a}_n^{(p+1)} \, e^{i \, T_t \, \omega_{kn}} . \qquad (11.3.7b)$$

which is at least formally solvable under the conditions assumed, as well as truncated at the desired term.

The regeneration of convergence when the original series is divergent is now trivial. In fact, the original divergence is due to the excessively large term V_{kn}. But these terms can be turned in isotopic images $\hat{V}_{kn}$ as small as desired via a suitable selection of the two isotopic operators G and T in isoeigenvalues (113.7), and this proves Lemma 10.3.2.

We can therefore conclude by stating that the regaining of convergence for strong interactions is indeed within technical reach. The *isodual isotopic perturbation theory* is a simple antiautomorphic image of the above theory. For the more general *genotopic perturbation theory* we refer the interested reader to the original derivation [4].

REFERENCES

1. R. M. SANTILLI, Hadronic J. $\underline{1}$, 574 (1978

2. R. M. SANTILLI, Hadronic J. Suppl. $\underline{4B}$, issue no. 2 (1989)

3. R. M. SANTILLI, Isotopies of quark theories, invited paper for Intern. J. Phys., in press

4. A. JANNUSSIS and R. MIGNANI, Physica $\underline{A187}$, 575 (1992)

5. W. S. C. WILLIAMS, *An Introduction to Elementary Particles*, Academic Press, New York (1971)

12: NONPOTENTIAL SCATTERING THEORY

12.1: STATEMENT OF THE PROBLEM

It is generally believed that the data elaboration of contemporary measures on strong interacting elaborated via the conventional *potential scattering theory* (see, e.g., ref.s [1]) have a final experimental character. Hadronic mechanics has stimulated a moment of reflection on this "experimental belief" because of the existence of a consistent *nonpotential scattering theory* which show *different numerical results in the data elaboration of the same experiments.*

The potential scattering theory was historically conceived, developed, and originally applied only for *action–at–a–distance interactions,* such as the scattering of alpha particles by nuclei, in which there is no physical "contact" between the two scattering objects.

Subsequently, because of its original successes, and according to a historical process not sufficiently investigated by historians, the same theory was developed up to the available degrees of diversification and was indiscriminately applied to *all* contemporary scattering experiments of hadrons, including deep inelastic scattering.

A comparison of the original historical conception of the potential scattering theory and its current use in high energy hadron physics reveals the following rather forceful aspects. On one side, the theory has remained of fundamentally potential/action–at–a–distance type, while, on the other side, deep inelastic scattering experiments are expected to imply contact–nonpotential interactions which are at variance with the very conception and structure of the potential scattering theory.

In fact, the conditions of these inelastic scatterings imply the deep mutual penetration and overlapping at high energy of extended charge distributions of hadrons with consequential expectation of nonlinear, nonlocal and noncanonical interactions which are structurally beyond the formulation, let alone treatment, of the contemporary scattering theory.

As we shall see in this chapter, hadronic mechanics does indeed imply a consistent, step-by-step isotopic generalization of the potential scattering theory for nonlinear–nonlocal–nonpotential interactions which can be expected to originate from its axiomatic structure (Ch.s II.3 and II.4).

The fundamental issue in which hadronic mechanics has stimulated a moment of reflection is that *the nonpotential scattering theory implies an apparent revision of at least some of the currently accepted experimental numbers in hadron scatterings*. In fact, *nonpotential internal effects imply a necessary alteration of the very definition of differential and total cross sections*. The possible revision of at least some of the currently accepted numerical interpretation of the experiments is a mere consequence.

Independently from the above considerations implying mutual penetration of hadrons, the potential scattering theory has shown clear limitations, e.g., in its inability to represent the different cross sections for the *elastic* scattering of spin $\frac{1}{2}$ hadrons with spins parallel and antiparallel [1]. This latter occurrence implies contact among hadrons without appreciable mutual overlap. As such, it focuses the attention on another restrictive character of the theory, the point–like approximation of particles as necessary from its local–differential topology.

Stated in different terms, singlet and triplet couplings are admitted with equal probability from the very axioms of quantum mechanics, thus resulting in the same cross sections for inelastic scattering with spins parallel and antiparallel which is contrary to experimental evidence. Any "phenomenological" model attempting the representation of the above occurrence deviates in one way or another from the basic axioms of the theory.

This is another arena of clear applicability of the covering nonpotential scattering theory. In fact, hadrons are represented with their actual nonspherical shape via the isounit of the theory. In turn, the admission of the extended character of particles implies the necessary *inequivalence* of singlet and triplet couplings under conditions of mutual contact. As an example, gears can only couple in singlet because their triplet coupling would imply each gear rotating *against* the rotation of the other.

The quantitative representation of the inequivalence of singlet and triplet couplings is then expected to permit the nonpotential scattering theory to reach the first quantitative–numerical representation on record of the experimental data in the field via a theory derivable from primitive axioms.

It is evident that the study of the above issues will likely continue into the next century, and cannot possibly be resolved in these first books in the field. Our objective is merely that of outlining the current status of our knowledge in the problem considered and identify the essential open issues for their resolution by interested researchers at some future time.

The idea to build a nonpotential scattering theory was submitted by this author in the original proposal of the hadronic mechanics [2], where the inequivalence of singlet and triplet coupling at small distances was established

via the so-called "gear model". The formal foundations of the nonpotential scattering theory were subsequently studied in detail by Mignani (see papers [3-6] and additional references quoted therein) and, for this reason, the theory has been here called *Mignani's nonpotential scattering theory*. Additional studies were conducted by this author in ref. [7]. The extension of the results to the superscattering framework has been recently done by Bartoli and Bergia [8]. No additional contribution is available in the field at this writing (summer 1994) to our best knowledge.

In essence, Mignani [loc. cit.] used the incomplete isoschrödinger's equation (II.2.3.5) with conventional plane-waves and conventional spherical coordinates over conventional fields, geometries and Hilbert spaces. This author [loc. cit.] studied the foundations of the theory via the full isoschrödinger's equations, related isoplane waves and isospherical coordinates defined on isofields, isogeometries and isohilbert spaces, including the first study of the partial wave expansions in terms of the isospherical harmonics (see App. 11.6.B).

These latter studies permitted the identification of deeper deviations from the potential scattering theory caused by nonlinear, nonlocal and nonpotential internal effects, with particular reference to numerical deviations in the *interpretation* of conventionally measured angles and cross sections.

This chapter is specifically devoted to the study of the *Lie-isotopic* formulation of the nonpotential scattering theory, which is the branch applicable to *scattering processes in their center-of-mass system* when considered as closed-isolated with conventional total conservation laws. The extension to the more general *Lie-admissible* formulation is straighforward from the analysis of the preceding chapters and will be left to the interested reader for brevity.[134]

The Lie-admissible theory applies to open-nonconservative scatterings, e.g., a beam of particles scattering on an external target in such a way to experience the *lack* of conservation of at least some quantity (e.g., the beam's energy or angular momentum). As such, the Lie-admissible nonpotential scattering theory is particularly suited to complete our axiomatization of the origin of irreversibility at the particle level studied throughout this volume.

At this point the studies of this chapter acquire a new dimension because applicable to a number of generalizations existing in the literature. In fact, the need for a *nonunitary* scattering matrix for irreversible processes has been known for some time (see, e.g., the article by Hawkins [9] on the nonunitary scattering matrix, the application of the theory to a nonunitary irreversible model of quasar dynamics by Ellis et al. [10] and references quoted therein).

As pointed out in App. 11.9.A, these models are afflicted by a number of problematic aspects which are common to all conventional formulations of nonunitary time evolutions, such as loss of form-invariance, loss of Hermiticity-observability at all times, loss of the measurement theory because of the lack of

---

[134] Most of the literature in the field has been written for the Lie-admissible case and reduced in this chapter to the simpler Lie-isotopic structure.

conservation of the basic unit, etc.

As proved by this author in ref. [7] (see Sect. II.12.4), *all nonunitary scattering matrices defined over conventional fields, geometries and Hilbert spaces always admit an identical reformulation for the closed-isolated case as isounitary scattering matrices in isofield, isogeometries and isohilbert spaces, and for the open-irreversible case as genounitary scattering matrices over genofields, genogeometries and genohilbert spaces.*

This result implies the regaining of unitarity at the abstract level with the consequential resolution of all problematic aspects mentioned earlier, i.e., form-invariance of the theory under the time evolution, preservation of Hermiticity-observability at all times, applicability of the measurement theory because of the invariance of the basic unit, etc.

In this chapter we shall present the theoretical foundations of Lie-isotopic nonpotential scattering theory. Applications to experiments will be presented in Vol. III. A technical knowledge of the virtual entirety of hadronic mechanics is necessary for a technical understanding of this chapter.

12.2: BASIC DIFFERENCES BETWEEN POTENTIAL AND NONPOTENTIAL SCATTERING THEORIES

It may be advantageous to point out up-front some of the basic differences between the potential and nonpotential scattering theories, which can thereafter be of guidance in the understanding of the new formulation.

When first exposed to the idea of a possible nonpotential generalization of the scattering theory a rather natural reaction is that there cannot be a generalization, because the scattering theory measures "numbers" of scattered particles, their "angle" of scattering and other quantities which cannot be changed by the potential or nonpotential nature of the interactions.

As the reader will soon see, such a simplistic attitude is essentially due to lack of technical knowledge of the isotopies. In fact, it is evident that the measurement of the scattering angle cannot be changed by any theory and, therefore, it is not the issue. The issue is instead the "interpretation" of that angle in term of the theory as an "experimental evidence" whenever dealing with scatterings implying actual contact among extended particles. In fact, the measured angle is that use in partial wave analysis. But the very notion of "angle" is inapplicable under nonpotential interactions (Appendix I.6.A), let alone the conventional trigonometric functions, or the conventional partial wave analysis. The questionable nature of the current use of the "potential" scattering theory for "nonpotential" scatterings is then beyond credible doubts.

In short, *when first exposed to the nonpotential scattering theory, the recommended attitude is that of expecting no alteration of current experimental*

measures, but expecting instead deviations from current theoretical elaborations of the same measures.

Consider the conventional theory as presented, e.g., in ref. [1], p. 68 and ff. The starting point is the notion of an incoming plane wave in vacuum in conventional Euclidean space $E(r,\delta,R)$

$$\psi^{in}(t, r) = N e^{i k r}. \tag{12.2.1}$$

The outgoing or scattered wave is usually written

$$\psi^{out} = \psi^{in} + e^{i k r}\frac{f(\theta)}{r}, \tag{12.2.2}$$

where θ is the polar angle between initial and final directions. To compute the scattering amplitude $f(\theta)$ it is customary to assume the original direction along the z-axis, and to develop the plane wave into eigenfunctions of the square of the angular momentum L^2, according to the familiar expression

$$\psi^{in}(t, r) = \sum_{L=0,\infty} (2L + 1) i^L P_L(\cos \theta) F_L(kr), \tag{12.2.3}$$

where: r is the distance between the particles; $F_0(kr) = [\sin (kr)] / kr$; the other expressions $F_L(kr)$ have certain known connections with Bessel functions; and $P_L(\cos \theta)$ represents the Legendre polynomials as eigenfunctions of L^2 with $L_z = 0$. Under these familiar assumptions, we have

$$f(\theta) = (2 i k)^{-1} \sum_L (2L + 1)[e^{2i\delta_L} - 1] P_L(\cos \theta), \tag{12.2.4}$$

where δ_L is the phase shift.

The *potential differential cross section* is then given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = k^{-2}|\sum_L(2L + 1) e^{i\delta_L} \sin \delta_L P_L(\cos \theta)|^2, \tag{12.2.5}$$

where Ω is the solid angle in steradians, and the *potential total cross section* is given by the familiar form

$$\sigma = \int_{-1}^{+1} (d\sigma / d\Omega) 2\pi d(\cos \theta) = 4\pi k^{-2} \sum_L (2L + 1) \sin^2 \delta_L. \tag{12.2.6}$$

Hadronic mechanics implies a unique generalization/covering of the above theory beginning from its geometric foundations. In this chapter we shall proceed in stages of generalizations and, therefore, of complexity. First, in this section we shall outline the generalization of the preceding structure worked out by the author in ref. [7]. Further developments will be studied thereafter.

As now familiar, the very notion of Euclidean space $E(r,\delta,R)$ over the reals $R(n,+,\times)$ is generalized for the interior of hadrons into the *isoeuclidean spaces*

$\hat{E}(R,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, $T > 0$ over the isofields $\hat{R}(\hat{n},+,*)$ with isounit $\hat{1} = T^{-1}$. The lifting $E(r,\delta,R) \rightarrow \hat{E}(r,\hat{\delta}\leftrightharpoons,\hat{R})$ then represent the transition from motion in the homogeneous and isotropic vacuum, as requested by the potential scattering among point particles, to motion within an inhomogeneous and anisotropic physical medium, as requested by the nonpotential scattering theory (see for more details Fig. I2.2.1).

It should be stressed that, according to the nonpotential scattering theory, *the conventional Euclidean geometry remains valid everywhere except in the scattering region where the covering isoeuclidean geometry applies.*

Since scattering measures are done outside of the scattering region, the isotopic element T must be averaged to constants, and we can write

$$T = \text{diag.}\,(b^{\circ}_1{}^2, b^{\circ}_2{}^2, b^{\circ}_3{}^2)\,, \quad b^{\circ}_k = \text{cost} > 0, k = x, y, z. \quad (12.2.7)$$

This implies the use of the *restricted isogalilean transforms* of Ch. II.7 (or of the *restricted isopoincaré transforms* of Ch. II.8 for relativistic extensions) which are known to be linear, thus preserving the conventional inertial character of the reference frame. Thus, despite their constancy, the b°_k quantities are average over highly nonlinear–nonlocal–nonpotential internal effects and, as such, they are nontrivial.

For *elastic* scattering of hadrons it is generally recommendable to introduce the condition

$$\text{Det}\,T = b^{\circ}_1{}^2\,b^{\circ}_2{}^2\,b^{\circ}_3{}^2 = I\,. \quad (12.2.8)$$

which essentially expresses the conservation of the volume of the scattering particles. For inelastic scatterings the above condition is generally unwarranted because of the compressibility of hadrons scattering at very high energies.

Needless to say, an explicit functional dependence of the isotopic element is indeed admissible $T = T(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, ...)$ with the clear understanding that we are now dealing with a representation *inside the scattering* area. The reader should be aware that in this case the *general isogalilean (or isopoincaré) transforms* apply with consequential loss of the inertial character of the admitted frames.

The first departure from the potential scattering theory implied by the isoeuclidean geometry with measurable consequences is the generalization of the conventional incoming plane wave into the *incoming isoplane waves*

$$\hat{\psi}^{in} = \hat{N} * e^{i\,k\,T\,r} = N\,e^{i\,k^i b^{\circ}_i{}^2\,r^i}, \quad (12.2.9)$$

essentially representing the transition from motion in the homogeneous and isotropic vacuum to inhomogeneous and anisotropic physical media.

The second departure with additional measurable consequences is the inapplicability of conventional spherical coordinates in favor of the *isospherical*

coordinates of Ch.s 11.5 and 11.6. In fact, as the reader will recall, the conventional spherical polar coordinates do not permit the separation in isospace between the radial and the angular components (Sect. 11.5.4).

The third initial departure is that the very notion of angle is inapplicable for the isoeuclidean geometry in favor of the *isoangles.* It is at this point where we begin to see real differences in the data elaborations of the same experiment via the potential and nonpotential scattering theories. In fact, the scattering angle θ is precisely that measured by experiments and, as such it is certainly valid up to the scattering region. According to the former theory, such angle applies also in the interior of the scattering region. According to the latter theory, a different angle $\hat{\theta}$, the isoangle of the isospherical coordinates, applies for the interior scattering region much along the conventional refraction of light.

Suppose that one preserves the original direction of the incident wave for the interior region along the z–axis of $\hat{E}(r,\hat{\delta},\hat{R})$. Suppose, for this first illustration, that such incident z–direction is also the z–axis of the target (see Fig. II.12.1). Then, under these conditions the isospherical coordinates imply the expression

$$k^i \, b^{\circ}_i{}^2 \, r^i \; = \; k \, b_3{}^2 z \; = \; k \, b^{\circ}_3{}^2 \, r \, b^{\circ}_3{}^{-1} \cos \hat{\theta} = k \, b_3 \, r \cos \hat{\theta}, \quad k = k^3 . \quad (12.2.10a)$$

$$\hat{\theta} \; = \; \theta \, b^{\circ}_3 , \qquad\qquad\qquad (12.2.10b)$$

where one should remember that the b°_3 factor in the argument of the trigonometric function can be different for different forms of the isospheric coordinates or, equivalently, for different orientations between the incident wave and the target.

The conventional scattered wave is then generalized into the expression

$$\hat{\psi}_{out} \; = \; e^{i \, k_i \, b_i{}^2 \, r_i} \; + \; e^{i \, k \, b_3 \, r} \; \frac{\hat{f}(\theta)}{r} \qquad\qquad (12.2.11)$$

where $\hat{\theta}$ is the isopolar angle (11.2.9b) between the original and final directions and $\hat{f}(\theta)$ is the *isoscattering amplitude* (see Sect. 11.12.10 for the full definition $\hat{f}(\hat{\theta}, \hat{\phi})$).

To determine the new expression $\hat{f}(\hat{\theta})$, called *nonpotential scattering amplitude* , we now assume the validity within the scattering region of the *hadronic angular momentum* according to the rules

$$\hat{L}^2 * \hat{Y}_{\hat{L}}{}^{\tilde{M}}(\hat{\theta}, \hat{\phi}) \; = \; D^{-2} \, \hat{L} \, (\, \hat{L} + 1) \, \hat{Y}(\hat{\theta}, \hat{\phi}), \qquad (12.2.12a)$$

$$\hat{L}_z * \hat{Y}_L{}^{\tilde{M}}(\hat{\theta}, \hat{\phi}) \; = \; D^{-1} \, b_3 \, \tilde{M} \, \hat{Y}_{\hat{L}}{}^{\tilde{M}}(\hat{\theta}, \hat{\phi}), \qquad (12.2.12b)$$

$$\hat{\psi}(r, \hat{\theta}, \hat{\phi}) \; = \; \hat{R}(\hat{r}) \, \hat{Y}_{\hat{L}}{}^{\tilde{M}}(\hat{\theta}, \hat{\phi}), \qquad \hat{R}(r) = N_{\hat{L}} \, J_{\hat{L}}(kr), \quad k' = (\, 2 \, m \, E \,)^{\frac{1}{2}}, \quad (12.2.12c)$$

$$D = (\, \det T \,)^{1/2} = b^{\circ}_1 \, b^{\circ}_2 \, b^{\circ}_3 = 1, \qquad (12.2.12d)$$

with isoeigenvalues

$$\tilde{L} \; = \; 0, 1, 2, ... , \quad \hat{M} \; = \; \hat{L}, \hat{L}-1, ..., -\hat{L}, \qquad (12.2.13a)$$

and isounit for isospherical coordinates

$$1_{\hat{\omega},\hat{\phi}} = (T_\omega T_\phi)^{-1} = (\text{Diag. } T)^{-1/2} = 1 \qquad (12.2.14)$$

where $\hat{Y}_{\hat{L}}^{\hat{M}}(\hat{\theta}, \hat{\phi})$ are the *regular isospherical harmonics* (11.6.15) and $J_{\hat{L}}(kr)$ are the conventional Bessel functions, although defined with respect to radial variables on isospace. We then have the *isoplane-wave expansion* (11.6.B.6) first studied by this author

$$e^{i\,\mathbf{kTr}} = \sum_{\hat{L}=0,\dots,\infty} \sum_{\hat{M}=L,\dots,-L} N_{\hat{L}\hat{M}} J_L(kr)\, \hat{Y}_{\hat{L}}^{\hat{M}}(\hat{\theta},\hat{\phi}), \qquad (12.2.15)$$

with the simplified form for the expansion along the polar axis

$$\hat{\psi}^{in} = e^{i\,kb_3^2 z} = e^{i\,kb_3 r \cos b_3\theta} = \sum_{\hat{L}} N_{\hat{L}} J_{\hat{L}}(kr)\, \hat{P}_{\hat{L}}[\cos(b_3\theta)]. \qquad 11.2.16)$$

The *phase isoshift*, that is, the phase shift due to potential and nonpotential interactions, can be defined in a variety of ways, e.g., in terms of the expression

$$\hat{\psi}^{out} = (2ikb_3^\circ r)^{-1} \sum_{\hat{L}} (2\hat{L}+1)\,(\,e^{ikb_3^\circ r + 2i\delta_L} - e^{-ikb_3^\circ r + i\pi L})\,\hat{P}_{\hat{L}}(\cos\hat{\theta}), \qquad (12.2.17)$$

The *nonpotential differential cross section* is then given by [7]

$$\frac{\hat{d}\hat{\sigma}}{\hat{d}\Omega} * |\hat{f}(\theta)|^2 = 1_{\theta\phi}\; \frac{\hat{d}\hat{\sigma}}{\hat{d}\Omega}|\hat{f}(\theta) * \hat{f}(\theta)| = (kb_3^\circ)^{-2}\,|\sum_{\hat{L}}(2\hat{L}+1)\,e^{i\delta_L}\sin\hat{\delta}_L\,\hat{P}_{\hat{L}}(\cos\theta)|^2$$

$$(12.2.18)$$

where $\hat{\Omega}$ is the solid angle on $\hat{E}(r,\hat{\delta},\hat{R})$ which coincides with Ω under the assumption $\text{Det}T = 1$. The *nonpotential total cross section* is then given by [7][135]

$$\sigma = \int_{-1}^{+1} (\hat{d}\hat{\sigma}/\hat{d}\Omega)\,2\pi\,\hat{d}(\cos\theta) = 4\pi\,(k\,b_3^\circ)^{-2}\sum_{\hat{L}}(2\hat{L}+1)\sin^2\hat{\delta}_{\hat{L}}. \qquad (12.2.19)$$

where one can use ordinary integrals from the identity $\int \hat{d}\hat{\Omega} = \int 1_{\theta,\phi}\, T_\theta T_\phi \hat{d}\hat{\sigma} = \int \hat{d}\hat{\sigma}$

Comparison with the corresponding expression of the potential scattering theory

$$\sigma = \int_{1}^{+1} (d\sigma/d\Omega)\,2\pi\,d(\cos\theta) = 4\pi\,k^{-2}\sum_{L}(2L+1)\sin^2\delta_L, \qquad (12.2.20)$$

---

[135] The first expression of the nonpotential cross section was derived by Mignani in ref. [4] of 1984 on conventional fields, geometries and Hilbert spaces which implied the appearance of the isotopic element T as a *factor* of the conventional differential and total cross sections. The identification of the alteration of the phase shift, scattering angles and cross section presented in this section was done by this author in ref. [7] of 1989. This occurrence shows the impact on the final numerical values of the proper use of hadronic mechanics on isofields, isogeometries and isohilbert spaces.

then clarifies the fundamental point of Sect. II.12.1, that *the nonpotential scattering theory implies no change in the total cross section σ which evidently remains as measured. However, the data elaborations of the same number σ are different for the potential and nonpotential scattering theories, because they imply numerically different phase shifts, numerically different values of the angular momentum, etc.*

From a geometric viewpoint, the transition from the conventional to the isotopic cross sections essentially represents the transition from the scattering of a plane wave on a perfectly spherical target represented by δ = diag. (1, 1, 1), to a nonspherical ellipsoidical target represented by the isometric $\hat{\delta}$

$$\delta = \text{diag.} (1, 1, 1) \quad \Rightarrow \quad \hat{\delta} = \text{diag.} (b°_1{}^2, b°_2{}^2, b°_3{}^2), \tag{12.2.21}$$

Condition (11.2.12d) then ensures the preservation of conventional magnitude of the angular momentum. Thus, the case herein considered essentially consists of the scattering of the wave on a target consisting of (see Fig. 12.2.1):

A) an oblate spheroidal ellipsoid with oblateness characterized by $b°_3 > 1$ and $b°_1 = b°_2 < 1$, in which case $\hat{\theta} = \theta \, b°_3 > \theta$, or

B) a prolate spheroidal ellipsoid with prolateness characterized by $b°_3 < 1$ and $b°_1 = b°_2 > 1$, in which case $\hat{\theta} = \theta \, b°_3 < \theta$.

A general rule of the isotopies, which implies their reconstruction of the exact rotational symmetry, is that *deformed angles recover perfectly spherical angles when multiplied by their corresponding isotopic element.*

By following ref. [7], we shall therefore denote θ_{QM} the angle measured for a perfectly spherical target, and $\hat{\theta}_{HM}$ the angle for a nonspherical target. The regeneration of the exact rotational symmetry under nonspherical targets is then based on identities (II.6.3.20),

$$\hat{\theta}_{HM} \equiv \theta_{QM} \, b_3 . \tag{12.2.22}$$

Thus, for waves penetrating nonspherical targets, *we expect a scattering angle $\hat{\theta}_{HM}$ which is bigger or smaller than the corresponding scattering angle θ_{QM} of waves penetrating within a perfectly spherical target depending on whether the target is prolate or oblate along the direction of the incident wave ,*

$$b°_3 < 1 : \qquad \hat{\theta}_{HM} > \theta_{QM} , \tag{12.2.23a}$$

$$b°_3 > 1 : \qquad \hat{\theta}_{HM} < \theta_{QM}. \tag{12.2.23b}$$

The conclusion of ref. [7] is therefore that *the alteration of the geometry of space in which the wave propagates, when represented via the isotopic methods, implies:*

I) The alteration of the phase shift from the numerical value δ_L predicted by the potential scattering theory to different values $\hat{\delta}_L$ depending on the selected medium in which motion occurs;

II) The alteration of the potential scattering angle θ used in the plane-wave expansions into the isotopic form $\hat{\theta} = \theta b°_3$ representing the nonspherical shape of consequential modifications of the differential and total cross sections.

THE CONCEPT OF POTENTIAL SCATTERING
Scattering of wave through a spherical target
xx + yy + zz = 1
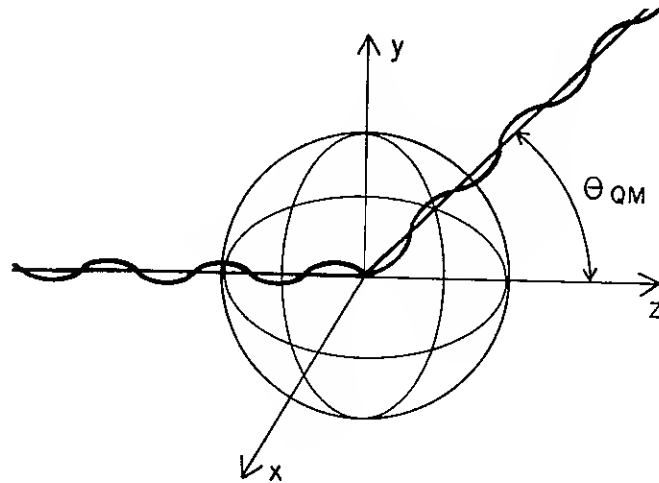
THE CONCEPT OF NONPOTENTIAL SCATTERING
Scattering of waves through spheroidal targets
$$xb°_1{}^2 x + y b°_2{}^2 y + z b°_3{}^2 z = 1$$
CASE 1: Prolate spheroid $b°_3 > 1$, $b°_1 = b°_2 < 1$, $\hat{\theta}_{HM} < \theta_{QM}$
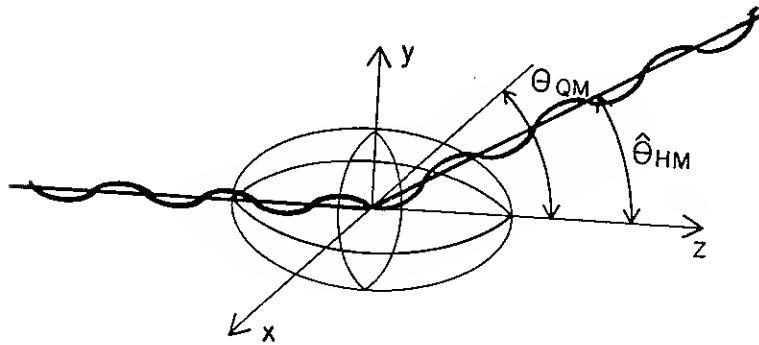
CASE B: Oblate spheroid $b^\circ_3 > 1, b^\circ_1 = b^\circ_2 < 1, \hat{\theta}_{HM} > \theta_{QM}$
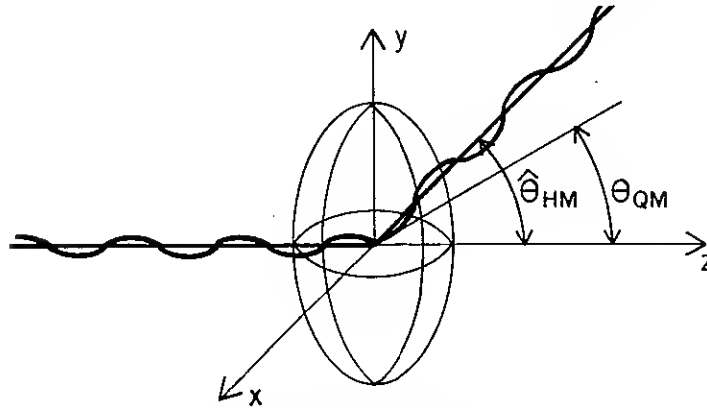
FIGURE 12.2.1: Even though not sufficiently stressed in contemporary textbooks in the field, the conventional *potential scattering theory* is centrally dependent on the point-like approximation of the scattering particles, as well as the the homogeneity and isotropy of the medium in which waves propagate, the vacuum. With the passing of time, the theory was applied to the data elaboration of all high energy inelastic collisions among hadrons with mutual penetrations of the charge distributions-wavepackets. But, in the physical reality, hadrons are not points. The admission of the actual extended character of hadrons at the level of first quantization implies that, while the potential scattering theory is indeed exact for scattering experiments at-a-distance, its exact character for the conditions of mutual penetrations of hadrons is questionable on a number of independent counts, the understanding being that its *approximate* validity is out of scientific doubts. In this figure we present a conceptual view of the scattering of plane waves on a perfect sphere, as in the potential scattering theory, and on ellipsoids as for Mignani's nonpotential scattering theory. The different scattering angles or, more properly, the different interpretations of a given scattering angle, are then evident.

12.3: NONPOTENTIAL SCATTERING OF ISOPARTICLES WITH SPIN

The presence of spin in the nonpotential scattering theory was first studied in ref. [7] to add a central aspect, the contribution to the scattering due to the anisotropy of the medium in which waves propagate (which is caused precisely by its intrinsic angular momentum).

The emerging theory is expected to provide the first interpretation on record of the differences between the elastic scatterings for hadrons with spins parallel and antiparallel which has not been interpreted via the conventional theory.

The simplest possible assumption is that the scattering produces no

mutation of the *individual* spins. This permits the acceptance of the isotopic $S\hat{U}(2)$ symmetry everywhere under the condition that the isoeigenvalues are conventional (i.e., we can use regular isoreps with Det T = 1 or standard isoreps of $S\hat{U}(2)$). Despite that, the nonpotential scattering theory shows the emergence of *hidden* degree of freedom much similar to those pointed out in Ch. 11.6.

A lesser restrictive condition is that the *total* spin of the incoming particle and of the target is conventionally quantized which, when treated via the isotopic $S\hat{U}(2)$ theory, implies possible individual departures from the individual quantum mechanical spin values. This deeper study apply for sufficiently higher energies and permits the identification of additional *hidden* degrees of freedom also absent in the conventional theory.

The isotopy of the conventional scattering of a spin zero by a spin $\frac{1}{2}$ particle (see, e.g., ref. [1], p. 76 and ff) is straightforward. Introduce a basis of the isopauli matrices (Sect. 11.6.8) with proper isonormalization to 1, and denote them with $\hat{b}(\frac{1}{2}, \pm\frac{1}{2})$. The incoming iso-plane wave expansion of the preceding section then becomes

$$e^{i k b^{\circ}_{3} z} \times \hat{b}(\tfrac{1}{2}, \pm\tfrac{1}{2}) = (4\pi)^{\frac{1}{2}} \sum_{\hat{L}} (2\hat{L} + 1)^{\frac{1}{2}} i^{\hat{L}} J_{\hat{L}} (k\hat{r}) Y_{\hat{L}0}(\cos b^{\circ}_{3} \theta) \times \hat{b}(\tfrac{1}{2}, \pm\tfrac{1}{2}) . \quad (12.3.1)$$

Tedious but straighforward calculations based on a step-by-step isotopy of the conventional case then yield the *nonpotential total cross section* with the minimal possible deviations

$$\sigma = 4 \pi \sum_{L = J - \frac{1}{2}, J + \frac{1}{2}} (J + \tfrac{1}{2}) | A_{JL} * A_{JL} | , \qquad (12.3.2a)$$

$$A_{JL} = [\eta_{JL} \exp (2 i \hat{\delta}_{JL}) - 1] / 2 i k b^{\circ}_{3} , \qquad (12.3.2b)$$

with progressively increasing deviations when the mutation of individual angular momenta is admitted.

The approach is easily extendable to the *elastic* scattering of a spin $\frac{1}{2}$ particle by a spin $\frac{1}{2}$ particle (ref. [1], p. 91 and ff) resulting in the *nonpotential scattering amplitude* with minimal possible deviations

$$\hat{f}_{1,2}(\theta) = (1 / 2 i k b^{\circ}_{3}) \sum_{\hat{L} \text{ even}} [4 \pi (2 \hat{L} + 1)]^{1/2} (e^{2 i \hat{\delta}_{\hat{L}}} - 1) \hat{Y}_{\hat{L}}(\theta) . \quad (12.3.3)$$

As we shall see in Vol. 111, *the above expression has different values depending on whether the scattering is for spins parallel or antiparallel,* thus producing the first interpretation on record of their differences.

12.4: ISOUNITARY NONPOTENTIAL SCATTERING MATRIX

We shall now review Mignani's [6] formal studies of the nonpotential scattering theory beginning with the case without spin, in which there is no consideration

regarding scattering angles, although reformulated as in ref. [7] for full compliance with the axioms of hadronic mechanics. The extension of studies to scatterings with spin in isospaces $\hat{E}(\hat{r}.\hat{\delta},\hat{R})$ will be considered thereafter.

Consider the Lie–isotopic branch of hadronic mechanics of Class I characterized by the isotopies $F \to \hat{F}_T$ (= $\hat{R}_T$ or $\hat{C}_T$), $\xi \to \hat{\xi}_T$ and $\mathfrak{K} \to \mathfrak{K}_G$. We initially assume different isotopic element T and G to better illustrate the origin of the differences between the potential and nonpotential scattering theories, in order to ascertain whether they originate from the enveloping operator algebra $\hat{\xi}_T$ or from the isohilbert space $\mathfrak{K}_G$.

In this section we follow ref. [5] as reinspected in ref. [7] for their use of the complete isoschrödinger's equations and the isotopic special functions. The objective is to study the N–body reaction

$$A_1 + A_2 + ... + A_n \Rightarrow B_1 + B_2 + ... + B_n, \quad n + m = N, \qquad (12.4.1)$$

under the following basic assumptions:

A: The covering nonpotential scattering theory applies whenever there is an appreciable penetration/overlapping of the charge distributions and/or wave–packets of at least some of the particles, with the contact interactions being represented via the isotopic elements $T \neq I$ and $G \neq I$, while all action–at–a–distance interactions are represented via the usual Hamiltonian H;

B: The conventional potential scattering theory [1] applies when all scatterings occur at a distance without any appreciable mutual penetration/overlapping of the particles, in which case we have the particular case $T \equiv G \equiv I$ with all possible interactions being represented by the Hamiltonian.

The above assumptions imply that the isotopic elements T and G are different than I only in regions of the order of I fm, namely, the conventional theory applies everywhere except for corrections occurring inside the scattering region.

The various nomenclature of the conventional theory [1] is preserved for the isotopic one. For instance, we have the arrangement of the reaction into *channels.* We are then studying the *multichannel nonpotential scattering theory.* Similarly, we have the *incoming and outgoing channels* depending on whether the channel considered refers to a particle before or after the scattering. A *cluster* is a properly partitioned subset of the N particles; a *cluster fragment* occurs when the particles are bound together whether before or after the scattering; and so on.

Definition 12.4.1 [5,6]: *The "nonpotential scattering matrix" is the isounitary operator $\hat{S} \in \hat{\xi}_T$ with elements on the isofield $\hat{C}_T$ acting*

on *the isohilbert space* $\mathcal{H}_G$ *which transforms the "incoming isostate"* $| \hat{\psi}^{in} \rangle$, *that before any scattering, into the "outgoing isostate"* $| \hat{\psi}^{out} \rangle$, *that after all scatterings, according to the isomodular expression*

$$| \hat{\psi}^{out} \rangle \; = \; \hat{S} * | \hat{\psi}^{in} \rangle \; = \; \hat{S} \, T(t, r, p, \dot{p}, \hat{\psi}, \hat{\psi}^{\dagger}, \partial\hat{\psi}, \partial\hat{\psi}^{\dagger}, \mu, \tau, n, ...) \, | \hat{\psi}^{in} \rangle \, .$$
$$(12.4.2)$$

with matrix elements

$$\hat{S}_{out|in} \; = \; \langle \hat{\psi}^{out} | \, o \, \hat{S} * | \hat{\psi}^{in} \rangle \; = \; \langle \hat{\psi}^{out} | \, G \, \hat{S} \, T | \hat{\psi}^{in} \rangle \, \hat{1} \, \epsilon \, \hat{C}_T. \quad (12.\,4.3)$$

The above defined $\hat{S}$-matrix is called "nonpotential" to denote the fact that it holds under the most general possible nonlinear, nonlocal and nonpotential interactions where, e.g., nonlinearity is referred not only to the wavefunction but also to its derivatives of arbitrary order, the nonlocality is referred to the integral dependence, in general, on all needed quantities, and the nonpotentiality indicates the violation, in general of the conditions of variational selfadjointness for the existence of a Hamiltonian.

The reader should remember from our assumptions that *the isotopic $\hat{S}$-matrix coincides with the conventional S-matrix for mutual distances of hadrons bigger than 1 fm (exterior scattering problem), and that appreciable differences exist for mutual distances of the scattering hadrons equal or smaller than 1 fm (interior scattering problem).*

The central issue of the nonpotential scattering theory is therefore that of ascertaining whether the nonlinear, nonlocal and noncanonical contributions in the interior problem have a measurable effect in the exterior problem. In order to initiate the study of the issue, let us introduce the *isotransition matrix* $\hat{\mathcal{F}}$ via an isotopy of the conventional expression [5]

$$\hat{\mathcal{F}} \; = \; (\hat{1} - \hat{S}) / 2\pi i \, , \qquad\qquad (12.4.4)$$

with matrix elements evidently given by

$$\hat{\mathcal{F}}_{fi} \; = \; \langle \hat{\psi}^{out} | \, o \, \hat{\mathcal{F}} * | \hat{\psi}^{in} \rangle \, . \qquad\qquad (12.4.5)$$

It is then easy to see that the $\hat{\mathcal{F}}$-matrix implies, as in the conventional case,

$$| \hat{\psi}^{scatt.} \rangle \; = \; - 2\pi i \, \hat{\mathcal{F}} * | \hat{\psi}^{in} \rangle \, . \qquad\qquad (12.4.6)$$

Let $\hat{W}_{fi}$ be the transition isoprobability for reaction (12.4.1) to be identified later on, and let $\hat{\Phi}_i$ be the incident flux. We then have the following

Definition 12.4.2 [5,6]: *The "total nonpotential cross section" is given by*

$$\hat{\sigma} = \frac{W_{fi}}{\Phi_i} = \frac{W_{fi}}{\Phi_i}\hat{\rho}(E)\,|\,T_{fi}\,|^2, \qquad (12.4.7)$$

where $\hat{\rho}(E)$ *is the isotopic density of the final states in the neighborhood of the energy E.*

Most of our subsequent studies are devoted to a deeper understanding of expression (1.4.7). The proof of the *isounitarity of the S-matrix,* first studied by Mignani [3–6], is then trivial because

$$\hat{S}^\dagger * \hat{S} = \hat{S} * \hat{S}^\dagger = \hat{1} = T^{-1}. \qquad (12.4.8)$$

It then follows that the $\hat{\mathcal{F}}$-*matrix is isohermitean*

$$\hat{\mathcal{F}}^\dagger = T^{-1}\,G\,\mathcal{F}^\dagger\,T\,G^{-1} \equiv \hat{\mathcal{F}}, \qquad (12.4.9)$$

as it is possible to prove in first order from the isounitarity of $\hat{S}$ and in general via the isoexponentiation.

Note that *in the conventional Hilbert space* $\mathcal{H}$ *the* $\hat{S}$-*matrix is in general nonunitary.* As a matter of fact, the departure from the conventional unitarity is a measure of the nonpotential forces contained in the theory.

Similarly, *the* $\hat{\mathcal{F}}$-*matrix is generally nonhermitean in* $\mathcal{H}$. However, $\hat{\mathcal{F}}$ is always conventionally Hermitean when $T = G$, although $\hat{S}$ remains conventionally nonunitary even when $T = G$.

Recently, following the studies on irreversible processes by Hawkins [9] and others, there has been a growing interest on *nonunitary* scattering amplitude. The following property then illustrates the significance of hadronic mechanics.

Lemma 12.4.1 [7]: *Any nonunitary scattering matrix* S *on a conventional Hilbert space* $\mathcal{H}$ *over* $C(c,+,\times)$ *can always be decomposed according to the rules*

$$S = \hat{S}\,T^{1/2}, \quad S\,S^\dagger = 1 \neq 0, \quad T = T^\dagger, \qquad (113.4.10)$$

yielding an isounitary scattering matrix $\hat{S}$ *on an isohilbert space* $\mathcal{H}_T$ *over the isofield* $\hat{C}_T(\hat{c},+,*)$ *according to the rules*

$$\hat{S} * \hat{S}^\dagger = \hat{S}^\dagger * \hat{S} = \hat{1}. \qquad (12.4.11)$$

The above reformulation is nontrivial inasmuch as it permits the removal of the problematic aspects of the conventional nonunitary formulation, such as lack of form invariance, loss of Hermiticity–observability at all times, loss of the

measurement theory because of the lack of an invariant unit, etc.

12.5: TIME-DEPENDENT NONPOTENTIAL SCATTERING THEORY

In the time–dependent theory, the incoming (outgoing) isostates describe the system of particles (II.12.4.1) in the infinite past (future), i.e., [5]

$$| \hat{\psi}^{in} > = | \hat{\psi}(-\infty) > , \qquad | \hat{\psi}^{out} > = | \hat{\psi}(+\infty) > . \qquad (12.5.1)$$

The nonpotential $\hat{S}$–matrix can then be defined via the expression

$$| \hat{\psi}(+\infty) > = \hat{S}*| \hat{\psi}(-\infty) > . \qquad (12.5.2)$$

In the conventional potential theory there is a direct connection between the S–matrix and the time evolution operator. The same connection evidently persists under isotopy. Introduce the isounitary time–evolution operator as per Postulate III (Ch. II.3)

$$| \hat{\psi}(t) > = \hat{U}(t, t_0)*| \hat{\psi}(t_0) > , \qquad (12.5.3a)$$

$$\hat{U}\dagger* \hat{U} = \hat{U}*\hat{U}\dagger = 1 , \quad \hat{U}(t_0, t_0) = 1 , \quad \hat{U}\dagger(t, t_0 = \hat{U}(t_0, t) . \qquad (12.5.3b)$$

Then, we have the following formal definition of the $\hat{S}$–matrix where all limits are hereon assumed to be strong as in the conventional case [5,6]

$$\hat{S} = Lim_{t_0 \Rightarrow -\infty} Lim_{t \Rightarrow +\infty} \hat{U}(t, t_0) . \qquad (12.5.4)$$

The differential equation obeyed by $\hat{U}$ has been derived in Sect. II.2.7 and it is given by the full isoschrödinger's equation[136]

$$i 1_t \frac{\partial}{\partial t} \hat{U} = H*\hat{U} = H T \hat{U} , \qquad (12.5.6)$$

where 1_t is the time isounit and H is the total Hamiltonian, which evidently implies the isoschrödinger's equation for the isostate of reaction (II.12.4.1)

$$i 1_t \frac{\partial}{\partial t} | \hat{\psi} > = H * | \hat{\psi} > , \qquad (12.5.7)$$

---

[136] Mignani [3–6] used the preceding version without the factor 1_t, with conventional plane–waves.

Under sufficient smoothness and regularity conditions, the property for $\hat{U}$ to constitute a Lie–isotopic group then implies

$$\hat{U}(t, t_0) = 1 - i \int_{t_0}^{t} dt' \, H * \hat{U}(t', t_0) , \qquad (12.5.8)$$

When H does not depend explicitly on time, we can write

$$\hat{U}(t, t_0) = e_{|\xi}^{-i H (t - t_0)} = \{ e^{-i H T (t - t_0)} \} 1 . \qquad (12.5.9)$$

The above solution, however, is mainly useful to verify the formal consistency of the theory, rather in practical model, owing to the general dependence of H on time. With this objective in mind, and by continuing to follow ref.s [5,6], it is easy to see that the "formal trick" for the conventional time evolution operator admits a consistent isotopic extension and we can write

$$\begin{aligned}
\hat{U}(t, -\infty) &= \mathrm{Lim}_{\eta \Rightarrow +0} \int_{-\infty}^{0} dt' \, e^{\eta t'} \hat{U}(t, t') , \\
\hat{U}(+\infty, t_0) &= \mathrm{Lim}_{\eta \Rightarrow +0} \int_{0}^{+\infty} dt' \, e^{-\eta t'} \hat{U}(t, t') , \\
\hat{U}(-\infty, t_0) &= \mathrm{Lim}_{\eta \Rightarrow +0} \int_{-\infty}^{0} dt' \, e^{\eta t'} \hat{U}(t, t') , \\
\hat{U}(t, +\infty) &= \mathrm{Lim}_{\eta \Rightarrow +0} \int_{0}^{+\infty} dt' \, e^{-\eta t'} \hat{U}(t, t') ,
\end{aligned} \qquad (12.5.10)$$

An isotopy of the standard approach then confirms the formal consistency of the nonpotential scattering theory.

To proceed further, we now introduce the *isomøller operators* for the general reaction (II.12.4.1) (see later on in this chapter for the same operator referred to a given channel)

$$\hat{U}(0, -\infty) = \hat{\Omega}^{+} , \quad \hat{U}(0, +\infty) = \hat{\Omega}^{-} . \qquad (12.5.11a)$$

$$\hat{U}(-\infty, 0) = \Omega \hat{\dagger}^{+} , \quad \hat{U}(+\infty, 0) = \Omega \hat{\dagger}^{-} . \qquad (12.5.11b)$$

Then, by again conducting an isotopy of conventional treatments, the limit definition (10.is formally consistent because can write

$$\hat{S} = \hat{U}(+\infty, -\infty) = \Omega \hat{\dagger}^{-} * \hat{\Omega}^{+} . \qquad (12.5.12)$$

A consistent isotopy of the remaining conventional aspects [1] then follows.

12.6: CONVERGENT PERTURBATIVE ISOEXPANSIONS

We now point out that conventional perturbative expansion techniques for the potential scattering theory admit a consistent isotopic covering of Class 1 and,

when conventionally divergent, there always exist an isotopy under which the series become convergent. The existence of consistent isotopies of conventional perturbative series of the scattering theory was first identified in ref. [3-6]. Their isotopic convergence was first studied in ref. [7].

Consider the *iso-Dyson-Feynman perturbative expansion* (see ref. [1] for the conventional case)

$$\hat{U}(t, t_0) = \sum_{n=0,\infty} \hat{U}_n(t, t_0), \qquad (12.6.1a)$$

$$\hat{U}_n(t, t_0) = (-i)^n \int_{t_0}^{t} \int_{t_0}^{t^{(n-1)}} \times \times \int_{t_0}^{t^{(n-1)}} dt^{(1)} \times\times\times dt^{(n)} H(t^{(1)}) *\times\times* H(t^{(n)}), (12.6.1b)$$

$$t_0 \leqq t^{(n)} \leqq t^{(n-1)} \leqq t^{(1)} \leqq t. \qquad (12.6.1c)$$

An intriguing property of the nonpotential scattering theory is the possibility of turning divergent perturbative series into convergent isotopic forms, which is a primary objective for the achievement of a convergent perturbative theory for strong interactions (Ch. II.11).

In fact, *when the conventional Dyson-Feynman perturbation series is divergent, there always exist an isotopic lifting under which it becomes convergent.* Intriguingly, this appears to be the case particularly for the isotopies of quark theories (Vol. III).

To properly handle the integrals in Eq.s (12.6.1), Mignani [loc. cit.] introduces the *isochronological ordering operator* defined for an arbitrary operator A(t) via the following isotopy of the conventional definition

$$\hat{P}\{A(t_1) * A(t_2)\} = \begin{cases} A(t_1) * A(t_2) & \text{if } t_1 > t_2 \\ A(t_2) * A(t_1) & \text{if } t_1 < t_2 \end{cases} \qquad (12.6.2)$$

Then we can formally write

$$\hat{U}(t, t_0) = \hat{P}\{\exp_{|\xi}\{-i\int_{t_0}^{t} dt' H(t')\}. \qquad (12.6.3)$$

The *isomagnus expansion* [6] is similarly given by

$$\hat{U}(t, t_0) = e_{|\xi}^{A(t, t_0)}, \qquad (12.6.4)$$

where

$$A(t, t_0) = (-i)\int_{t_0}^{t} dt' H(t') + (-i)^2/2 \int_{t_0}^{t} dt' \int_{t_0}^{t''} dt'' [H(t')\hat{,}H(t'')] +$$

$$+ (-i)^3/4 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' [H(t')\hat{,}[H(t')\hat{,}H(t''')]] +$$

$$+ (-i)^3/12 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \int_{t_0}^{t''} dt''' [[H(t')\hat{,}H(t'')]\hat{,}H(t''')] + \dots , (12.6.5)$$

where the commutator is evidently the isotopic form ATB − BTA. The truncation of the isomagnus series at any order then confirms the isounitarity of the time-evolution operator. Again, if the conventional Magnus series diverges, there

always exist isotopies under which it becomes convergent.

Another useful series is the *iso-Weil–Norman expansion* [6]. Suppose that the Hamiltonian admits the finite expansion

$$H(t) = \sum_{k=1,m} \hat{a}(t) * H_k \tag{12.6.6}$$

where H_k is independent of time and the $\hat{a}$'s are certain isoscalar functions.

Then, the isotopic time evolution operator can be written

$$\hat{U}(t, t_o) = \prod{}^{*}_{k=1,n} \exp_{|\xi} \{ - i\, g_k(t) * H_k \}, \tag{12.6.7}$$

where the H's are generators of an n–dimensional Lie–isotopic symmetry and the g's are certain nonlinear functions.

The importance of the perturbative expansions for the isotopic time evolution operator is that they permit the identification of corresponding expansion for the $\hat{S}$–matrix,

$$\hat{S} = \sum_k \hat{S}_n = \sum_n \hat{U}_n(+\infty, -\infty) \tag{12.6.8}$$

The selection of isotopic elements such that $| T | \ll 1$ then permits the truncation of the series to the initial terms, thus permitting meaningful approximations also for strong interactions.

12.7: MULTI-CHANNEL TIME-DEPENDENT NONPOTENTIAL SCATTERING THEORY

Consider any channel of reaction (II.12.4.1), denote it with the letter a and suppose that it is composed of two clusters C_1 and C_2. Let R_a be the center–of–mass coordinates of the clusters C_1 and C_2. We then suppose that the total Hamiltonian H can be split into two part, one $H^0{}_a$ when the two fragments C_1 and C_2 are sufficiently far apart to be noninteracting, and the remainder H'_a,

$$H^0{}_a = \mathrm{Lim}_{|R_1 - R_2| \Rightarrow \infty} H, \quad \dot{H}''_a = H - H^0{}_a. \tag{12.7.1}$$

Consider now the total isohilbert space $\mathfrak{H}$ as that characterized by the isoschrödinger's equation in H, with the isohilbert subspace $\mathfrak{H}_a$ being characterized by the particular equation

$$i\,\hat{1}_t \frac{\partial}{\partial t} | \hat{\psi}_a(t) > = H^0{}_a * | \hat{\psi}_a(t) >, \tag{12.7.2}$$

with underlying isoinner product[137]

---

[137] We should indicate that the isotopic elements T and G are interchanged in ref. [6].

$$< \hat{\psi}_a(t) \, | \, o \, | \, \hat{\psi}_b(t) > \hat{1} \, \epsilon \, \hat{C}_T \,, \tag{12.7.3}$$

where the isotopic element T and G can be decomposed into tensorial products of the type $T = T^o_a \times T'$ whenever needed. Note that different isohilbert spaces $\mathcal{JC}_a$ and $\mathcal{JC}_b$ corresponding to different channels a and b are not necessarily isoorthogonal.

The isoprojection operator P_a of $\mathcal{JC}$ into $\mathcal{JC}_a$ (Sect. 1.6.4) is characterized by

$$P_a * \mathcal{JC} = \mathcal{JC}_a, \quad P_a * P_a = P_a \quad P\dagger_a = P_a, \tag{12.7.4}$$

and it is given explicitly by

$$P_a = \sum_k |\hat{a}_k> <\hat{a}_k| G \, T^{-1} \,, \tag{12.7.5}$$

where $\hat{a}_k$ is an iso–orthogonal basis of $\mathcal{JC}_a$. Note that when the channel a coincides with the entire reaction and $\hat{a}$ is the isobasis of the entire space, we have $P = T^{-1} \equiv \hat{1}$.

Following Mignani [6], we now introduce the *retarded* $\hat{G}^-_a$ and *advanced* $\hat{G}^+_a$ *isogreen functions* or *isopropagators* for the a–channel via the equations

$$(i \hat{1}_t \frac{\partial}{\partial t} - H^o_a*) \hat{G}^\pm_a = \hat{\delta}_1(t), \tag{12.7.6a}$$

$$\hat{G}^+_a(t) = 0, \quad t < 0, \quad \hat{G}^-_a(t) = 0, \quad t > 0 \,, \tag{12.7.6b}$$

where $\hat{\delta}_1(t)$ is the isodelta function of the first type as per Definition 1.6.3.1.[138]

The isopropagators are isotopic time evolution operators in the sense that

$$|\hat{\psi}_a(t')> \, = \, + (-) \, i \, \hat{G}^{+(-)}_a(t' - t) * |\hat{\psi}_a(t)> \,. \tag{12.7.7}$$

By continuing the isotopy of the conventional case, we introduce now the *incoming* and *outgoing isostates* in the a–channel via the expression

$$|\hat{\psi}^{in}_a(t) > \, = \, \mathrm{Lim}_{t' \Rightarrow -\infty} \, i \, \hat{G}^+_a(t - t') * |\hat{\psi}_a(t')> \,. \tag{12.7.8a}$$

$$|\hat{\psi}^{out}_a(t) > \, = \, \mathrm{Lim}_{t \Rightarrow +\infty} \, - i \, G^-_a(t - t') * |\hat{\psi}_a(t')> \,. \tag{12.7.8b}$$

Denote with $\hat{\psi}^+$ and $\hat{\psi}^-$ the solutions of the general isoschrödinger's equation of reaction (11.12.4.1) for the infinite past and future, respectively. An isotopy of the conventional treatment then leads to the property

$$|\hat{\psi}^\pm(t) > \, \equiv \, |\hat{\psi}_a(t) > + \int_{-\infty}^{+\infty} dt' \, \hat{G}^\pm_a(t - t') * \hat{H}_a * |\hat{\psi}^\pm(t')> \,. \tag{12.7.9}$$

---

[138] We should note that the isopropagators $\hat{G}^\pm_a$ and isodelta function $\hat{\delta}_1(t)$ used in this volume differ than those of ref. [6].

To reach a formal solution of the preceding equation, we introduce the expressions

$$| \hat{\Phi}^{+}(t) > \; = \; \mathrm{Lim}_{t' \Rightarrow -\infty} \; i \, \hat{G}^{+}_{a}(t - t') * | \hat{\Phi}^{in}_{a}(t') > \, , \qquad (12.7.10a)$$

$$| \hat{\Phi}^{-}(t) > \; = \; \mathrm{Lim}_{t \Rightarrow +\infty} \; - i \, G^{-}_{a}(t - t') * | \hat{\Phi}^{out}_{a}(t') > \, . \qquad (12.7.10b)$$

Then, we can formally write

$$| \hat{\Phi}^{+}(t) > \; \equiv \; | \hat{\Phi}^{in}_{a}(t) > + \int_{-\infty}^{+\infty} dt' \; \hat{G}^{+}_{a}(t - t') * \hat{H}_{a} * | \hat{\Phi}^{in}(t') > \, . \quad (12.7.11)$$

$$| \hat{\Phi}^{-}(t) > \; \equiv \; | \hat{\Phi}^{out}_{a}(t) > + \int_{-\infty}^{+\infty} dt' \; \hat{G}^{-}_{a}(t - t') * \hat{H}_{a} * | \hat{\Phi}^{\pm}(t') > \, . \quad (12.7.12)$$

Approximate, convergent solution can then be found via the isoexpansions of the preceding section.

12.8: MULTICHANNEL ISOMØLLER OPERATOR

By continuing to follow ref. [6], the *isomøller operators* for the a–channel can be defined by

$$| \hat{\Phi}^{\pm}(t) > \; = \; \hat{\Omega}^{\pm}_{a} * | \hat{\Phi}_{a}(t) > \, . \qquad (12.8.1)$$

Note that $\hat{\Omega}^{\pm}_{a}$ acts in $\mathcal{H}_{a}$ and, therefore,

$$\hat{\Omega}^{\pm}_{a} * P_{a} \; = \; \hat{\Omega}^{\pm}_{a} \, . \qquad (12.8.2)$$

By again conducting the isotopy of the conventional case, the preceding expressions allow the explicit forms

$$\hat{\Omega}^{+(-)}_{a} \; = \; P_{a} - (+) \int_{-\infty}^{+\infty} dt \; \hat{G}^{+(-)}(-t) * H'_{a} * \hat{G}^{-(+)}_{a}(t) * P_{a} \, , \qquad (12.8.3)$$

and

$$\hat{\Omega}^{+}_{a} \; = \; \mathrm{Lim}_{t \Rightarrow -\infty} \; \hat{G}^{+}(-t) * \hat{G}^{-}_{a}(t) \, , \qquad (12.8.4a)$$

$$\hat{\Omega}^{-}_{a} \; = \; \mathrm{Lim}_{t \Rightarrow +\infty} \; \hat{G}^{-}(-t) * \hat{G}^{+}_{a}(t) \, . \qquad (12.8.4b)$$

Denote with $\hat{\mathfrak{R}}^{\pm}_{a}$ the range of the corresponding isomøller's operators, and let $Q^{\pm}_{a}$ be the corresponding projection operators. Then, via an isotopy of the corresponding conventional properties, the following isotopic expressions hold

$$Q^{\pm}_{a} * Q^{\pm}_{a} \; \equiv \; 0 \, , \qquad \hat{\Omega}^{\dagger \, \pm}_{a} * \hat{\Omega}^{\pm}_{a} \; = \; P_{a} \, , \qquad (12.8.6a)$$

$$\hat{\Omega}^{\pm}_{a} * \hat{\Omega}^{\dagger \, \pm}_{a} \; = \; Q_{a} \, , \qquad \hat{\Omega}^{\dagger \pm}_{a} * \hat{\Omega}^{\pm}_{b} \; = \; P_{a} * \hat{\delta}_{ab} \, , \qquad (12.8.6b)$$

where $\hat{\delta}_{ab}$ is the isokronecker delta.

A further property of the isomøller operators is that

$$H * \Omega^{\pm}_a = \Omega^{\pm}_a * H^0_a .$$
(12.8.7)

Introduce now the projection operator $\hat{C}$ i to a given cluster orthogonal to a. Then

$$Q^{\pm}_a * \hat{C} = 0,$$
(12.8.8)

and

$$\hat{C} + \sum_a Q^{\pm}_a = 1 = T^{-1} .$$
(12.8.9)

By using the preceding properties, the (strong) limit then follow

$$Lim_{t\Rightarrow+\infty} | \hat{\psi}^{+}(t) > = | \hat{\psi}^{out}(t) > ,$$
(12.8.10)

We reach in this way the following additional expression for the nonpotential scattering matrix

$$\sum \hat{\psi}^{out}(t) > = \sum_b \hat{S}_{ba} | \hat{\psi}_a(t) > ,$$
(12.8.11)

$$\hat{S}_{ba} = \hat{\Omega}\hat{\dagger}^{-}_b * \Omega^{+}_a ,$$

with the properties

$$\hat{S}_{ba} * H^0_a = H^0_b * \hat{S}_{ba} .$$
(12.8.12)

The elements of the $\hat{S}$–matrix as defined above are the n given by

$$\hat{S}_{fi} = < \hat{\psi}_{b,} (0) | G \hat{S}_{ba} T | \hat{\psi}_{a,i}(0) > .$$
(12.8.13)

In different terms, the multichannel nonpotential scattering theory has a number of scattering operators $\hat{S}_{ba}$ much along the conventional case. The set of these scattering operators can be arranged into a matrix, but they are not the component of a true matrix because the eigenstates are not necessarily orthogonal.

Despite that, the overall scattering operator is indeed isounitary. This can be verified via the expressions

$$\sum_b \hat{S}_{ba} * \hat{S}\dagger_{cb} = \hat{\Omega}\hat{\dagger}^{-}_a * (1-\hat{C}) * \Omega^{-}_c = \hat{\Omega}\hat{\dagger}^{-}_a * \Omega^{-}_c = P_a * \hat{\delta}_{ab} ,$$
(12.8.14a)

$$\sum_b \hat{S}\hat{\dagger}_{ba} * \hat{S}_{cb} = \hat{\Omega}\hat{\dagger}^{+}_a * (1-\hat{C}) * \Omega^{+}_c = \hat{\Omega}\hat{\dagger}^{+}_a * \Omega^{+}_c = P_a * \hat{\delta}_{ab} ,$$
(12.8.14b)

from which the isounitarity readily follows.

12.9: TIME INDEPENDENT NONPOTENTIAL SCATTERING THEORY

By following again Mignani [6], in this section we shall review the foundations of the time–independent multichannel theory which can be constructed from the time–dependent one via an isotopy of conventional treatments [1].

The fundamental notion is the *time–independent isogreen function* or *propagator* for the a–channel

$$\hat{G}^{\pm}_a = \text{Lim}_{\eta \Rightarrow +0} (\hat{E}^{\pm} - H^o_a)^{-1}, \qquad (\hat{E} - H^o_a) * \hat{G}^{\pm}_a = 1, \qquad (12.9.1)$$

where

$$\hat{E}^{\pm} = (E \pm \eta) 1 \in \hat{\xi}_T, \qquad (12.9.2)$$

with corresponding expression for the full isotopic operator

$$\hat{G}^{\pm} = \text{Lim}_{\eta \Rightarrow +0} (\hat{E}^{\pm} - H)^{-1}, \qquad (\hat{E} - H) * \hat{G}^{\pm} = 1. \qquad (11.9.3)$$

Let $| \hat{\psi}_a >$ and $| \hat{\psi} >$ the solution for the stationary isoscrödinger's equation. Then, we can introduce the *iso–Lippmann–Schwinger equations* for the a–channel

$$| \hat{\psi}^{\pm} > = | \hat{\psi}_a > + \hat{G}^{\pm}_a * H'_a * | \hat{\psi}^{\pm} > = | \hat{\psi}_a > + \hat{G}^{\pm} * H'_a * | \hat{\psi}_a >, \qquad (12.9.4)$$

where $| \hat{\psi}^{\pm} >$ are the time–independent incoming and outgoing isostates.

The operator $\mathfrak{F}_{ba}$ for the transition from the a–channel to the b–channel can then be expressed by the forms

$$\mathfrak{F}_{\overline{b}a} = H'_b + H'_b * \hat{G}^+ * H'_a = H'_b + \mathfrak{F}_{\overline{b}a} * \hat{G}^+ * H'_a =$$

$$H'_b + H'_b * \hat{G}^+_a * \mathfrak{F}_{aa} = H'_b + \mathfrak{F}_{bb} * \hat{G}^+_b * H'_a, \qquad (12.9.5a)$$

$$\mathfrak{F}_{b\overline{a}} = H'_a + H'_b * \hat{G}^+ * H'_a = \mathfrak{F}_{\overline{b}a} + (H'_a - H'_b) = H'_a + H'_b * \hat{G}^+_b * \mathfrak{F}_{b\overline{a}}, \qquad (12.9.5b)$$

with corresponding matrix elements

$$\mathfrak{F}_{fi} = < \hat{\psi}_{b,f} | G \mathfrak{F}_{\overline{b}a} T | \hat{\psi}_{a,i} > =$$

$$= < \hat{\psi}_{b,f} | G H'_b T | \hat{\psi}^+_i > = < \hat{\psi}^-_f | G H'_a T | \hat{\psi}_{a,i} >. \qquad (12.9.6)$$

As it was the case for the multichannel $\hat{S}$–"matrix", the $\mathfrak{F}$–operator too can be arranged into a matrix form but not necessarily is a true matrix.

If a Born series expansion is needed, we have the isotopy of the conventional forms

$$\mathfrak{F}_{\overline{b}a} = H'_b * \sum_k (\hat{G}^+_a * H'_a)^k, \qquad (12.9.7)$$

which, again, can be convergent when the original conventional series diverge.

For the formulation of the theory via two isohilbert spaces and other aspects, we refer the interested reader to ref. [6].

12.10: NONPOTENTIAL CROSS SECTIONS

We now use the results of the preceding sections to provide a more detailed formulation of the differential and total nonpotential cross sections. The analysis was first conducted by Mignani in ref. [4,5,6] without isoplane–waves, and then reformulated by this author in ref. [7] with the notion of isoplane–waves and the use of the full iscoschrödinger's equation.

Let us now assume that the term H' represents a conventional potential in isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{\Re})$

$$ H' = V(r) . \tag{12.10.1} $$

Then, the isogreen functions can be expressed in the form

$$ \hat{G}^{+}{}_{0}(r, r') = \exp_{|\xi} [i k (r - r')] / 2 \pi (r, r') = $$

$$ = \hat{1} \exp [i k T (r - r')] / 2 \pi (r - r') , \quad (\hbar = 1) , \tag{12.10.2} $$

by therefore resulting to have the structure of an isoplane–wave, as necessary for the $\hat{G}$–quantities to be solutions of (11.12.9.4).

The outgoing isostate has then the structure

$$ | \hat{\psi}^{+}(r) > = \hat{\psi}^{0}(r) - (m/2\pi) \int d^3r' \frac{\exp_{|\xi}[i k (r - r')}{r - r'} * V(r') * \hat{\psi}^{+}(r') = $$

$$ = \hat{\psi}^{0}(r) - (m/2\pi) \int d^3r' \frac{\exp[i kT(r - r')]}{r - r'} V(r') * \hat{\psi}^{+}(r') $$

$$ \cong \hat{\psi}^{0}(r) - (m/2\pi) \frac{\exp (i kTr)}{r} \int d^3r' \, \exp (-i kT r') V(r') * \hat{\psi}^{+}(r') . \tag{12.10.3} $$

Introduce now the isospherical coordinates in $\hat{E}(r,\hat{\delta},\hat{\Re})$

$$ x = r \, b_1^{-1} \cos \hat{\theta} \text{ con } \hat{\phi}, \quad y = r \, b_2^{-1} \cos \theta \sin \hat{\phi}, \quad z = r \, b_3^{-1} \sin \hat{\theta} , \tag{12.10.4a} $$

$$ \hat{\phi} = \phi \, b_1 \, b_2 \quad \hat{\theta} = \theta \, b_3 . \tag{12.10.4b} $$

We the have the following

> **Definition 12.10.1 [6,7]:** *The "nonpotential scattering amplitude" in isospherical coordinates on isospace* $Ê(r,ŝ,Ř)$ *is given by*[139]

$$\hat{T}(θ, \hat{φ}) = - (m/2π) \int d^3r' \exp(ikTr') V(r') T \hat{ψ}(r') =$$

$$= - (m/2π) < \hat{ψ} \mid G \, V(r) \, T \mid \hat{ψ}^+ > , \qquad (12.10.5)$$

$$< \hat{ψ} \mid G \mid \hat{ψ} > = 1 .$$

The generalization of the treatment of Sect. II.12.2 is now evident. In fact, we have the full isospherical dependence indicating the deviations from both standard scattering angles $θ$ and $φ$. Note that, since the measures are external to the scattering region, the isotopic element T must be averaged to constant $T°$, and we can write

$$\hat{T}(θ, \hat{φ}) = - (m/2π) \, T° \int d^3r' \exp(i \, k \, T \, r') \, V(r') \, \hat{ψ}(r') . \qquad (12.10.6)$$

This illustrates how nonlinear–nonlocal–nonpotential effects which are internal to the scattering region, i.e. at mutual distances of the order of 1 fm, do indeed affect the numerical values of the measures at macroscopic distances.

The elements of the $\mathfrak{F}$–matrix are then given by

$$\mathfrak{F}_{k, k_0} = < \hat{ψ}°_k \mid G \, V \, T \mid \hat{ψ}^+_{k_0} = - (2π/m) \, \hat{T}(k, k - k_0) \hat{1} , \qquad (12.10.7)$$

where the arguments k and $k - k_0$ are now defined in isospace $Ê(r,ŝ,Ř)$.

The total nonpotential cross section can then be computed with any of the isotopies of the various conventional methods [1].

Extension to the nonpotential superscattering theory can be found in ref. [8]. Applications of the novel scattering theory are presented in Vol. III.

---

[139] Quantity (12.10.5) was first defined in ref. [6] but according to the expression in *conventional* spherical polar coordinates expressed with respect to the conventional plane–waves $\exp(i \, k \, r)$,

$$\hat{T}(θ, φ) = (m/2π) \int d^3r' \exp(ikr') V(r') T \hat{ψ}^+(r') , \qquad (a)$$

thus implying

$$\hat{T}(θ, φ) = T \, f(θ, φ), \qquad (b)$$

where $f(θ, φ)$ is the usual expression. Note, by comparison that in Definition 12.10.1 $\hat{T}(θ,\hat{φ}) \neq T \, f(θ, φ)$.

REFERENCES

1. W. S. C.WILLIAMS, *An Introduction to Elementary Particles,* Academic Press, New York (1971); E. PRUGOVECKI, *Quantum Mechanics in Hilbert space,* Academic Press, New York (1981)
2. R. M. SANTILLI, Hadronic J. 1, 574 (1978)
3. R. MIGNANI, Lett. Nuovo Cimento 39, 406 (1984)
4. R. MIGNANI, Lett. Nuovo Cimento 43, 355 (1985)
5. R. MIGNANI, Toward a nonpotential scattering theory, Univ. of Rome preprint no. 454 (1985)
6. R. MIGNANI, Hadronic J. 9, 103 (1986)
7. R. M. SANTILLI, Hadronic J. Suppl. 4B, issue no. 2 (1989)
8. A. BARTOLI and S. BERGIA, Nuovo Cimento 108B, 227 (1993)
9. S. HAWKINS, Comm. Math. Phys .87, 395 (1982)
10. J. ELLIS, N. E. MAVROMATOS and D. V. NANOPOULOS, CERN preprint TH.7195/94 (revised version), and in *Erice Summer School, 31-st Course: From Superstrings to the Origin of Space-time,* Erice, July 1993, World Scientific, Singapore

INDEX

ABOUT THE AUTHOR

Ruggero Maria Santilli was born and educated in Italy where he received his Ph. D. in theoretical physics in 1966 from the University of Torino. In 1967 he moved with his family to the USA where he held academic positions in various institutions including the Center for Theoretical Physics of the University of Miami in Florida, the Department of Physics of Boston University, the Center for Theoretical Physics of the Massachusetts Institute of Technology, the Lyman Laboratory of Physics and the Department of Mathematics of Harvard University. He is currently President and Professor of Theoretical Physics at The Institute for Basic Research, which operated in Cambridge from 1983 to 1991 and then moved to Florida. Santilli has visited numerous academic institutions in various Countries. He is currently a Honorary Professor of Physics at the Academy of Sciences of the Ukraine, Kiev, and a Visiting Scientist at the Joint Institute for Nuclear Research in Dubna, Russia. Besides being a referee for various journals, Santilli is the founder and editor in chief of the *Hadronic Journal* (sixteen years of regular publication), the *Hadronic Journal Supplement* (nine years of regular publication) and *Algebras, Groups and Geometries* (eleven years of regular publication). Santilli has been the organizer of the five *International Workshops on Lie-admissible Formulations* (held at Harvard), the co-organizer of five *International Workshops on Hadronic Mechanics* (held in the USA, Italy and Greece) and of the *First International Conference on Nonpotential Interactions and their Lie-Admissible Treatment* (held at the Université d'Orléans, France). He is the author of over one hundred and fifty articles published in numerous physics and mathematics journals; he has written nine research monographs published by Springer-Verlag (in the prestigious series of "Texts and Monographs in Physics"), the Academy of Sciences of Ukraine and other publishers; he has been the editor of over twenty conference proceedings; he is the originator of new branches in mathematics and physics, some of which are studied in these books; he has received research support from the U. S. Air Force, NASA and the Department of Energy; and he has been the recipient of various honors, including the *Gold Medals for Scientific Merits* from the Molise Province in Italy and the City of Orléans, France and the nomination by the Estonian Academy of Sciences among the most illustrious applied mathematicians of all times. Santilli has been nominated for the Nobel Prize in Physics by various senior scholars since 1985.

ABOUT THE BOOKS

These are the first books written on *Hadronic Mechanics*, which is an axiom-preserving generalization of quantum mechanics for the study of strongly interacting particles (called *hadrons*) with nonlinear, nonlocal, and nonpotential contributions due to the overlapping of wavepackets and charge distributions of the hyperdense hadrons at distances smaller than their size, as necessary to activate strong interactions. After being proposed by Santilli at Harvard University in 1978 under D.O.E. support, the new mechanics has been developed by numerous scholars, discussed at several international meetings and studied in numerous papers in physics and mathematics journals in various countries. The new mechanics is based on a generalization of the mathematical structure of quantum mechanics called of *isotopic type* in the sense of being axiom-preserving. Hadronic mechanics therefore provides a more general, nonlinear, nonlocal and noncanonical *realization* of conventional quantum mechanical axioms which can be interpreted as a form of *completion* of quantum mechanics as suggested by Einstein, Podolsky and Rosen. The unrestricted functional character of the isotopies renders the new mechanics "directly universal" for all interactions considered; the mathematical consistency of hadronic mechanics is assured by the preservation of the quantum axioms; and the significance for new applications is expressed by the broader representational capabilities, which are simply absent in quantum mechanics.

VOLUME I contains nonlinear, nonlocal and nonpotential isotopies of numbers, fields, spaces, algebras, groups, symmetries, geometries and functional analysis. VOLUME II presents a step-by-step isotopic generalization of quantum mechanical physical laws, Heisenberg's and Schroedinger's representations, Galilei's and Einstein's relativities, including the isotopies of perturbation theory, and of the scattering theory for inelastic collisions with nonlinear, nonlocal and nonhamiltonian internal effects due to mutual penetrations of hyperdense particles. VOLUME III presents applications and experimental verifications in nuclear physics, particle physics, statistical physics, astrophysics, gravitation, cosmology, antimatter physics, superconductivity and other fields such as theoretical biology and conchology. These books are recommended to all scientists.

ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE FOR THEORETICAL PHYSICS